BA4201 QUANTITATIVE TECHNIQUES FOR DECISION MAKING **L T P C**
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COURSE OBJECTIVE:

- To apply quantitative techniques in modeling and solving business related problems.

UNIT I INTRODUCTION TO LINEAR PROGRAMMING (LP) 9

Relevance of quantitative techniques in management decision making. Linear Programming formulation, solution by graphical and simplex methods (Primal - Penalty, Two Phase), Special cases. Sensitivity Analysis.

UNIT II LINEAR PROGRAMMING EXTENSIONS 9

Transportation Models (Minimising and Maximising Problems) – Balanced and unbalanced Problems – Initial Basic feasible solution by N-W Corner Rule, Least cost and Vogel's approximation methods. Check for optimality. Solution by MODI / Stepping Stone method. Case of Degeneracy. Transhipment Models.

Assignment Models (Minimising and Maximising Problems) – Balanced and Unbalanced Problems. Solution by Hungarian and Branch and Bound Algorithms. Travelling Salesman problem. Crew Assignment Models.

UNIT III DECISION AND GAME THEORIES 9

Decision making under risk – Decision trees – Decision making under uncertainty. Game Theory-Two-person Zero sum games-Saddle point, Dominance Rule, Convex Linear Combination (Averages), methods of matrices, graphical and LP solutions.

UNIT IV INVENTORY AND REPLACEMENT MODELS 9

Inventory Models – EOQ and EBQ Models (With and without shortages), Quantity Discount Models. Replacement Models-Individual replacement Models (With and without time value of money) – Group Replacement Models.

UNIT V QUEUING THEORY AND SIMULATION 9

Queuing Theory - single and multi-channel models – infinite number of customers and infinite calling source. Monte Carlo simulation – use of random numbers, application of simulation techniques

TOTAL: 45 PERIODS

COURSE OUTCOMES:

To understand the applications of

1. Linear programming in product mix decisions
2. Transportation and assignment in logistics and job allocation scenarios
3. Game theory and heuristics of decision making in real time decisions
4. Inventory management and replacement models in manufacturing context
5. Queuing and simulation in real time scenario optimization.

REFERENCES:

1. N. D Vohra, Quantitative Techniques in Management, Tata Mcgraw Hill, 2010.
2. G. Srinivasan, Operations Research – Principles and Applications, 2^{nd} edition, PHI, 2011.
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5. Bernard W.Taylor III, Introduction to Management Science, 9th Edition, Pearson Ed.
6. Frederick & Mark Hillier, Introduction to Management Science – A Modeling and case studies approach with spreadsheets, Tata Mcgraw Hill, 2010.
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Unit 1

Introduction to Linear Programming

1.1. QUANTITATIVE TECHNIQUES /OPERATIONS RESEARCH

1.1.1. Introduction

Quantitative Techniques (QT) is a discipline that deals with the application of advanced analytical methods to help in make better and improved decision. It is a systematic study of basic structure, characteristics, functions and relationships of an organisation, and provides a basis to managers for improved decision-making.

QT takes a scientific approach to best decide, how to design and operate man-machine systems, for industrial use. In other words, QT deals with optimal resource allocation. Most of the actual work is done using analytical and numerical techniques that helps to develop and manipulate mathematical models of organisational systems.

Quantitative Techniques is also known as **Decision Science** or **Operations research.** Unlike many other disciplines, that focuses on technology, OR is an interdisciplinary mathematical study that focuses on the effective use of technology by organisations.

Quantitative Techniques arrives at optimal or near-optimal solutions to complex decision-making problems, by employing techniques like mathematical modelling, statistical analysis, and mathematical optimisation. QT is basically helpful enhancement to judgment and intuition. Quantitative techniques assess planning factors and alternatives as and when they arise rather than suggest courses of action.

Quantitative techniques may be defined as those techniques which provide the decision maker with a systematic and powerful means of analysis and help, based on quantifiable data, in exploring policies for achieving pre-determined goals. Quantitative techniques are mainly appropriate to problems of complex business enterprises.

QT can be considered as the scientific approach to managerial decision making. This approach starts from raw data and after manipulation or processing, information is produced which is valuable for making decision. The main aim of quantitative analysis is the processing and manipulating of raw data into meaningful information. For various use of quantitative analysis, computer can be used as an instrument.

According to C.R. Kothari "Quantitative Techniques may be defined as those technique which provide the decision maker with a systematic and powerful means of analysis and help, based on quantity in exploring policies for achieving pre-determined goals".

Quantitative Techniques are the devices developed on the basis of mathematical and statistical models.

1.1.2. Features of Quantitative Techniques

The broad features of quantitative approach to any decision problem are summarized as follows:

1) **Decision-Making:** Decision-making or problem solving constitutes the major working of operations research. Managerial decision-making is considered to be a general systematic process of operations research (OR).

2) **Scientific Approach:** Like any other research, operations research also emphasises on the overall approach and takes into account all the significant effects of the system. It understands and evaluates them as a whole. It takes a scientific approach towards reasoning. It involves the methods defining the problem, its formulation, testing and analysing of the results obtained.

3) **Objective-Oriented Approach:** Operations Research not only takes the overall view of the problem, but also endeavours to arrive at the best possible (say optimal) solution to the problem in hand. It takes an objective-oriented approach.

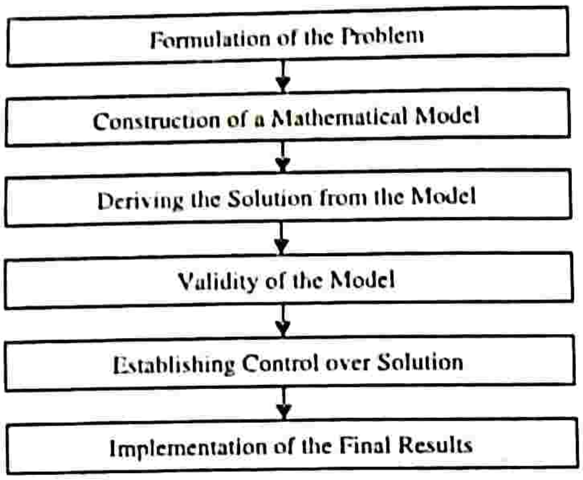
   To achieve this, it is necessary to have a defined measure of effectiveness which is based on the goals of the organisation. This measure is then used to make a comparison between alternative solutions to the problem and adopt the best one.

4) **Inter-Disciplinary Approach:** No approach can be effective, if applied individually. OR is also inter-disciplinary in nature. Problems are multi-dimensional and approach needs a team work.

   For example, managerial problems are affected by economic, sociological, biological, psychological, physical and engineering aspect. A team that plans to arrive at a solution, to such a problem, needs people who are specialists in areas such as mathematics, engineering, economics, statistics, management, etc.

1.1.3. Quantitative Analysis Process

Following are the six steps towards problem solving (figure below):



Step 1: Formulation of the Problem: First of all, a manager should be able to form an appropriate model of the problem, so as to arrive at a solution.

Step 2: Construction of a Mathematical Model: The second step is to build a mathematical model, which represents the system under study using variables.

Step 3: Deriving the Solution from the Model: An effective computation of all the decision variables that constitutes the problem is needed to maximise or minimise an objective function is required. Such solution is called an **optimal solution.**

Step 4: Validity of the Model: Every model needs a validation for accuracy. A model can be valid or accurate if:

1) All the objectives, constraints and decision variables included in the model and are relevant to the problem or are a part of it, and
2) It has valid functional relationships.

Step 5: Establishing Control over Solution: The immediate next step after arriving at a solution, is to exercise and establish control over it. It requires enforcing feedback on those variables, which actually have tendency to deviate from the acceptable regime considerably.

Step 6: Implementation of the Final Results: In the end, the final results of the model are put to work. Careful explanation of the adopted solution, and its relationship with the functional realities should be considered.

1.1.4. Relevance of Quantitative Techniques in Management Decision Making

The quantitative approach to decision making is assuming an increasing degree of importance in the theory and practice of management. The factors that are responsible for this development are:

1) Decision problems of management are so complex that only a conscious, systematic and scientifically based analysis can yield realistic solution,
2) Availability of well-structured quantitative models and methods that are available for solving these complex managerial problems,
3) Attitude of accumulating scientific knowledge in the management of organisations, and
4) Availability of computer software to apply quantitative models to real-life problems.

Hence, if decision-makers are totally or fully utilise the potentials of quantitative models, then the decision problem be defined, analysed and solved in a concision, rational, logical, systematic and scientific manner based on the data, facts, information and logic and not on guess.

The quantitative approach does not preclude the qualitative or judgemental elements that always put a substantial influence on managerial decision making. Rather, the quantitative approach must build upon, be modified by and continually benefit from the experiences and creative insights of business' managers. The quantitative approach attempts to cultivate a managerial style that demands a conscious, systematic, and scientific analysis – and resolution of decision problems.

The symbolic relationship between qualitative and quantitative models is shown in **figure 1.1:**

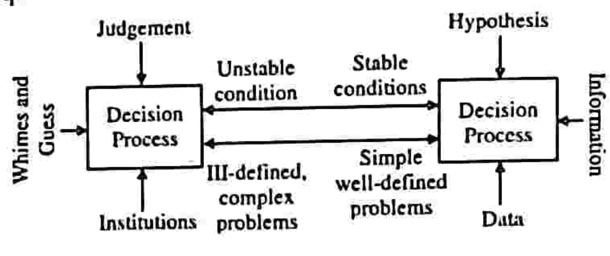


Figure 1.1: Symbiotic Relationship between Quantitative and Qualitative Models

In general, while solving a real life problem, the decision-maker must examine it both from quantitative as well as qualitative perspective. Information about the problem from both these perspectives needs to be brought together and assessed in the context of the problem. Based on some mix of the two sources of information, a decision should be taken by the decision-maker. **Figure 1.2** illustrates this in a simplistic way:

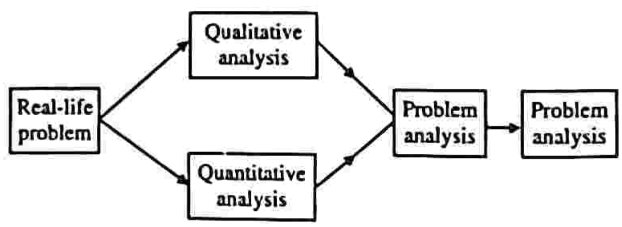


Figure 1.2: Decision making

Figure 1.3 shows a flow chart of scientific procedure to arrive at a decision where it is possible to follow only the left-hand path (i.e., describe the data) in order to gather data in a qualitative manner.

For example, instead of claiming that most customers would buy the product if it is advertised on TV, it would be better to ask how many items customers would buy if the product price is also advertised on TV. Such questions implies that we intend to measure quantities (i.e., number of items bought at a certain price) and are using quantitative methods. Consequently, observations can be described in much detail. Also, we may follow, right-hand path (i.e., analyse the data) in order to gather data in quantitative manner to understand that a large number of people buy the product only if it is advertised on TV. As it is known quantitative methods not only assist in decision-making but also help in arriving at a better decision. The quantitative approach to decision-making uses concepts and or tools (methods) of mathematics and statistics. The commonly used terms for quantitative approach to decision making are **Operations Research, Management Science, Decision Analysis and Decision science.**

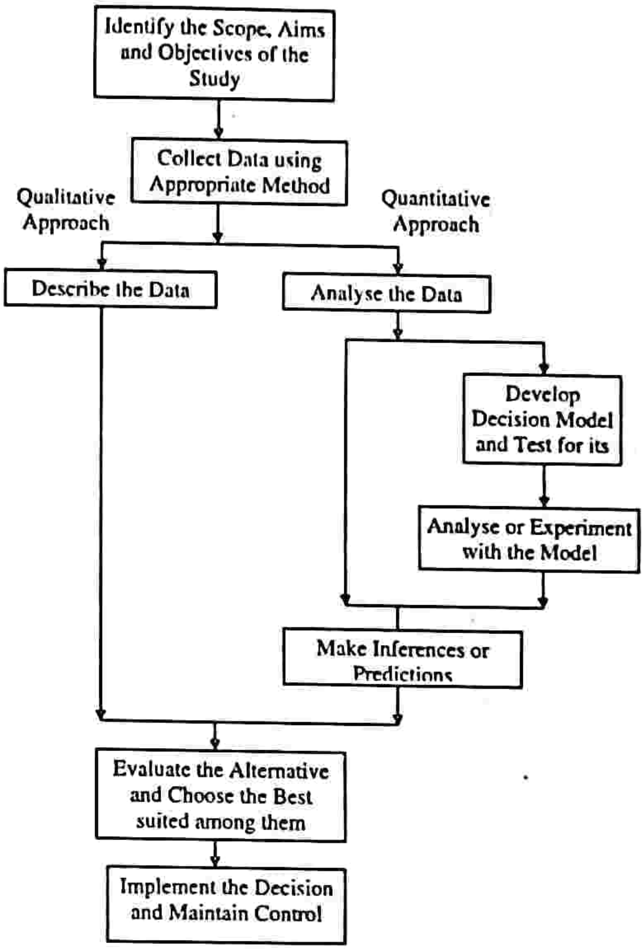


Figure 1.3: An Iconic Model of Data Analysis

The major roles of quantitative techniques in decision making are as below:

1) It provides a tool for scientific analysis.
2) It offers solutions for various business problems
3) It enables proper deployment of resources.
4) It supports in minimising waiting and servicing costs.
5) It helps the management to decide when to buy and what is the procedure of buying.
6) It helps in reducing the total processing time necessary for performing a set of jobs.

1.1.5. Advantages of Quantitative Techniques/Operations Research

1) **Better Control:** For large organisations, it is practically impossible to continuously supervise every routine work. OR approach comes handy and gives an analytical and quantitative basis to identify the problem area. OR approach is most frequently adopted with production scheduling and inventory replenishment.
2) **Better Systems: For example,** Problems identifying the best location for factories or decision on whether to open a new warehouse, etc., are often been studied and analysed by OR approach. This approach helps to improve the existing system such as, selecting economical means of transportation, production scheduling, job sequencing, or replacing old machinery.
3) **Better Decisions:** OR models help in improved decision-making and thereby reduce the risk of wrong decisions. OR approach gives the executive an improved insight into the problem and thereby improves decision-making.
4) **Better Coordination:** OR models help in coordination of different or various divisions of an organisation.

1.1.6. Disadvantages of Quantitative Techniques/Operations Research

1) **Dependence on an Electronic Computer:** OR approach is mathematical in nature. OR techniques try to find out an optimal solution to a problem by taking all the factors into consideration. The need of computers become unavoidable because these factors are enormous (huge), it requires huge calculations to express them in quantity and to establish relationships among them.
2) **Non-Quantifiable Factors:** One of the drawbacks of OR techniques is that they provide a solution only when all the elements related to a problem are quantified. Since all relevant variables may not be quantified, they do not find a place in OR models.
3) **Wrong Estimation:** Certain assumptions and estimates are made for assigning quantitative values to factors involved in OR, so that a quantitative analysis can be done. If such estimates are wrong, the result can be misleading.
4) **Involves Time and Cost:** Operations research is a costly affair. An organisation needs to invest time, money and effort into OR to make it effective. Professionals need to be hired to conduct constant research. For better research outcomes, these professionals must constantly review the rapidly changing business scenarios.
5) **Implementation:** The complexities of human relations and behaviour must be taken into account while implementing OR decisions, as it is a very delicate task.

1.1.7. Scope/Applications of Quantitative Techniques/Operations Research

Operation Research can be applied to different areas of business such as:

1) **Industry:** Industrial management deals with a series of problems, starting right from the purchase of raw materials till the dispatch of final products. The management is ultimately interested in overall understanding of the methods, of optimising profits. Therefore, to take decision on scientific basis, operations research team has to think about various alternative methods, to produce goods and obtaining returns in each case.

   Not only this, the operations research study should also suggest possible changes in the overall structure like installation of a new machine or introduction to automation, etc., for optimising the results. Many industries have gained immensely by applying operations research in various tasks. **For example,** operations research can be used in the fields of manufacturing and production, blending and product mix, inventory management, for forecasting demand, sale and purchase, for repair and maintenance jobs, for scheduling and sequencing planning, and also for scheduling and control of projects.

2) **Developing Economics:** OR is applicable to both developing and developed economies. But a lot of scope exists in developing economies, for building up an operations research approach towards planning. The basic idea is to orient the planning, to achieve maximum growth per capita income in minimum time; considering the goals and restrictions of the country. Poverty and hunger are the core problems faced by many countries of Asia and Africa. Therefore, people like statisticians, economists, technicians, administrators, politicians and agriculture experts can work in conjunction, to solve this problem with an operations research approach.

3) **Agriculture Industry:** Operations research approach has a huge scope in agriculture sector. Population explosion has led to scarcity of food. Optimum allocation of land for various crops in accordance with climatic conditions is a challenge for many countries. Also, each developing country is facing the problem of optimal distribution of water from several water bodies. These areas of concern hold a great scope for scientific research.

4) **Organisation:** Organisation, big or small, can adopt operations research approach effectively. Operational productivity of organisations have improved by using quantitative techniques. Techniques of operations research, can be applied to minimise cost, and maximise benefit for decisions. **For example,** a departmental store faces problem like, employing additional sales girls, or purchasing an additional van, etc.

5) **Business and Society:** Businesses and society can directly be benefited from operations research. **For example,** hospitals, clinics etc., Operations research methods can be applied directly to solve administrative problems such as minimising the waiting time of outdoor patients. Similarly, the business of transport can also be benefited by applying simulation methods. Such methods, can help to regulate train arrivals and their running timings. Queuing theory, can be applied to minimise congestion and passengers waiting time. These methods are increasingly being applied in L.I.C. workplaces. It helps in deciding the premium rates of various policies. Industries such as petroleum, paper, chemical, metal processing, aircraft, rubber, mining and textile have been extremely benefited by its use.

1.2. LINEAR PROGRAMMING (LP)

1.2.1. Introduction

Linear programming is a two-word phrase. It means:

1) The word **linear** refers to any **linear** relationship among variables in a model, means any change in one variable will result into a proportional change in other variables.

2) **Programming** refers to any problem that can be modelled and solved mathematically. There are various alternate strategies to achieve the desired objective. But in programming, economic allocation of limited resources and choosing a particular course of strategy helps in solving the problem.

An analysis of problems represented in linear function with a number of variables that is to be optimised, when subjected to a number of restraints in the form of inequalities, is called Linear Programming. A problem can be approached and solved with different strategies. Linear Programming selects the best possible strategy from the available alternatives.

Strategy is selected on the basis of maximisation or minimisation of some required output, i.e., it has to be optimal. **For example,** maximisation of output/profit or minimisation of production cost.

According to William M Fox, "Linear programming is a planning technique that permits some objective functions to be minimized or maximised within the framework of given situational restrictions."

Linear programming is a technique that works on a mathematical basis for determining the optimal solution. It takes into consideration the alternative uses of resources like man, machine, money, material, etc., to attain a particular objective. Linear programming can be applied on various problems. However, this technique can be ideally used for solving maximisation and/or minimisation problems, subject to some assumptions. **For example,** maximisation of profit/ sales or minimisation of cost.

1.2.2. Mathematical Model/General Structure of LPP

Let $x_1, x_2 \ldots x_n$ be n decision variables. Then the mathematical form of linear programming problem is as follows:

Optimize (Maximize or Minimize)

$Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$ (Objective function)

Subject to the constraints;

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \ (\leq, =, \geq) \ b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \ (\leq, =, \geq) \ b_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \ (\leq, =, \geq) \ b_n \end{array}\right\} \text{(Constraints)}$$

and $x_1, x_2, \ldots, x_n \geq 0$ (Non-negativity condition)

Where a_{ij}, b_j, c_j are constants and x_j is decision variable.

By using the symbol 'Σ', i.e., the 'sum' of notation, the above formulation may be put in the following compact form:

$$\text{Optimise (Max. or Min.) } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_j x_j$$

(Objective function) ...(1)

Subject to the linear constraints:

$$\sum_{j=1}^{n} a_{ij}x_j \ (\leq, =, \geq) \ b_j \ ; i = 1, 2, \ldots, m \qquad \text{(Constraints)} \quad \ldots(2)$$

$x_j \geq 0; j = 1, 2, \ldots, n$ (Non-negativity) ...(3)

Some Useful Definitions

Definition 1: A **feasible Solution** to the linear programming problem is a vector $X = (x_1, x_2, x_3, \ldots x_n)$ which satisfies conditions (2) and (3).

A feasible solution to a linear program is a solution that satisfies all constraints.

Definition 2: A basic solution to equation (2) is a solution obtained by setting (n – m) variables equal to zero and solving for the remaining m variables, provided that the determinant of the coefficients of these m variables is non-zero. The m non-zero variables are called **basic variables** and (n – m) zero variables are called **non-basic variables.**

For examples,

1) **Maximization Case**

Maximise $Z = 40x_1 + 35x_2$ (Profit)

Subject to

$2x_1 + 3x_2 \leq 60$ (First constraint)

$4x_1 + 3x_2 \leq 96$ (Second constraint)

$x_1, x_2 \geq 0$ (Non-negativity condition)

2) **Minimization Case**

Minimize $Z = 40x_1 + 24x_2$ (Cost)

Subject to

$20x_1 + 50x_2 \geq 4800$ (First constraint)

$80x_1 + 50x_2 \geq 7200$ (Second constraint)

$x_1, x_2 \geq 0$ (Non-negativity condition)

1.2.3. Components of Linear Programming Problem

Following are the components of a linear programming problem (LPP):

1) **Objective Function:** LPP has a component called Objective Function or Criterion function. It is a linear function and can work either by maximisation or minimisation process. It includes all possible components of a problem to optimise the solution.

2) **Decision Variables:** The objective function uses variables to reach at a solution. The variables that help to decide the outcome are called **'Decision Variables'** or **'Activity Variables'**. The level of activity for each variable is specified. However, it can specify a zero value of some variable but not a negative value. Decision (or choice) variables can be $x_1, x_2, \ldots, x_n$.

3) **Constraints:** Constraints are a part of every practical situation. The mathematical program stated in an algebraic form will also specify constraints, i.e., the limited availability of resources, if it is a maximisation problem or a minimum quality or composition, if it is a minimisation problem. Thus, every problem will come with its limitations and accordingly, a strategy or an approach to a problem is to be developed. This is called a Constraint.

Constraint is expressed as (<, =, >) follows:

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1$

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geq b_1$ Or

$-a_{11}x_1 - a_{12}x_2 - \ldots - a_{1n}x_n \leq b_1$

1.2.4. Assumptions of Linear Programming Problem

Four basic assumptions that are important for all linear programming problems are as follows:

1) **Certainty:** All the variables or parameters in an LP model should be assumed to be constant or known. For the best optimal solution, parameters such as availability and consumption of resources or the profit/cost contribution variable should be assumed to be certain. Certainty in an LP model is concerned with the coefficients in the objective function c_j, the coefficients in the functional coefficients a_{ij} and the right hand sides of the functional constraints b_i. The values of the coefficients of each variable are to be constant or certain. This means that values of c_j, a_{ij}, b_i are fixed and known with certainty.

2) **Divisibility (or Continuity):** It is assumed that decision variables and resources have solution values which are either whole numbers (integers) or mixed numbers (integer and fractional). The integer programming method may be applied to get the desired values if only integer variables are desired, e.g., machines, employees, etc. However, in an LPP model, decision variables can have any value that satisfy the functional and non-negative constraints. It can also take a non-integer value.

3) **Additive:** This means, the function value is the sum of the contributions of each term, i.e., when two or more activities are used, the total product is equal to the sum of the individual products where there is no interaction effect between the activities. **For example,** the sum of the profits earned separately from A and B must be equal to the total profit earned by the sale of two products A and B. Similarly, the sum of resources used for A and B individually must be equal to the amount of a resource consumed by A and B.

4) **Linearity:** Linear programming requires linearity in the equations, i.e., the relationship in both objective function and constraints must be linear. Linearity requires that a change in a variable should result in proportionate change in that variable's contribution to the value of the function. **For example,** the resource i consumes $5a_{ij}$, if decision variable x = 5 but the consumption will be $10a_{ij}$ if x = 10, and so on. Where a_{ij} represents the amount of resource i used for an activity j (decision variable).

1.2.5. Advantages of Linear Programming

1) **Improves Quality Decision:** LP improves the quality of the decisions. There might be other constraints operating outside the problem. Hence, linear programming gives possible and practical solution.
2) **Cost-Benefit Analysis:** LP is a versatile technique that helps in representing real-life business situations. It is a very cost-effective technique and is helpful in planning and executing the policies of the top management in a cost effective manner.
3) **Flexibility:** LP provides better tools for meeting the changing conditions. Re-evaluation for changing conditions can be done, even after the plans are prepared. LPP is a very effective technique under such changing circumstances.
4) **Number of Possible Solutions:** Management problems are complex, but with LPP technique, managers can arrive at the best alternative solution to the problem. LPP helps in assuring that the manager is considering the best optimal solution.
5) **Use of Productive Factors:** The Linear Programming technique helps in making the best possible use of the available productive resources such as time, labour, machine etc. A decision-maker can employ his productive factors effectively by selecting and distributing these elements.
6) **Scientific Approach:** LP is effective as it highlights the bottlenecks in the production process. This means that an LP presents a clear picture of the problem. Hence, it becomes easy to deal with the problem.

1.2.6. Disadvantages of Linear Programming

1) **Linear Relationship:** A primary requirement of an LP is that the objective function and every constraint must be linear. However, in real life situations, many business problems can only be expressed in a non-linear form. In such situations, LP technique is not applicable.
2) **Coefficients are Constraints:** LP assumes that all values of coefficients of decision variables are stated with certainty. Due to this restriction, LP cannot be applied to a wide variety of problems.
3) **Fractional Solutions:** Many times the solution to a problem may not be an integer but a fraction. Solution in fractions may not remain optional in rounding off.
4) **Complexity:** There are computational difficulties when it comes to large problems. LP model is a mathematical formulation which becomes complex when there are large number of variables and constraints.
5) **Possibility of More than One Objective:** LP deals with the problems with single objective. But in real life situations there are more than one objective. LP faces limitations in such situations.
6) **Time Effect:** The effect of time is not considered in linear programming model.

1.2.7. Applications of Linear Programming

Linear programming models can be applied to a variety of business problems. Following areas make use of this technique:

1) **Finance:** LP techniques are very important for the finance sector. It is very useful in profit planning and control. LP can be used to maximise the profit margins by making an optimum use of financial resources.
2) **Industrial:** Production costs can be put under check with the help of linear programming. LP finds the best solution under prevailing constraints and helps maximising output and minimising costs.
3) **Administrative:** The administrative tasks become easier and efficient with the use of LP techniques. It helps the manager to choose and decide the best method out of various managerial methods for achieving the desired output.
4) **Defence:** LP is extensively used in military operations. In the defence sector, LP can be used to utilise the optimal limited defence resources to achieve the desired goal.
5) **Trade:** Linear programming can be used very widely in this sector. It helps in the estimation of demand and supply, price and cost etc.
6) **Transport:** In the transport sector, LP models are employed to determine the optimal distribution system. It also helps in ascertaining that minimum cost is incurred on transportation.

1.2.8. Formulation of LP Problem

Mathematical model use for LP is known as Formulation of Linear Programming Problem.

Steps of LPP Formulation

LP requires the formulation of a model in a mathematical form using various symbols. The basic steps in formulating a linear programming model are as follows:

Step 1: Identifying the decision variables (values of which are found by solving the L.P.P.) and assigning the symbols $x_1, x_2, \ldots$ or $x, y, \ldots$ to them.

Step 2: Identify the objective function to be optimised (minimised or maximised) and expressed in terms of pre-defined decision variables as below:
$z = c_1x_1 + c_2x_2 + \dots c_nx_n$

Step 3: Identify all the constraints in a given problem, which restrict the operation at a given point of time and express them as linear equations and/or inequalities in terms of pre-defined decision variables as below:
$a_{i1}x_1 + a_{i2}x_2 + \dots (\leq, =, \geq) b_i$, where i = 1, 2, ...m

Step 4: Since the negative values of decision variables do not have any valid physical interpretation, write the non-negativity conditions as:
$x_1 \geq 0, x_2 \geq 0, \dots$ or $x_j \geq 0$, where j = 1, 2, ...n

Step 5: In the last stage, form a linear programming problem model by putting the objective function, linear constraints and non-negativity conditions together in the form of an equation.

Example 1: A firm uses lathes, milling machines and grinding machines to produce two machine parts. Table given below represents the machining times required for each part, the machining times available on different machines and the profit on each machine part.

Type of machine	Machining Time Required for the Machine Part (minutes)		Maximum Time Available per Week (minutes)
	M_1	M_2	
Lathes	12	6	3,000
Milling machines	4	10	2,000
Grinding machines	2	3	900
Profit per unit	₹40	₹100	

Formulate the problem so that the number of parts I and II to be manufactured per week to maximize the profit.

Solution: Formulation of L.P. Model

1) **Objective Functions:** The first major requirement of an LPP that we should be able to identify the objective functions. It will be maximised or minimised. Mathematically the objective function relates the variables which we are dealing in the problem. In this problem we could be obtained by producing and selling two machine parts (M_1) and (M_2). Let x_1 and x_2 represent the number of time required for producing and selling M_1 and M_2 respectively. Now, Machine part (M_1) is obtained profit ₹40 and Machine part (M_2) is obtained profit ₹100 respectively. So, objective is to maximise the profit.

   Max (Z) = $40x_1 + 100x_2$ is the objective function.

2) **Constraints/Conditions (Constraints are on the Time Available on Each Machine):** The mathematical relationship which is used to explain the inequality in the variables. The inequalities can be expressed in terms of less than or equal (≤) and in the terms of greater than or equal (≥). Each part of machine (M_1) required 12 minutes for Lathes and Machine (M_2) require 6 minutes for Lathes. The total maximum time is available of Lathes machine is 3,000 minutes.

   We can express the constraints for Lathes as:
   $12x_1 + 6x_2 \leq 3{,}000$

   Each of machine part (M_1) require 4 minutes and machine part (M_2) require 10 minutes for milling machines and total maximum time available is 2,000 minutes. We can express the constraints for milling machines as:
   $4x_1 + 10x_2 \leq 2{,}000$

   Each of machine part (M_1) will require 2 minutes and machine part (M_2) will require 3 minutes for grinding machine and total maximum time available is 900 minutes. We can express the constraints for grinding machine as follows:
   $2x_1 + 3x_2 \leq 900$

   Therefore, the subjective function is:

For Lathes,	$12x_1 + 6x_2 \leq 3{,}000$
For milling machines	$4x_1 + 10x_2 \leq 2{,}000$
For grinding machine	$2x_1 + 3x_2 \leq 900$

3) **Non-Negative Conditions:** x_1 and x_2 are the number of machine parts produced by machine (M1) and machine (M_2) and they cannot have negative value. The non-negative condition is expressed as:
   $x_1, x_2 \geq 0$

Now, we can write the problem in complete form of LPP as follows:

Max (Z) = $40x_1 + 100x_2$ (Objective functions)

Subject to,

$12x_1 + 6x_2 \leq 3{,}000$
$4x_1 + 10x_2 \leq 2{,}000$ (Subjective functions)
$2x_1 + 3x_2 \leq 900$

and non-negative conditions is $x_1, x_2 \geq 0$.

Example 2: A firm is engaged in producing two products P_1 and P_2. Each unit of product P_1 requires 2kg of raw material and 4 labour hours for processing. Whereas each unit of product P_2 requires 5kg of raw material and 3 labour hours of the same type. Every week the firm has the availability of 50kg of raw material and 60 labour hours. One unit of product P_1 sold earn profit ₹20 and unit of product P_2 sold gives ₹30 as a profit.

Formulate this problem as linear programming to determine as to how many units of each of the products should be produced per week so that the firm can earn maximum profit, assume all units produced can be sold in the market?

Solution: Let the company produced two types of product $P_1(x_1)$ and $P_2(x_2)$. Now the problem is formulated as follows:

	Products		
Resources	$P_1(x_1)$	$P_2(x_2)$	**Availability**
Raw Material	2kg	5kg	50kg
Labour	4 hours	3 hours	60 hours
Profit	₹20	₹30	

1) **Objective Functions:** Let, x_1 and x_2 represent the number of units of product P_1 and P_2 respectively.
   Max (Z) = $20x_1 + 30x_2$ is the objective function.

2) **Constraints/Conditions:** Each unit of product P_1 require 2kg of raw material while each unit of product P_2 requires 5kg. The total availability of raw material is 50kg.

Now, we can express the constraint or condition as:

$$2x_1 + 5x_2 \le 50$$

Similarly that a unit of product P_1 requires 4 labour hours for its production and one unit of product P_2 requires 3 labour hours with an availability of 60 labour hours. The labour constraint or condition will be expressed as mathematical equation:

$$4x_1 + 3x_2 \le 60$$

Therefore, the subjective function is as follows:

$$2x_1 + 5x_2 \le 50$$
$$4x_1 + 3x_2 \le 60$$

3) **Non-Negative Conditions:** x_1 and x_2 are the number of units produced of product P_1 and P_2 cannot have negative value. The non-negative condition is expressed as follows:

$$x_1, x_2 \ge 0$$

Now, we can write the problem in complete form of LPP as follows:

Max (Z) = $20x_1 + 30x_2$

Subject to,

$$2x_1 + 5x_2 \le 50$$
$$4x_1 + 3x_2 \le 60$$
and $x_1, x_2 \ge 0$

Example 3: Mohan-Meaking Breweries Ltd. has two bottling plants, one located at Solan and the other at Mohan Nagar. Each plant produces three drinks, whisky, beer and fruit juices named A, B and C respectively. The numbers of bottles produced per day are as follows:

	Plant at	
	Solan(S)	Mohan Nagar (M)
Whisky, A	1,500	1,500
Beer, B	3,000	1,000
Fruit Juices, C	2,000	5,000

A market survey indicates that during the month of April, there will be a demand of 20,000 bottles of whisky, 40,000 bottles of beer and 44,000 bottles of fruit juices. The operating costs per day for plants at Solan and Mohan Nagar are 600 and 400 monetary units. Formulate the problem so that each plant be run in April so as to minimize the production cost, while still meeting the market demand?

Solution: Formulation of L.P. Model

1) **Objective Function:** Key decision is to determine the number of days for which each plant must be run in April. Let the plants at Solan and Mohan Nagar be run for x_1 and x_2 days.

Objective is to minimize the production cost.
i.e., minimise $Z = 600x_1 + 400x_2$

2) **Constraints:** Constraints are on the demand.
i.e., for whisky, $1{,}500\, x_1 + 1{,}500\, x_2 \ge 20{,}000$,
For beer, $3{,}000\, x_1 + 1{,}000\, x_2 \ge 40{,}000$,
For fruit juices, $2{,}000\, x_1 + 5{,}000\, x_2 \ge 44{,}000$

3) **Non-Negative Conditions:** Feasible alternatives are sets of values of $x_1 \ge 0$ and $x_2 \ge 0$ which meet the objective.

Example 4: A chemical company produces two products, X and Y. Each unit of product X requires 3 hours on operation I and 4 hours on operation II, while each unit of product Y requires 4 hours on operation I and 5 hours on operation II. Total available time for operations I and II is 20 hours and 26 hours respectively. The production of each unit of product Y also results in two units of a by-product Z at no extra cost.

Product X sells at profit of ₹10/unit, while Y sells at profit of ₹20/unit. By product Z brings a unit profit of ₹6 if sold; in case it cannot be sold, the destruction cost is ₹4/unit. Forecasts indicate that not more than 5 units of Z can be sold. Formulate the problem so that the quantities of X and Y to be produced, keeping Z in mind, so that the profit earned is maximum.

Solution: Formulation of L.P. Model

Step 1: The key decision to be made is to determine the number of units of products X, Y and Z to be produced.

Step 2: Let the number of units of products X, Y and Z produced be x_1, x_2, x_3, where

x_3 = number of units of Z produced
= number of units of Z sold + number of units of Z destroyed = $x_3 + x_4$ (say).

Step 3: Feasible alternatives are sets of values of x_1, x_2, x_3 and x_4, where $x_1, x_2, x_3, x_4 \ge 0$.

Step 4: Objective is to maximise the profit. Objective function (profit function) for products X and Y is linear because their profits (₹10/unit and ₹20/unit) are constants irrespective of the number of units produced. A graph between the total profit and quantity produced will be a straight line. However, a similar graph for product Z is non-linear since it has slope + 6 for first part, while a slope of –4 for the second. However, it is piece-wise linear, since it is linear in the regions (0, 5) and (5, Y). Thus splitting x_z into two parts, viz. the number of units of Z sold (x_3) and number of units of Z destroyed (x_4) makes the objective function for product Z also linear.

Thus the objective function is maximise $Z = 10x_1 + 20x_2 + 6x_3 - 4x_4$.

Step 5: Constraints are:
On time available on operation I: $3x_1 + 4x_2 \le 20$,
On time available on operation II: $4x_1 + 5x_2 \le 26$,
On the number of units of product Z sold: $x_3 \le 5$,
On the number of units of product Z produced:
$2x_2 = x_3 + x_4$ or $-2x_2 + x_3 + x_4 = 0$ and
$x_1, x_2, x_3, x_4, x_5 \ge 0$

Example 5: A person wants to decide the constituents of a diet which will fulfil his daily requirements of proteins, fat and carbohydrates at the minimum cost. The choice is to be made from four different types foods. The yields per unit of these foods are given below:

Food Type	Yield per Unit			Cost per Unit
	Proteins	Fats	Carbohydrates	
1	3	2	6	45
2	4	2	4	40
3	8	7	7	85
4	6	5	4	65
Minimum Requirement	800	200	700	

Formulate the LPP for the problem.

Solution: Formulation of Linear Programming Model

Step 1) Let consider that x_1, x_2, x_3 and x_4 are the number of units of food of type 1, 2, 3 and 4 to be used.

Step 2) The sets of values x_j show the feasible alternatives. Here x_j represents the number of units of food type j to be used. The values of j are 1, 2, 3 and 4.

Also $x_j \geq 0$ ...(1)

This shows non-negativity condition.

Step 3) The main objective is to minimise the total cost of foods, thus we have the following objective function:

Min $Z = ₹(45x_1 + 40x_2 + 85x_3 + 65x_4)$ ...(2)

Step 4) Daily requirements are fulfilled by various constraints which are given follows:

$3x_1 + 4x_2 + 8x_3 + 6x_4 \geq 800$(Proteins)

$2x_1 + 2x_2 + 7x_3 + 5x_4 \geq 200$(Fats) ....(3)

$6x_1 + 4x_2 + 7x_3 + 4x_4 \geq 700$(Carbohydrates)

Thus the objective of the linear programming models is to find the number of units x_1, x_2, x_3 and x_4 for minimising the objective function(equation 2) subject to constraints (equation 3) and follow non-negatively condition (equation1).

Example 6: A truck company requires the following number of drivers for its trucks during 24 hours.

Time	No. Required
00-04 hr	5
04-08 hr	10
08-12 hr	20
12-16 hr	12
16-20 hr	22
20-24 hr	8

According to the shift schedule a deriver may join for duty at midnight, 04, 08, 12, 16, 20 hours and work continuously for 8 hours. Formulate the problem as L.P. problem for optimal shift plan.

Solution: Let x_1, x_2, x_3, x_4, x_5, and x_6 denote the number of drivers joining duty at 00, 04, 08, 12, 16, 20 hours, respectively.

Hence, the objective function is to minimise the number of drivers required. This means that,

Minimise $Z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$

Drivers who join duty at 00 hours and 04 hours shall be available between 04 and 08 hours. As the number of drivers required during this interval is 10, we have the constraint:

$x_1 + x_2 \geq 10$; $x_2 + x_3 \geq 20$; $x_3 + x_4 \geq 12$

$x_4 + x_5 \geq 22$; $x_5 + x_6 \geq 8$ $x_6 + x_1 \geq 5$

Hence the L.P. problem is as below:

Minimise $Z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$

Subject to constraints,

$x_1 + x_2 \geq 10$

$x_2 + x_3 \geq 20$

$x_3 + x_4 \geq 12$

$x_4 + x_5 \geq 22$

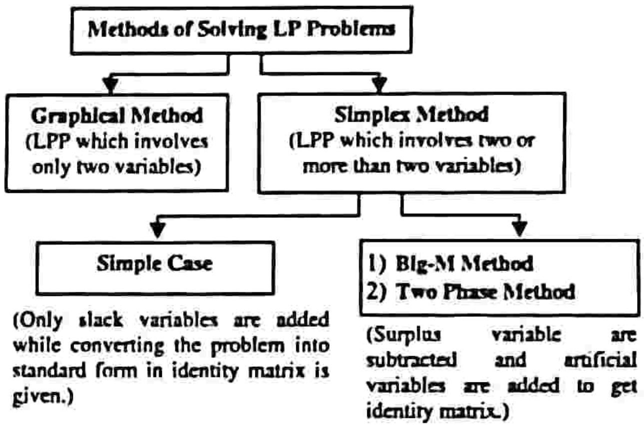
$x_5 + x_6 \geq 8$

$x_6 + x_1 \geq 5$

Where $x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

1.3. SOLUTIONS OF LINEAR PROGRAMMING PROBLEM(LPP)

Linear Programming problem (LPP) can be basically solved by two methods. This is shown in the figure below:



1) **Graphical Method:** Graphical method is a very basic method of solving linear programming problems. Equations are represented in a graphical form and are solved using the intersection points and finding the values of the bounded area.

2) **Simplex Method:** The most general and powerful technique to solve linear programming problem is Simplex method. It is an iterative method which solves the LPP in a limited number of steps or indicates that the problem has unbounded solutions. Simplex method is designed to solve a number of linear equations simultaneously with more/less unknown variables. The computational procedure is repeated until an optimal solution is determined.

1.4. GRAPHICAL METHODS

1.4.1. Introduction

Graphical method indicates the constraints and determines the 'feasible region' on the graph. The **feasible region** is the area that contains all possibly feasible solutions to the problem, i.e., those solutions which satisfy all the constraints of the problem.

The graphical method is applicable to solve the linear programming problem which involves two decision variables. One can arrive at an optimal solution to LPP by evaluating the value of the objective function at each vertex of the feasible region. It will only occur at one of the extreme points.

For example, let consider the following LP problem:

Maximize $Z = 4x + 3y$

Subject to the restrictions,

$x + y \le 5$

$x + 2y \le 8$ and $x, y \ge 0$

In the **figure 1.4** points B, C, D, E is vertices for the feasible region shaded area.

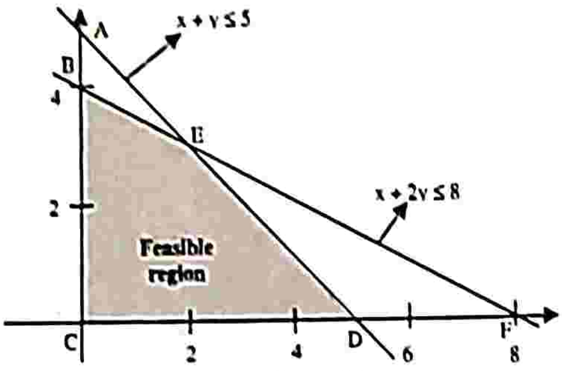


Figure 1.4

1.4.2. Steps of Graphical Solution

Steps involved in graphical method of LPP are as follows:

Step 1) Formulate the linear programming problem.

Step 2) Inequality in the constraints is converted as equality.

Step 3) Constraint lines are constructed as equations.

Step 4) Then 'feasible' solution region is identified.

Step 5) Then corner points of the feasible region are located.

Step 6) On the corner point, the value of the objective function is calculated.

Step 7) The point where the objective function has optimal value is selected.

Step 8) On the same graph paper, objective function is also constructed by assuming some arbitrary value of Z.

Step 9) The value of the objective function can be calculated if the optimal values occur at the corner points of the feasible region and the one which gives the optimal solution is selected.

1.4.3. Maximisation Linear Programming Problems

Example 7: Use the graphical method to solve the following LP problem.

Maximise $Z = 3x_1 + 2x_2$

Subject to the restrictions,

$2x_1 + x_2 \le 40$

$x_1 + x_2 \le 24$

$2x_1 + 3x_2 \le 60$ and $x_1, x_2 \ge 0$

Solution: For the purpose of plotting the above equation on the graph, one needs to convert inequalities into equalities and find out the point of line.

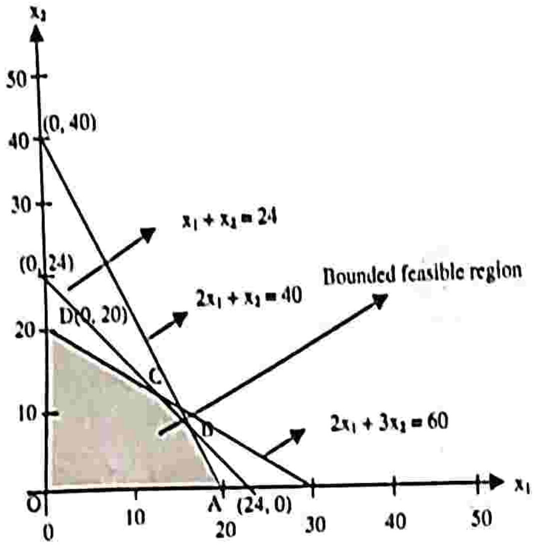
$2x_1 + x_2 = 40$ ...(1)

$x_1 + x_2 = 24$ ...(2)

$2x_1 + 3x_2 = 60$ ...(3)

	Point I	Point II
$2x_1 + x_2 = 40$	$x_1 = 0, x_2 = 40$	$x_1 = 20, x_2 = 0$
$x_1 + x_2 = 24$	$x_1 = 0, x_2 = 24$	$x_1 = 24, x_2 = 0$
$2x_1 + 3x_2 = 60$	$x_1 = 0, x_2 = 20$	$x_1 = 30, x_2 = 0$

All these lines are plotted in graph below:



The feasible region is given by OABCD.

The value of the objective function at each of these extreme points is as follows:

Extreme Point	Coordinates	Objective Function Value $Z = 3x_1 + 2x_2$
O	(0, 0)	$3 \times 0 + 2 \times 0 = 0$
A	(20, 0)	$3 \times 20 + 2 \times 0 = 60$
B	(16, 8)	$3 \times 16 + 2 \times 8 = 64$
C	(12, 12)	$3 \times 12 + 2 \times 12 = 60$
D	(0, 20)	$3 \times 0 + 2 \times 20 = 40$

The maximum value of the objective function Z = 64 occurs at the extreme point B (16, 8). Hence optimal solution to the given LP problem is:

$x_1 = 16, x_2 = 8$, Max $Z = 64$

Example 8: Solve the following LPP

Maximise $Z = 2x_1 + x_2$

Subject to: $x_1 + 2x_2 \le 10$, $x_1 + x_2 \le 6$,

$x_1 - x_2 \le 2$, $x_1 - 2x_2 \le 1$, and $x_1, x_2 \le 0$.

Solution: For the purpose of plotting the above equation on the graph, one needs to convert inequalities into equalities and find out the point of line.

$x_1 + 2x_2 = 10$ ...(1)

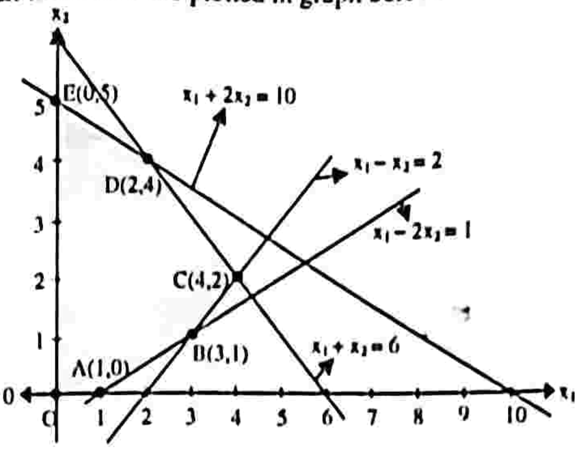
$x_1 + x_2 = 6$ ...(2)

$x_1 - x_2 = 2$ ...(3)

$x_1 - 2x_2 = 1$ ...(4)

	Point I	Point II
$x_1 + 2x_2 = 10$	$x_1 = 0, x_2 = 5$	$x_1 = 10, x_2 = 0$
$x_1 + x_2 = 6$	$x_1 = 0, x_2 = 6$	$x_1 = 6, x_2 = 0$
$x_1 - x_2 = 2$	$x_1 = 0, x_2 = -2$	$x_1 = 2, x_2 = 0$
$x_1 - 2x_2 = 1$	$x_1 = 0, x_2 = -0.5$	$x_1 = 1, x_2 = 0$

All these lines are plotted in graph below:



The feasible region is given by OABCDE.

The value of the objective function at each of these extreme points is as follows:

Extreme Point	Coordinates	Objective Function Value $Z = 2x_1 + 2x_2$
O	(0, 0)	$2 \times 0 + 2 \times 0 = 0$
A	(1, 0)	$2 \times 1 + 2 \times 0 = 2$
B	(3, 1)	$2 \times 3 + 2 \times 1 = 7$
C	(4, 2)	$2 \times 4 + 2 \times 2 = 10$
D	(2, 4)	$2 \times 2 + 2 \times 4 = 8$
E	(0, 5)	$2 \times 0 + 1 \times 5 = 5$

The maximum value of the objective function Z = 10 occurs at the extreme point C(4, 2). Hence optimal solution to the given LP problem is:

$x_1 = 4,\ x_2 = 2,\ \text{Max } Z = 10$

Example 9: A firm manufactures two products A and B on which the profits earned per unit are ₹ 3 and 4 respectively. Each product is processed on two machines M_1 and M_2. Product A requires one minute of processing time on machine M_1 and two minutes on M_2, while B requires one minute on M_1 and one minute on M_2. Machine M_1 is available for not more than 7 hours 30 minutes while machine M_2 is available for 10 hours during any working day. Formulate this as linear programming problem so as to maximize the total profit and solve by graphical method.

Solution: Let the firm decide to manufacture x_1 units of product A and x_2 units of product B. To produce these units of products A and B, it requires $x_1 + x_2$ hours of processing time on M_1, $2x_1 + x_2$ hours of processing time on M_2. But the availability of these two machines M_1 and M_2 are 450 minutes (7 hours and 30 minutes) and 600 minutes (10 hours), respectively, therefore the constraints are:

$x_1 + x_2 \le 450$

$2x_1 + x_2 \le 600$

$x_1, x_2 \ge 0$

Since the profit from product A is ₹ 3 per unit and from product B is ₹ 4 per unit the total profit is ₹ $3x_1 + 4x_2$. The objective is to maximize the profit ₹$3x_1 + 4x_2$. Hence, the LPP is:

Maximise $Z = 3x_1 + 4x_2$

Subject to: $x_1 + x_2 \le 450$

$2x_1 + x_2 \le 600$

$x_1, x_2 \ge 0$

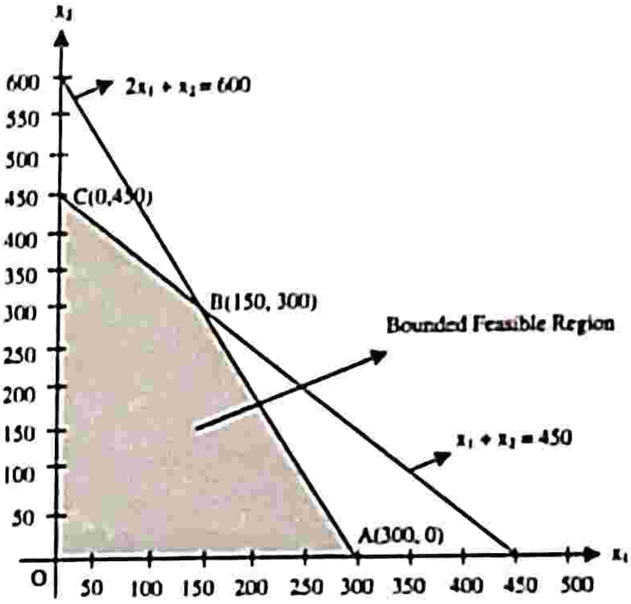
For the purpose of plotting the above equation on the graph, one converts the inequalities into equalities and find out the point of line:

$x_1 + x_2 = 450$ .....(1)

$2x_1 + x_2 = 600$ .....(2)

	Point I	Point II
$x_1 + x_2 = 450$	$x_1 = 0, x_2 = 450$	$x_2 = 0, x_1 = 450$
$2x_1 + x_2 = 600$	$x_1 = 0, x_2 = 600$	$x_2 = 0, x_1 = 300$

All these lines are plotted in graph below:



The feasible region is given by OABC.

The values of the objective function at each of these extreme points are as follows:

Extreme Points	Coordinates	Value of Objective Function $Z = 3x_1 + 4x_2$
O	(0, 0)	0
A	(300, 0)	900
B	(150, 300)	1,650
C	(0, 450)	1,800

Since the problem is of maximisation and the maximum value of Z is attained at a single vertex, hence this problem has a unique optimal solution. The optimal solution is $x_1 = 0$, $x_2 = 450$ and maximum value of Z = 1800.

Example 10: A carpenter makes two products, chair and tables. The manufacturing of these two products is done on two machines A and B. A chair requires two hours on machine A and six hours on machine B. A table requires five hours on machine A and three hours on machine B.

Profits from a chair is ₹20 and from table is ₹50. Machine A is available for 50 hours and machine B is available for 54 hours in a week. Solve the problem graphically to maximise the profit if not more than 9 tables are to be produced.

Solution: Let consider that the carpenter produces the following products: x_1 units of chair and x_2 units of table. It is given that there is a profit of ₹20 on chair and ₹50 on chair. Hence, Total Profit $Z = 20x_1 + 50x_2$

For Machine A

The machine A requires 2 hours to produce one chair. Thus, total hours required for making x_1 chairs is $2x_1$ hours. In the same way, machine A requires 5 hours for making one table. So the total hours needed for making x_2 tables is $5x_2$ hours. But the total machine hours for machine A is 50.

Hence, $2x_1 + 5x_2 \le 50$ .....(1)

For Machine B

Machine B requires 6 hours for producing one chair. For producing x_1 chairs, the total hours required is $6x_1$ hours. In the same way, machine B requires 3 hours for making one table. Hence for producing x_2 table, the total hours needed is $3x_2$ hours. But the total machine hours for machine B is 54.

Hence, $6x_1 + 3x_2 \le 54$ .....(2)

As chairs and tables can never be negative, so $x_1 \ge 0$ and $x_2 \ge 0$.

Hence the LPP is defined as:

Max $Z = 20x_1 + 50x_2$

Subject to $2x_1 + 5x_2 \le 50$

$6x_1 + 3x_2 \le 54$

and $x_1 \ge 0, x_2 \ge 0$

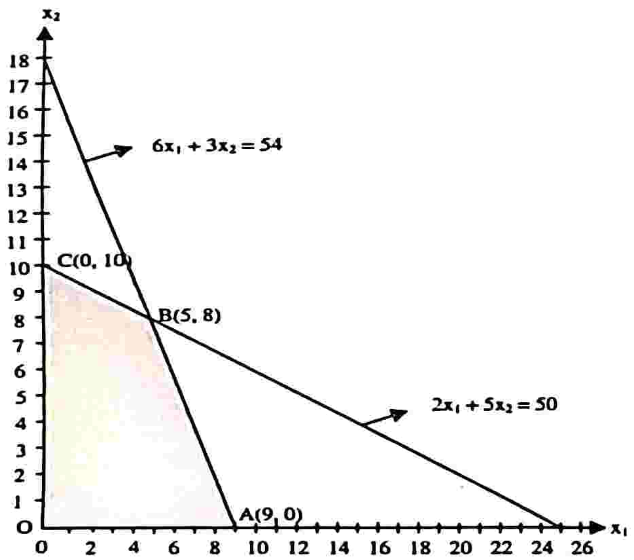
For the purpose of plotting the above equation on the graph we convert inequalities into equalities and find out the point of line.

$2x_1 + 5x_2 = 50$ .....(3)

$6x_1 + 3x_2 = 54$ .....(4)

	Point I	Point II
$2x_1 + 5x_2 = 50$	$x_1 = 0, x_2 = 10$	$x_1 = 25, x_2 = 0$
$6x_1 + 3x_2 = 54$	$x_1 = 0, x_2 = 18$	$x_1 = 9, x_2 = 0$

All these lines are plotted in graph below:



The feasible region is given by OABC. The values of objective function at each of these extreme points are as follows:

Extreme Point	Coordinates	Objective Function Value $Z = 20x_1 + 50x_2$
O	(0, 0)	0
A	(9, 0)	180
B	(5, 8)	500
C	(0, 10)	500

Since according to the question, table should not be more than 9, hence the optimal solution:

$x_1 = 5, x_2 = 8$, Max $Z = 500$.

Example 11: Solve the following LPP graphical method.

Maximum $z = x_1 + x_2$

Subject to, $x_1 - x_2 \ge 0$

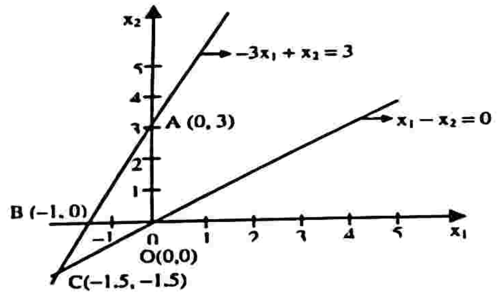
$-3x_1 + x_2 \ge 3$

And $x_1, x_2, \ge 0$

Solution: For the purpose of plotting the equation on the graph, we have to convert inequalities into equation and find-out the point of line:

	Point I	Point II
$x_1 - x_2 = 0$	$x_1 = 0, x_2 = 0$	$x_1 = 0, x_2 = 0$
$-3x_1 + x_2 = 3$	$x_1 = 0, x_2 = 3$	$x_1 = -1, x_2 = 0$

All these lines are plotted on the following figure:



There are three extreme points as follows:

Extreme Points	Coordinates	Objective Function Value $z = x_1 + x_2$
O	(0, 0)	0
A	(0, 3)	3
B	(-1, 0)	-1

The maximum value of the objective function z = 3 occurs at the extreme points (0, 3). Hence the optimal solution to the given LPP is $x_1 = 0, x_2 = 3, z = 3$.

Example 12: Old hen can be bought at ₹2 each and young ones at ₹5 each. The old hens lay 3 eggs per week and the young ones lay 5 eggs per week. Each egg being worth 30 paise. A hen costs ₹1 per week to feed. Mr. X has only ₹80 to spend for hens.

How many of each kind should Mr. X buy to give a profit of atleast ₹6 per week? Assuming that Mr. X cannot house more than 20 hens. Solve the LPP graphically.

Solution: Let suppose Mr. X purchases x_1 old hens and x_2 hens. As old hens lay 3 eggs per week and the young ones lay 5 eggs per week, the total number of eggs Mr. X have per week is $3x_1 + 5x_2$. Thus, each egg have 30 paise, total income per week is ₹0.3 $(3x_1 + 5x_2)$.

Also, the costs for feeding $(x_1 + x_2)$ hens, at the rate of ₹1 per hen per week = ₹ $(x_1 + x_2)$.

Thus, the total profit earned per week is shown below:

$Z = 0.3\,(3x_1 + 5x_2) - (x_1 + x_2) = 0.5x_2 - 0.1x_1$

The price of one old hen is ₹2 and one young hen is ₹5. Mr. X have only ₹80 to spend for hens, hence:

$2x_1 + 5x_2 \le 80$

Since Mr. X cannot house more than 20 hens, hence:

$x + y \le 20$

As the number of old ones or young ones, can never be negative. Therefore, $x \ge 0$ and $y \ge 0$

Thus the LPP formulated for the given problem is:

Maximise $Z = 0.5y - 0.1x$

Subject to the constraints $2x + 5y \le 80$

$x + y \le 20$

and $x \ge 0, y \ge 0.$

For the purpose of plotting the above equation on the graph we convert inequalities into equalities and find out the point of line.

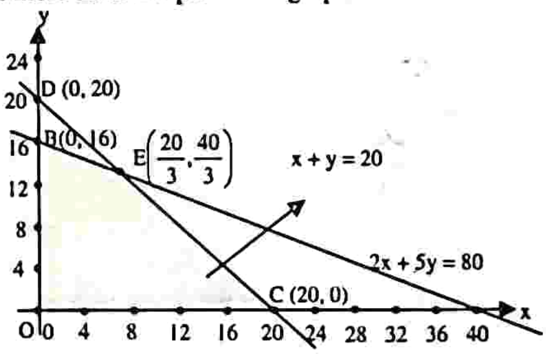
$2x + 5y = 80$ .....(1)

$x + y = 20$ .....(2)

Hence points of intersection of lines are as below:

	Point I	Point II
$2x + 5y = 80$	$x = 0, y = 16$	$y = 0, x = 40$
$x + y = 20$	$x = 0, y = 20$	$y = 0, x = 20$

All these lines are plotted in graph below:



The feasible region is given by OBEC. The value of the objective function at each of these extreme points is as follows:

Extreme Points	Coordinates	Value of the Objective Function $Z = 0.5y - 0.1x$
O	(0, 0)	$0.5 \times 0 - 0.1 \times 0 = 0$
B	(0, 16)	$0.5 \times 16 - 0.1 \times 0 = 8$
E	$\left(\frac{20}{3}, \frac{40}{3}\right)$	$0.5 \times \frac{40}{3} - 0.1 \times \frac{20}{3} = 6$
C	(20, 0)	$0.5 \times 0 - 0.1 \times 20 = -2$

Thus, the maximum value of Z is 8 and is attained when x = 0 and y = 16.

1.4.4. Minimisation Linear Programming Problem

Example 13: Solve graphically the following LPP:

Minimise $Z = 40x + 30y$

Subject to contraints,

$2x + 6y \ge 9$

$4x + y \ge 6$

$x, y \ge 0$

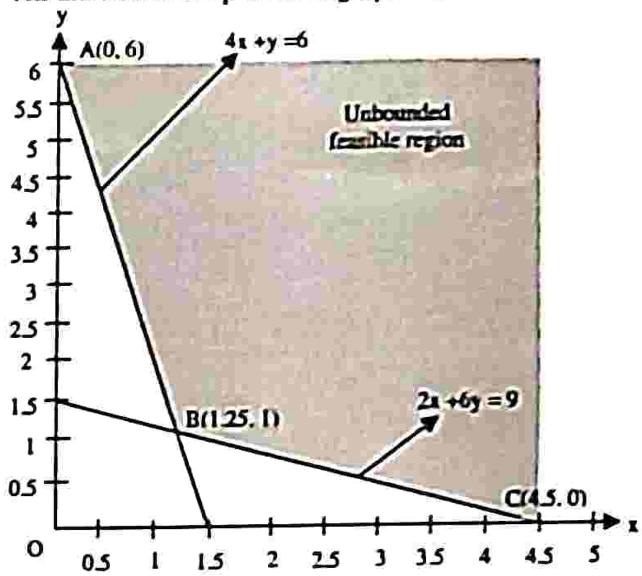
Solution: For the purpose of plotting the above equation on the graph, we convert inequalities into equalities and find-out the point of line:

$2x + 6y = 9$ .....(1)

$4x + y = 6$ .....(2)

	Point I	Point II
$2x + 6y = 9$	$x = 0, y = 1.5$	$y = 0, x = 4.5$
$4x + y = 6$	$x = 0, y = 6$	$y = 0, x = 1.5$

All these lines are plotted in graph below:



The region is unbounded upwards and to the right of ABC and is shown in shaded area in the figure above.

The values of the objective function at each of these extreme points are as follows:

Extreme Points	Coordinates	Objective Function Value $Z = 40x + 30y$
A	(0, 6)	180
B	$\left(\frac{27}{22}, \frac{12}{11}\right)$	$\frac{27}{22}(40) + \frac{12}{11}(30) = \frac{900}{11}$
C	(4.5, 0)	180

The minimum value of the objective function $Z = \frac{900}{11}$, occurs at the extreme points $\left(\frac{27}{22}, \frac{12}{11}\right)$. So, the optimal solution to the given LP problem is:

$x = \frac{27}{22}, y = \frac{12}{11}$ and minimum value of $Z = \frac{900}{11} = 81.82$

Example 14: A farmer is engaged in breeding pigs. The pigs are fed on various products grown on the farm. Because of the need to ensure nutrient constituents, it is necessary to buy additional one or two products, which we shall call A and B.

The nutrient constituents (vitamins and proteins) in each of the product are given below:

Nutrient Constituents	Nutrient in the Product		Minimum Requirement of Nutrient Constituents
	A	B	
X	36	06	108
Y	3	12	036
Z	20	10	100

Product A costs ₹20 per unit and product B costs ₹40 per unit. Determine how much of products A and B must be purchased so as to provide the pigs nutrients not less than the minimum required, at the lowest possible cost. Solve graphically.

Solution: Mathematical formulation of the above:

Minimise $Z = 20x_1 + 40x_2$ (Cost function)

Subject to Constraints:
$36x_1 + 6x_2 \geq 108$
$3x_1 + 12x_2 \geq 36$
$20x_1 + 10x_2 \geq 100$
(Nutrient constraints)

and $x_1, x_2 \geq 0$ (Non-negativity)

For the purpose of plotting the above equations on the graph, we convert inequalities into equalities and find out the point of line.

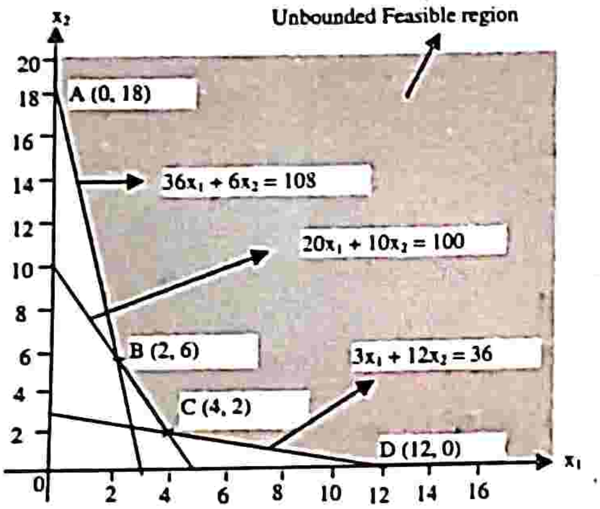
$36x_1 + 6x_2 = 108$ .....(1)

$3x_1 + 12x_2 = 36$ .....(2)

$20x_1 + 10x_2 = 100$ .....(3)

	Point I	Point II
$36x_1 + 6x_2 = 108$	$x_1 = 0, x_2 = 18$	$x_2 = 0, x_1 = 3$
$3x_1 + 12x_2 = 36$	$x_1 = 0, x_2 = 3$	$x_2 = 0, x_1 = 12$
$20x_1 + 10x_2 = 100$	$x_1 = 0, x_2 = 10$	$x_2 = 0, x_1 = 5$

All these lines are plotted in graph below.



Since each of them happened to be greater than or equal to type, constraints the two points x_1, x_2 satisfying them all will lie in the region that falls towards right of each of these lines. The region is unbounded upwards and to the right of ABCD and is shown in shaded area in the **figure above**.

The value of the objective function at each of these extreme points is as follows:

Corner Point	Coordinates	Value of Objective Functions $Z = 20x_1 + 40x_2$
A	0,18	720
B	2,6	280
C	4,2	**160**
D	12,0	240

Here, one find that minimum cost of ₹160 is at point C(4, 2).

So, $x_1 = 4, x_2 = 2, Z = 160$

Thus, the optimum product mix is to purchase 4 units of product A and 2 units of product B in order to maintain minimum costs of ₹160.

Example 15: Solve the following LPP

Minimise: $Z = 3x_1 + 2x_2$

Subject to:
$5x_1 + x_2 \geq 10,$
$x_1 + x_2 \geq 6,$
$x_1 + 4x_2 \geq 12,$
$x_1, x_2 \geq 0.$

Solution: For the purpose of plotting the above equation on the graph, we convert inequalities into equalities and find-out the point of line:

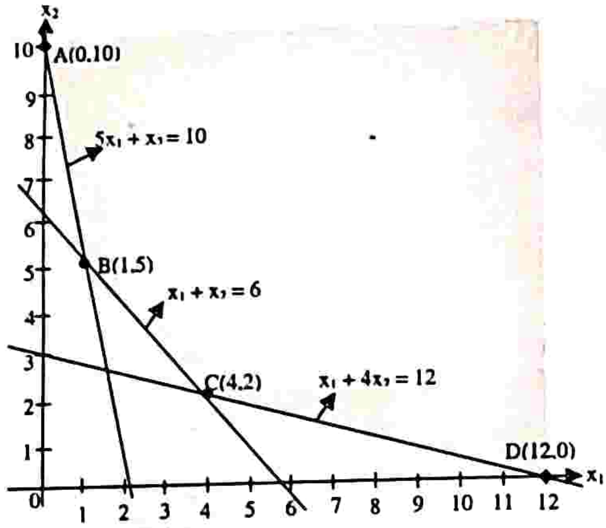
$5x_1 + x_2 = 10$ .....(1)

$x_1 + x_2 = 6$ .....(2)

$x_1 + 4x_2 = 12$ .....(3)

	Point I	Point II
$5x_1 + x_2 = 10$	$x_1 = 0, x_2 = 10$	$x_2 = 0, x_1 = 2$
$x_1 + x_2 = 6$	$x_1 = 0, x_2 = 6$	$x_2 = 0, x_1 = 6$
$x_1 + 4x_2 = 12$	$x_1 = 0, x_2 = 3$	$x_2 = 0, x_1 = 12$

All these lines are plotted in graph below:



The region is unbounded upwards and to the right of ABC and is shown in shaded area in the **figure above**. The value of the objective function at each of these extreme points is as follows:

Extreme Point Coordinates (x_1, x_2)	Objective Function Value $z = 3x_1 + 2x_2$
A (0, 10)	3(0) + 2(10) = 20
B (1, 5)	3(1) + 2(5) = 13
C (4, 2)	3(4) + 2(2) = 16
D (12, 0)	3(12) + 2(0) = 36

The minimum value of objective function z = 13 occurs at the extreme point (1, 5). Hence, the optimal solution to the given LP problem is: $x_1=1$, $x_2=5$ and min z = 13.

Example 16: A Sport Club is interested in deciding a mix of two types of balanced foods for its players in such a way that the cost of food per player is minimum and it contains atleast 8 units of vitamin A and 10 units of vitamin C. First type of food contains 2 units of vitamin A per kg and 1 unit of vitamin C per kg.

Second type of food contains 1 unit of vitamin A per kg and 2 units of vitamin C per kg. Cost of first type of food is ₹ 50 per kg and of second type of food is ₹ 70 per kg. Formulate above problem to determine how much of food of type first and second should be served to each player.

Solution: From the above, data the following table can be derived:

Food	Vitamin A	Vitamin C	Cost Per kg.
1	2	1	50
2	1	2	70
Minimum Requirement	8	10	

The mathematical formulation of the problem is as under:

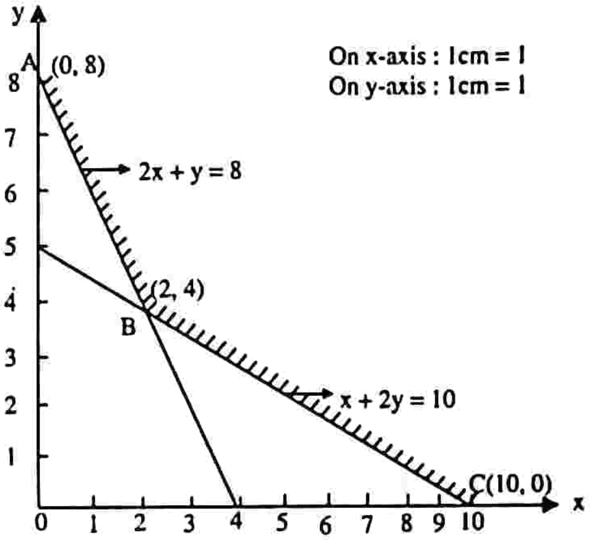
Min. $Z = 50x + 70y$

Subject to $2x + y \geq 8$; $x + 2y \geq 10$ and $x, y \geq 0$

For the purpose of plotting the above equation on the graph, we convert inequalities into equation and find out the point of line.

	Point I	Point II
$2x + y = 8$	$x = 0, y = 8$	$x = 4, y = 0$
$x + 2y = 10$	$x = 0, y = 5$	$x = 10, y = 0$

The feasible region is given by ABC.



The values of objective function at each of these extreme points are as follows:

Extreme Points	Coordinates	Objective Function Value ($z = 50x + 70y$)
A(0, 8)	(0, 8)	560
B(2, 4)	(2, 4)	380
C(10, 0)	(10, 0)	500

The minimum value of the objective function Z = 380 occurs at extreme point (2, 4).

Hence the optimal solution to the given problem is:

First type food = 2kg, Second type food = 4kg

Example 17: Find the minimum value of:

$Z = 5x_1 - 2x_2$

Subject to $2x_1 + 3x_2 \geq 1$,

where $x_1 \geq 0$ and $x_2 \geq 0$.

Solution: The solution space satisfying the given constraint and meeting the non-negativity restrictions $x_1 \geq 0$ and $x_2 \geq 0$ is shown illustrated in **figure 1.5**. Any point in the shaded region is a feasible solution to the above problem.

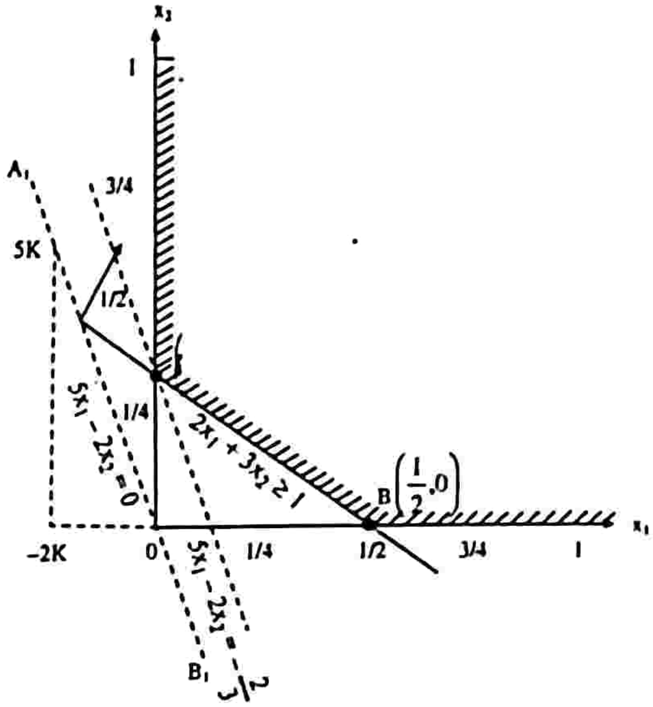


Figure 1.5

The coordinates of the two vertices of the unbounded convex region are shown below:

$$A\left(0, \frac{1}{3}\right) \text{ and } B\left(\frac{1}{2}, 0\right).$$

Values of the objective function $Z = 5x_1 - 2x_2$ at these vertices are shown below:

$$Z(A) = -\frac{2}{3}, Z(B) = \frac{5}{2}.$$

Since the minimum value of Z is $-2/3$, which occurs at the vertex A(0, 1/3), the solution to the given problem is

$$x_1 = 0, x_2 = \frac{1}{3}, Z_{min} = -\frac{2}{3}$$

Example 18: Solve the following linear programming problems graphically:

Minimise $z = 3x_1 + 2x_2$

Subject to $8x_1 + x_2 \geq 8$

$2x_1 + x_2 \geq 6$

$x_1 + x_2 \geq 6$

$x_1 + 6x_2 \geq 8$

$x_1, x_2 \geq 0$

Solution: For the purpose of plotting the above equation on the graph, we convert inequalities into equalities and find-out the point of line:

$8x_1 + x_2 = 8$ ....(1)

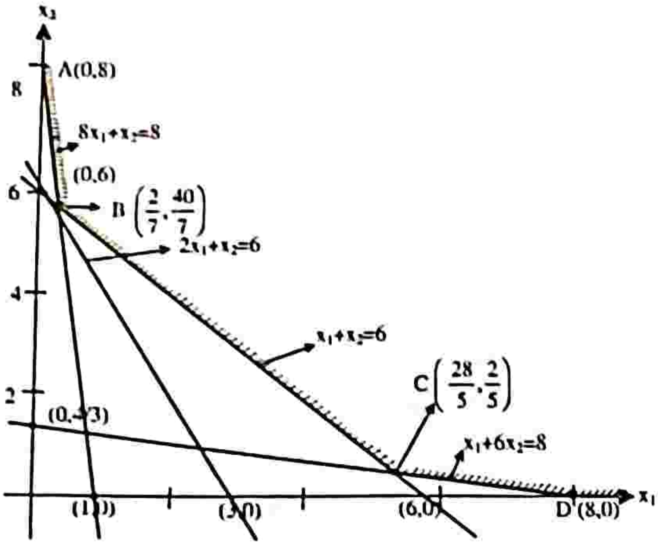
$2x_1 + x_2 = 6$ ....(2)

$x_1 + x_2 = 6$ ....(3)

$x_1 + 6x_2 = 8$ ....(4)

	Point I	Point II
$8x_1 + x_2 = 8$	$x_1 = 0, x_2 = 8$	$x_2 = 0, x_1 = 1$
$2x_1 + x_2 = 6$	$x_1 = 0, x_2 = 6$	$x_2 = 0, x_1 = 3$
$x_1 + x_2 = 6$	$x_1 = 0, x_2 = 6$	$x_2 = 0, x_1 = 6$
$x_1 + 6x_2 = 8$	$x_1 = 0, x_2 = 4/3$	$x_2 = 0, x_1 = 8$

All these lines are plotted in graph below:



The region is unbounded upwards and to the right of ABCD and is shown in shaded area in the **figure**.

The value of the objective function at each of these extreme points is as follows:

Extreme Points	Coordinates	Objective Function Value $Z = 3x_1 + 2x_2$
A	(0,8)	3(0)+2(8)=16
B	$\left(\frac{2}{7},\frac{40}{7}\right)$	$3\left(\frac{2}{7}\right)+2\left(\frac{40}{7}\right)=\frac{86}{7}$
C	$\left(\frac{28}{5},\frac{2}{5}\right)$	$3\left(\frac{28}{5}\right)+2\left(\frac{2}{5}\right)=\frac{88}{5}$
D	(8,0)	3(8)+2(0)=24

The minimum value of the objective function $Z = \frac{86}{7}$, occurs at the extreme points $B\left(\frac{2}{7},\frac{40}{7}\right)$. So, the optimal solution to the given LP problem is:

$$x_1 = \frac{2}{7},\ x_2 = \frac{40}{7} \text{ and minimum value of } Z = \frac{86}{7}.$$

1.4.5. Mixed Constraints Linear Programming Problem

Example 19: Use the graphical method to solve the following LP problem:

Minimise $Z = 20x_1 + 10x_2$

Subjected to, $x_1 + 2x_2 \le 40$

$3x_1 + x_2 \ge 30$

$4x_1 + 3x_2 \ge 60$ and

$x_1, x_2 \ge 0$

Solution: For the purpose of plotting the above equations on the graph, we convert inequalities into equalities and find out the point of line.

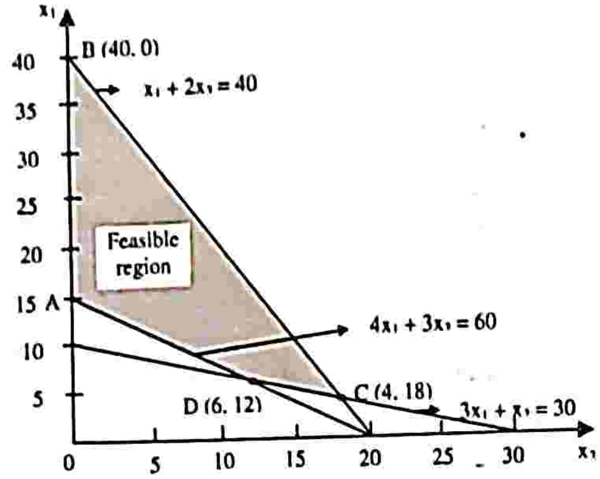
$x_1 + 2x_2 = 40$ .....(1)

$3x_1 + x_2 = 30$ .....(2)

$4x_1 + 3x_2 = 60$ .....(3)

	Point I	Point II
$x_1 + 2x_2 = 40$	$x_1 = 0, x_2 = 20$	$x_2 = 0, x_1 = 40$
$3x_1 + x_2 = 30$	$x_1 = 0, x_2 = 30$	$x_2 = 0, x_1 = 10$
$4x_1 + 3x_2 = 60$	$x_1 = 0, x_2 = 20$	$x_2 = 0, x_1 = 15$

All these lines are plotted in **figure below:**



The feasible region is given by ABCD.

The value of the objective function at each of these extreme points is as follows:

Extreme Points	Coordinates (x_1, x_2)	Objective Function Value $Z = 20x_1 + 10x_2$
A	(15, 0)	300
B	(40, 0)	800
C	(4, 18)	260
D	(6, 12)	240

The minimum value of the objective function Z = 240 occurs at the extreme point D (6, 12). Hence the optimal solution to the given LP problem is:

$$x_1 = 6,\ x_2 = 12 \text{ and Min } Z = 240$$

Example 20: Using graphical method, find the maximum value of:

Maximise $Z = 7x + 10y$

Subject to, $x + y \le 30000$

$y \le 12000$

$x \ge 6000$

$x \ge y$ and $x, y \ge 0$

Solution: For the purpose of plotting the above equations on the graph, we convert inequalities into equalities and find out the point of line.

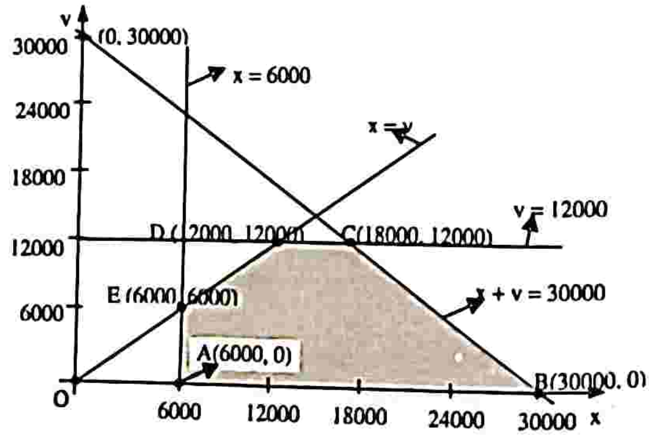
$x + y = 30000$ .....(1)

$y = 12000$ .....(2)

$x = 6000$ .....(3)

	Point I	Point II
x + y = 30000	x = 0, y = 30000	y = 0, x = 30000

All these lines are plotted in graph below:



The feasible region is given by ABCDE. The value of the objective function at each of these extreme points is as follows:

Extreme Points	Coordinates (x, y)	Objective Function Value $Z = 7x + 10y$
A	(6000, 0)	42000
B	(30000, 0)	210000
C	(18000, 12000)	246000
D	(12000, 12000)	204000
E	(6000, 6000)	102000

The maximum value of the objective function Z = 246000 occurs at the extreme point D (18000, 12000).

Example 21: Solve the following LPP graphically;

Minimise $Z = 3x + 2y$;

Subject to $x - y \leq 1$,

$x + y \geq 3$

and $x, y \geq 0$

Solution: In order to draw the above equation on the graph, we first convert the inequalities into equalities and then determine the point of line as follows:

$x - y = 1$ .....(1)

$x + y = 3$ .....(2)

Equations of Lines	Point I	Point II
$x - y = 1$	$x = 0, y = -1$	$y = 0, x = 1$
$x + y = 3$	$x = 0, y = 3$	$y = 0, x = 3$

The above two lines are drawn on the graph as shown in **figure 1.6**:

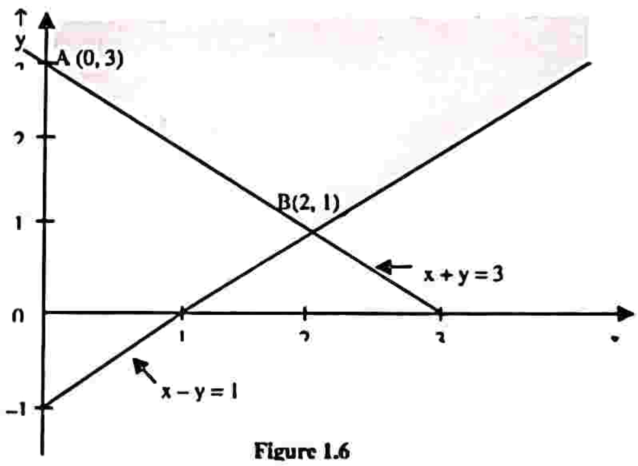


Figure 1.6

At every extreme point, the value of the objective function can be calculated as follows:

Extreme Points	Coordinates (x, y)	Objective Function Value $Z = 3x + 2y$
A	(0, 3)	3(0) + 2(3) = 6(min)
B	(2, 1)	3(2) + 2(1) = 8

From the above table, it is shown that minimum value of Z = 6 which occurs at the point A(0, 3).

Thus we have the following optimum solution for the given LP problem:

$x = 0, y = 3$ and min $Z = 6$

Example 22: Solve the following LPP using graphical method:

Maximise $z = 100x_1 + 80x_2$

Subject to $5x_1 + 10x_2 \leq 50$

$8x_1 + 2x_2 \geq 16$

$3x_1 - 2x_2 \geq 6$ and $x_1, x_2 \geq 0$

Solution: For the purpose of plotting the above equation on the graph one convert inequalities into equalities and find out the point of line.

$5x_1 + 10x_2 = 50$ ....(1)

$8x_1 + 2x_2 = 16$ .....(2)

$3x_1 - 2x_2 = 6$ .....(3)

	Point I	Point II
$5x_1 + 10x_2 = 50$	$x_1 = 0, x_2 = 5$	$x_1 = 10, x_2 = 0$
$8x_1 + 2x_2 = 16$	$x_1 = 0, x_2 = 8$	$x_1 = 2, x_2 = 0$
$3x_1 - 2x_2 = 6$	$x_1 = 0, x_2 = -3$	$x_1 = 2, x_2 = 0$

All these lines are plotted in **figure 1.7**:

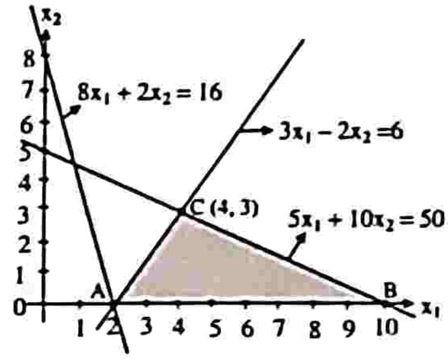


Figure 1.7: Graphical Method

The feasible region is given by ABC. The value of the objective function at each of these extreme points is as follows:

Extreme Point	Coordinates	Objective Function Value $Z = 100x_1 + 80x_2$
A	(2, 0)	100 × 2 + 80 × 0 = 200
B	(10, 0)	100 × 10 + 80 × 0 = 1000
C	(4, 3)	100 × 4 + 80 × 3 = 640

The z value is maximum for the corner point B. Hence, the corresponding solution is: $x_1 = 10, x_2 = 0$, z (optimum) = 1000.

Example 23: A company manufactures animal feed must produce 500kgs of a mixture daily. The mixture consists of two ingredients F_1 and F_2. Ingredient F_1 costs ₹ 5 per kg. and ingredient F_2 costs ₹ 8 per kg. Nutrient considerations dictate that the feed contains not more than 400kgs of F_1 and a minimum of 200kgs of F_2 Formulate the LPP and find the quantity of each ingredient used to minimize cost.

Solution: Let us assume that we have:

x_1 = No. of units of ingredient F_1

x_2 = No. of units of ingredient F_2

Objective function:

Min $Z = 5x_1 + 8x_2$

Subject to constraints, $x_1 + x_2 = 500$

$x_1 \leq 400$

$x_2 \geq 200$

$x_1, x_2 \geq 0$

For the purpose of plotting the above equation on the graph we convert inequalities into equalities and find out the point of line.

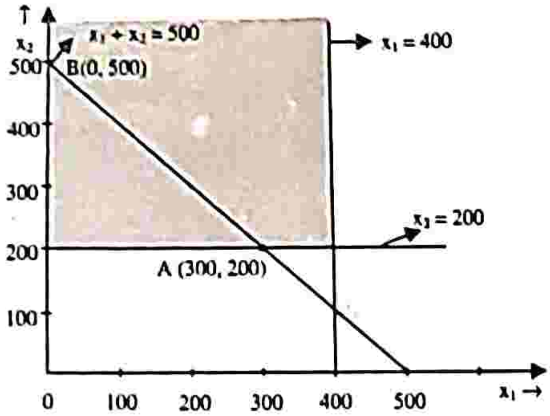
$x_1 + x_2 = 500$

$x_1 = 400$

$x_2 = 200$

	Point I	Point II
$x_1 + x_2 = 500$	$x_1 = 0, x_2 = 500$	$x_1 = 500, x_2 = 0$

All the lines are plotted in graph below.



Two extreme points are as below:

Extreme Points	Coordinates	Objective Function Value $Z = 5x_1 + 8x_2$
A	(300, 200)	1500 + 1600 = 3100
B	(0, 500)	0 + 4000 = 4000

The minimum value of the objective function Z = 3100, occurs at the extreme points (300, 200). So the optimal solution to the given problem is:

$x_1 = 300, x_2 = 200$ and Min Z = 3100

Example 24: Use the graphical method to solve the following LP problem.

Maximize $Z = 2x_1+3x_2$

Subject to the constraints

$x_1+ x_2 \leq 30$

$x_2 \geq 3$

$0 \leq x_2 \leq 12$

$x_1 \leq 20$

$x_1 - x_2 \geq 0$

Solution: Let us consider that every equation is in a linear equation. Then use inequality condition of each constraint to make the feasible region as shown in **figure 1.8.**

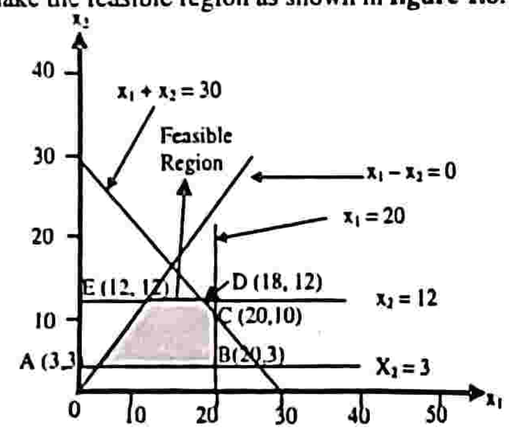


Figure 1.8: Graphical Solution of LP Problem

The coordinates of the extreme points of the feasible region are: A = (3, 3), B = (20, 3), C = (20,10), D = (18, 12) and E = (12,12).

The value of the objective function at each of these extreme points is as follows:

Extreme Point	Coordinates (x_1, x_2)	Objective function value $Z = 2x_1 + 3x_2$
A	(3,3)	2(3) + 3(3) = 15
B	(20,3)	2(20) + 3(3) = 49
C	(20,10)	2(20) + 3(10) = 70
D	(18,12)	2(18) + 3(12) = 72
E	(12,12)	2(12) + 3(12) = 60

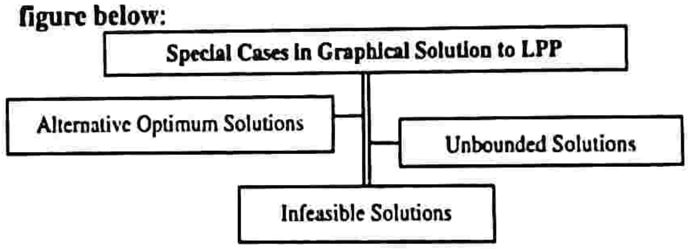
The maximum value of the objective function Z = 72 occurs at the extreme point D (18, 12). Hence, the optimal solution to the given LP problem is: $x_1 = 18, x_2 = 12$ and Max Z = 72.

1.4.6. Limitations of Graphical Method

1) **Linear Relationship:** Graphical method is applicable only when the objective functions and constraints are linear functions. But in real-life problems, this method cannot be applied as both objective functions and constraints are not in linear form.
2) **Coefficients are Constraints:** Since all the coefficients are constraints and defined with certainty in linear programming model, hence it is not efficient method.
3) **Fractional Solutions:** Many times there is a rounding off problem because solution is in fractions.
4) **Not a Powerful Tool:** This method is only useful for two variable problems. But in practical problems, there are more than two variables. Therefore this method is not a powerful tool.

1.4.7. Special Cases in Graphical Solution to LPP

There are some linear programming's problems which do not have unique optimal solutions. These are illustrated in **figure below:**



1.4.7.1. Alternative Optimum Solutions

The solution to a linear programming problem shall always be unique if the slope of the objective function is different from the slopes of the constraints. In case the objective function has slope which is same as that of a constraint, then multiple optimal solutions might be exist.

Example 25: Solve the LPP by graphical method.

Maximize $Z = 100x_1 + 40x_2$

Subject to $5x_1 + 2x_2 \leq 1000$

$3x_1 + 2x_2 \leq 900$

$x_1 + 2x_2 \leq 500$

and $x_1 + x_2 \geq 0$

Solution: For the purpose of plotting the above equations on the graph, we convert inequalities into equalities and find out the point of line.

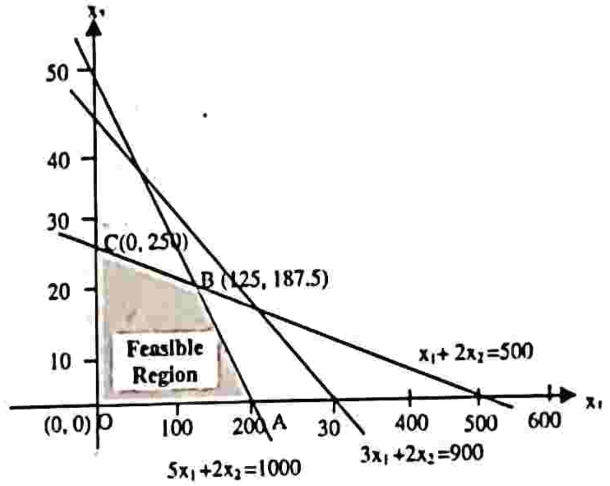
$5x_1 + 2x_2 = 1000$ .....(1)

$3x_1 + 2x_2 = 900$ .....(2)

$x_1 + 2x_2 = 500$ .....(3)

	Point I	Point II
$x_1 + 2x_2 = 40$	$x_1 = 0, x_2 = 20$	$x_2 = 0, x_1 = 40$
$3x_1 + x_2 = 30$	$x_1 = 0, x_2 = 30$	$x_2 = 0, x_1 = 10$
$4x_1 + 3x_2 = 60$	$x_1 = 0, x_2 = 20$	$x_2 = 0, x_1 = 15$

All these lines are plotted in **figure below:**



The feasible region is given by OABC. The value of the objective function at each of these extreme points is as follows:

Extreme Points	Coordinates (x_1, x_2)	Objective Function Value $Z = 100x_1 + 40x_2$
O	(0,0)	0
A	(200, 0)	20,000 (Max)
B	(125, 187.5)	20,000 (Max)
C	(0, 250)	10,000

∴ The maximum value of Z occurs at two vertices A and B

Since there are infinite number of points on the line joining A and B gives the same maximum value of Z. Thus, there are infinite numbers of optimal solutions for the LPP.

Example 26: Solve the following problem graphically:

Maximise $Z = -x_1 + 2x_2$

Subject to $x_1 - x_2 \leq -1$

$-0.5x_1 + x_2 \leq 2$

$x_1, x_2 \geq 0$

Solution: The right hand side of first constraint is negative. Multiplying both sides of the constraint by – 1, it takes the following form;

$-x_1 + x_2 \geq 1$

For the purpose of plotting the above equations on the graph, we convert inequalities into equalities and find out the point of line.

$-x_1 + x_2 = 1$

$-0.5x_1 + x_2 = 2$

	Point I	Point II
$-x_1 + x_2 = 1$	$x_1 = 0, x_2 = 1$	$x_2 = 0, x_1 = -1$
$-0.5x_1 + x_2 = 2$	$x_1 = 0, x_2 = 2$	$x_2 = 0, x_1 = -4$

All these lines are plotted in **figure 1.9.**

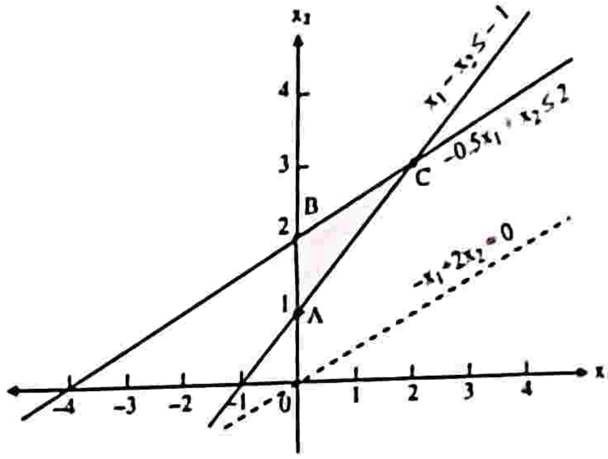


Figure 1.9

Values of the objective function at the vertices of the closed region ABC are as follows:

Extreme Point (Vertex)	Coordinates	Value of Z
A	(0, 1)	2
B	(0, 2)	4
C	(2, 3)	4

Thus both points B and C give the same maximum value of Z = 4. It follows that every point between B and C on the line BC also gives same value of Z. Therefore, the problem has multiple optimal solutions and $Z_{max} = 4$.

1.4.7.2. Unbounded Solutions

The solution is said to be unbounded when the value of decision variables in linear programming can increase infinitely without violating the feasibility condition. Though, the value of the objective function can be increased infinitely, the unbounded feasible region may yield some definite value.

Example 27: Using graphical method to solve the following LPP:

Maximize $Z = 4x_1 + 5x_2$

Subject to constraints,

$x_1 + x_2 \geq 1$

$-2x_1 + x_2 \leq 1$

$4x_1 - 2x_2 \geq 1$

and $x_1, x_2 \geq 0$

Solution: For the purpose of plotting the above equations on the graph, we convert inequalities into equalities and find out the point of line.

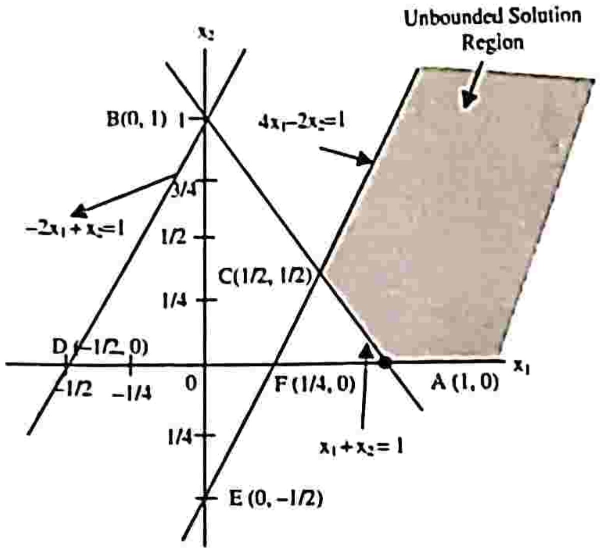
$x_1 + x_2 = 1$ .....(1)

$-2x_1 + x_2 = 1$ .....(2)

$4x_1 - 2x_2 = 1$ .....(3)

	Point I	Point II
$x_1 + x_2 = 1$	$x_1 = 0, x_2 = 1$	$x_2 = 0, x_1 = 1$
$-2x_1 + x_2 = 1$	$x_1 = 0, x_2 = 1$	$x_2 = 0, x_1 = -1/2$
$4x_1 - 2x_2 = 1$	$x_1 = 0, x_2 = -1/2$	$x_2 = 0, x_1 = 1/4$

All these lines are plotted in **figure below:**



The shaded area above the points C and B indicate the feasible region. However, it is unbounded on the upper side. Since the feasible region is unbounded, it is not possible to indicate the optimal solution though one may exist.

Example 28: Use the graphical method to solve the following LP problem:

Maximise $Z = 3x_1 + 4x_2$

subject to the constraints

$x_1 - x_2 = -1$;

$-x_1 + x_2 \leq 0$; and

$x_1, x_2 \geq 0$.

Solution: Plot on graph each constraint by first treating it as a linear equation. Then use the inequality condition of each constraint to mark the feasible region as shown in **figure 1.10.**

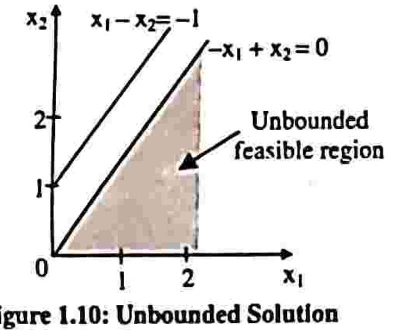


Figure 1.10: Unbounded Solution

From the above figure, it is given that there exist an infinite number of points in the convex region for which for which the value of the objective function increase as we move from extreme point (origin), to the right. That is, both the variables x_1 and x_2 can be made arbitrarily large and the value of objective function Z is also increased. Thus, the problem has an unbound solution.

1.4.7.3. Infeasible Solution

LP problem is said to have an infeasible or inconsistent solution if a feasible solution that satisfies all the constraints is not found. Infeasibility of solution has no relation with the objective function and is solely dependent on the constraints.

Example 29: Solve the following LPP.

Maximise $Z = x_1 + x_2$

Subject to the constraints $x_1 + x_2 \leq 1$

$-3x_1 + x_2 \geq 3$ and

$x_1, x_2 \geq 0$

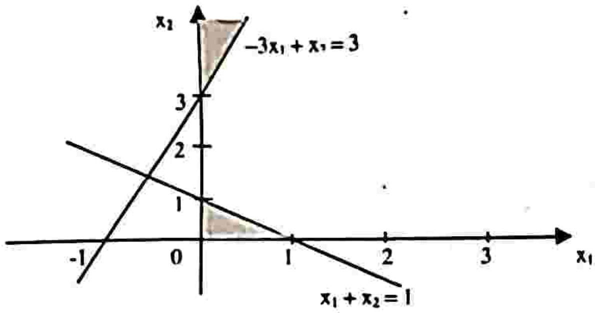
Solution: For the purpose of plotting the above equation on the graph, we convert inequalities into equalities and find out the point of line.

$x_1 + x_2 = 1$ .....(1)

$-3x_1 + x_2 = 3$ .....(2)

	Point I	Point II
$x_1 + x_2 = 1$	$x_1 = 0, x_2 = 1$	$x_2 = 0, x_1 = 1$
$-3x_1 + x_2 = 3$	$x_1 = 0, x_2 = 3$	$x_2 = 0, x_2 = -1$

All these lines are plotted in graph below:



There being no point (x_1, x_2) common to both the shaded regions. We cannot find a feasible region for this problem. So the problem cannot be solved. Hence, the problem has no feasible solution.

Example 30: Maximise $Z = 3x + 2y$

Subject to $-2x + 3y \leq 9$;

$3x - 2y \leq -20$;

$x, y \geq 0$

Solution: For the purpose of plotting the above equation on the graph, we convert inequalities into equalities and find out the point of line.

$-2x + 3y = 9$ .....(1)

$3x - 2y = -20$ .....(2)

	Point I	Point II
$-2x + 3y = 9$	$x = 0, y = 3$	$y = 0, x = -4.5$
$3x - 2y = -20$	$x_1 = 0, y = 10$	$y = 0, x = -20/3$

All these lines are plotted in graph below:

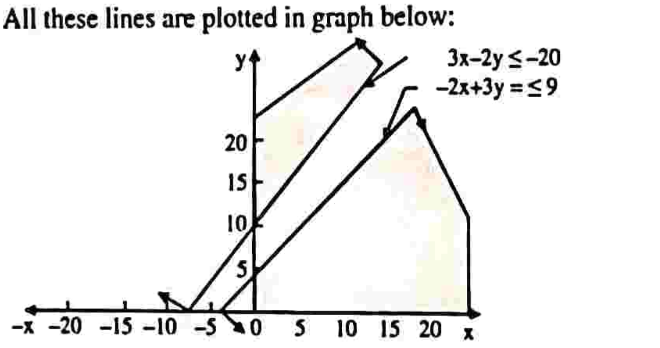


Figure 1.11

There is no point (x, y) common to both the shaded regions of first quadrant as shaded regions do not overlap. The problem cannot be solved graphically, i.e., the feasible solution to the problem does not exist or the problem has infeasible solution.

1.5. SIMPLEX METHOD

1.5.1. Introduction

For solving a linear programming problem which involves two or more variables, simplex method can be used. It is an iterative method and is based on the assumption of obtaining an optimal solution to the problem by the extreme point of the feasible region.

1.5.2. Principles of Simplex Method

1) Simplex method is based on the assumption of obtaining an optimum solution to LPP by the extreme point of the feasible region.
2) It is an iterative process for searching a better corner point until the value of objective function is improved by shifting to another corner.

1.5.3. Steps of Simplex Method

The steps involved in simplex method are as follows:

Step 1) First the linear programming problem is formulated.

Step 2) Initial simplex table with slack variables in maximisation solution is set-up.

Step 3) Inclusion of decision variables is determined.

Step 4) The variables that are to be replaced are determined.

Step 5) Then the new raw values for new variables are calculated.

Step 6) Next, the remaining rows are revised.

Until an optimum solution is reached, steps 3 to 6 are to be repeated.

1.5.4. Slack, Surplus and Artificial Variables

In Simplex Method, the inequalities with $\leq$ or $\geq$ signs are changed to equalities using slack, surplus and artificial variables. Let us understand these variables:

Table 1.1

Types of Constraint	Extra Variable to be Added	Coefficient of Extra Variables in Objective Function		Presence of Variables in Initial Solution
		Max Z	Min Z	
Less than or equal to ($\leq$)	Add only slack variable	0	0	Yes
Greater than or equal to ($\geq$)	Subtract surplus variable and add Artificial variable	0 – M	0 +M	No Yes
Equal to (=)	Add only artificial variable	– M	+M	Yes

Comparison between Slack, Surplus and Artificial Variables

1) **Slack Variables:** Slack variables convert $\leq$ inequalities into equality. For example,

$$a_1 x + a_2 y \leq b_1$$

can be redefined as, $a_1x + a_2y + S_1 = b_1$

Here, S_1 is the slack variable.

Slack variable = Total resource – Used resource

Slack variables are the unallocated portion of the given limited resources and are non-negative in nature. These variables help in a more detailed economic understanding of the solution.

2) **Surplus Variables:** Surplus variables convert $\geq$ inequalities into equalities. For example,

$$a_3X + a_4 Y \geq b_2$$

can be redefined as, $a_3x + a_4y - S_2 = b_2$

Here, S_2 is a surplus variable.

Surplus variable = Production – Requirement

Surplus variables are the additional amount over the required minimum level and are also non-negative in nature.

3) **Artificial Variables and Its Uses:** In many cases, constraints are represented in a combination of $\leq$, $\geq$ and = in equations. Sometimes, this problem is not solved even after introducing surplus variable in the equation. In such a condition, artificial variables are introduced to arrive at a feasible solution. These artificial variables do not have any physical identity and are completely fictitious. Artificial variables are designated as – M for maximisation problems and + M for minimisation problems where M per unit is a very large penalty assigned in objective function.

The summary of extra variables needed to add in the given Linear Programming Problem to convert it into standard form is given in table 1.2.

Table 1.2: Comparison of Variables

Particulars	Slack Variable	Surplus Variable	Artificial Variable
Mean	Unused Resources of Idle Resources.	Excess Amount of Resources utilised.	No physical or economic meaning. It is fictitous.
When to be used	'$\leq$' inequality	'$\geq$' inequality	'$\geq$' and ' = ' constraints
Co-efficient in the constraint	+1	–1	+1
Co-efficient in the objective function Z	0	0	+ M for minimisation and –M for maximization
Use as Initial program variable	Used as starting point (Initial Table or tableau i)	Cannot be used since unit matrix condition is not satisfied	Initially used but later on eliminated
Presence in the Optimal Table	Helps to interpret idle and key resources	–	Indicates Infeasible Solution

1.5.5. Simplex Algorithm (Maximisation Case)

The steps for the simplex algorithm for obtaining an optimal solution to an LPP are as follows:

Step 1: Linear program is converted into a standard form.

Step 2: All right-hand side values of the constraints are checked for non-negativity. In case of a negative value, the corresponding value in equation of the constraints is multiplied by −1 to get the non-negative values.

Step 3: The inequalities of the constraints are converted into equalities by using slack or surplus variables. Costs of these variables are put equal to zero.

Step 4: An initial solution to the problem is obtained and is put in first column of the simplex table.

Step 5: Net evolutions $\Delta_j = Z_j - C_j$ ($j = 1, 2, \ldots n$) is computed by using the relation

$$Z_j - C_j = C_B X_j - C_j.$$

Examine the Sign

1) If all the net evolutions are non-negative then it can be said that the initial basic feasible solution is an optimum solution.
2) In case of atleast one negative net evolution, we have to move to the next step.

Step 6: In case of more than one negative net evolution, choose the most negative of them. The corresponding column is called entering column.

1) In the entering column, if all the values are ≤ 0, then it can be said that the solution to the problem is unbounded.
2) If atleast one variable has a value > 0 then it can be said that the corresponding variable enters the basis.

Step 7: Ratio of X_B/Entering column is computed and the lowest of these ratios is selected. The row that corresponds to this lowest or minimum of ratios is known as leaving row. The common element that is present in both, the entering column and the leaving row is known as the key element or pivotal element.

Step 8: This key element is divided by the heading element to convert it to unity. All the other elements are divided using elementary row transformation to convert them to zeros.

Step 9: The computational procedure in step 5 is repeated till an optimum solution is reached or an unbounded solution to the problem is indicated.

Example 31: Solve the following LPP by simplex method:

Maximise $Z = 100x + 60y + 40z$,

Subject to contraints

$$x + y + z \le 100,$$
$$10x + 4y + 5z \le 600,$$
$$2x + 2y + 6z \le 300,$$

and $x \ge 0, y \ge 0$, and $z \ge 0$.

Solution: To convert it into an equation, we add a slack variable S_1, S_2, and S_3 on the left hand side to get

$$x + y + z + S_1 = 100$$
$$10x + 4y + 5z + S_2 = 600$$
$$2x + 2y + 6z + S_3 = 300$$

The problem can now be expressed as follows:

Max $Z = 100x + 60y + 40z + 0S_1 + 0S_2 + 0S_3$.

Subject to constraints

$$x + y + z + S_1 + 0S_2 + 0S_3 = 100$$
$$10x + 4y + 5z + 0S_1 + S_2 + 0S_3 = 600$$
$$2x + 2y + 6z + 0S_1 + 0S_2 + S_3 = 300$$

Simplex Table I: Non Optimal Solution

Basis		x	y	z	S_1	S_2	S_3	b_i	b_i/a_{ij}
S_1	0	1	1	1	1	0	0	100	100
S_2	0	<u>10</u>	4	5	0	1	0	600	60 (Minimum Value) →
S_3	0	2	2	6	0	0	1	300	150
c_j		100	60	40	0	0	0		
z_j		0	0	0	0	0	0		
$z_j - c_j$		−100	−60	−40	0	0	0		

↑ Incoming variable

Since all value of $z_j - c_j \le 0$, [max case]. Hence it is not optimum solution.

New entry =

$$\text{Old value} - \frac{(\text{Corresponding value of key row}) \times (\text{Corresponding value of key column})}{\text{Key number}}$$

For S_1 row	For S_3 row
$1 - 10 \times \frac{1}{10} = 0$	$2 - 10 \times \frac{2}{10} = 0$
$1 - 4 \times \frac{1}{10} = 3/5$	$2 - 4 \times \frac{2}{10} = 6/5$
$1 - 5 \times \frac{1}{10} = 1/2$	$6 - 5 \times \frac{2}{10} = 5$
$1 - 0 \times \frac{1}{10} = 1$	$0 - 0 \times \frac{2}{10} = 0$
$0 - 1 \times \frac{1}{10} = -1/10$	$0 - 1 \times \frac{2}{10} = -1/5$
$0 - 0 \times \frac{1}{10} = 0$	$1 - 0 \times \frac{2}{10} = 1$
$100 - 600 \times \frac{1}{10} = 40$	$300 - 600 \times \frac{2}{10} = 180$

Simplex Table II: Non Optimal Solution

Basis		x	y	z	S_1	S_2	S_3	b_i	b_i/a_{ij}
S_1	0	0	<u>3/5</u>	1/2	1	−1/10	0	40	66.675 (Minimum Value) →
x	100	1	2/5	1/2	0	1/10	0	60	150
S_3	0	0	6/5	5	0	−1/5	1	180	150
c_j		100	60	40	0	0	0		
z_j		100	40	50	0	10	0		
$z_j - c_j$		0	−20	10	0	10	0		

↑ Incoming variable

Since value of $z_2 - c_2 = -20$ (< 0), [max case], Hence it is not optimum solution.

New entry

$$= \text{Old value} - \frac{(\text{Corresponding value of key row}) \times (\text{Corresponding value of key column})}{\text{Key number}}$$

For Row x

$1 - 0\times\frac{2}{5}\times\frac{5}{3} = 1$

$2/5 - \frac{3}{5}\times\frac{2}{5}\times\frac{5}{3} = 0$

$1/2 - \frac{1}{2}\times\frac{2}{5}\times\frac{5}{3} = 1/6$

$0 - 1\times\frac{2}{5}\times\frac{5}{3} = -2/3$

$1/10 - \left(-\frac{1}{10}\right)\times\frac{2}{5}\times\frac{5}{3} = 1/6$

$0 - 0\times\frac{2}{5}\times\frac{5}{3} = 0$

$60 - 40\times\frac{2}{5}\times\frac{5}{3} = 100/3$

For Row S_3

$0 - 0\times\frac{6}{5}\times\frac{5}{3} = 0$

$6/5 - \frac{3}{5}\times\frac{6}{5}\times\frac{5}{3} = 0$

$5 - \frac{1}{2}\times\frac{6}{5}\times\frac{5}{3} = 4$

$0 - 1\times\frac{6}{5}\times\frac{5}{3} = -2$

$-1/5 - \left(-\frac{1}{10}\right)\times\frac{6}{5}\times\frac{5}{3} = 0$

$1 - 0\times\frac{6}{5}\times\frac{5}{3} = 1$

$180 - 40\times\frac{6}{5}\times\frac{5}{3} = 100$

Simplex Table III: Optimal Solution

Basis		x	y	z	S_1	S_2	S_3	b_i
y	60	0	1	5/6	5/3	–1/6	0	200/3
x	100	1	0	1/6	– 2/3	1/6	0	100/3
S_3	0	0	0	4	– 2	0	1	100
c_j		100	60	40	0	0	0	
z_j		100	60	200/3	100/3	20/3	0	
$z_j - c_j$		0	0	160/6	100/3	20/3	0	

Since $\Delta_j = z_j - c_j \geq 0$.

Thus, the optimal solution is x = 100/3, y = 200/3, z = 0

Max Z = 100x + 60y + 40z

$= 100\times\frac{100}{3} + 60\times\frac{200}{3} + 40\times 0 = 3333.33 + 4000 + 0$

= 7333.33

Example 32: Solve the following LPP using simplex method.

Max $Z = 15x_1 + 6x_2 + 9x_3 + 2x_4$

Subject to $2x_1 + x_2 + 5x_3 + 6x_4 \leq 20$

$3x_1 + x_2 + 3x_3 + 25x_4 \leq 24$

$7x_1 + 7x_4 \leq 70$

$x_1, x_2, x_3, x_4 \leq 0$

Solution: Rewrite the inequality of the constraint into an equation by adding slack variables S_1, S_2 and S_3 the standard form of LPP becomes:

Max $Z = 15x_1 + 6x_2 + 9x_3 + 2x_4 + 0S_1 + 0S_2 + 0S_3$

Subject to constraints

$2x_1 + x_2 + 5x_3 + 6x_4 + S_1 + 0S_2 + 0S_3 = 20$

$3x_1 + x_2 + 3x_3 + 25x_4 + 0S_1 + S_2 + 0S_3 = 24$

$7x_1 + 0x_2 + 0x_3 + 7x_4 + 0S_1 + 0S_2 + S_3 = 70$

$x_1, x_2, x_3, x_4, S_1, S_2, S_3 \geq 0$

The initial basic feasible solution is $S_1 = 20$, $S_2 = 24$, $S_3 = 70$, ($x_1 = x_2 = x_3 = x_4 = 0$ non basic)

The Initial simplex table is given by

Simplex Table I: Non Optimal Solution

Basis		x_1	x_2	x_3	x_4	S_1	S_2	S_3	b_i	b_i/a_{ij}
S_1	0	2	1	5	6	1	0	0	20	20/2 = 10
S_2	0	<u>3</u>	1	3	25	0	1	0	24	24/3 = 8(Minimum Value) →
S_3	0	7	0	0	7	0	0	1	70	70/7 = 10
c_j		15	6	9	2	0	0	0		
z_j		0	0	0	0	0	0	0		
$z_j - c_j$		–15	–6	–9	–2	0	0	0		

↑ Incoming variable

∴ As some of $z_j - c_j \leq 0$ the current basic feasible solution is not optimum. $z_1 - c_1 = -15$ is the most negative value and hence x_1 enters the basis and the variable S_2 leaves the basis.

Simplex Table II: Non Optimal Solution

Basis		x_1	x_2	x_3	x_4	S_1	S_2	S_3	b_i	b_i/a_{ij}
S_1	0	0	<u>1/3</u>	3	– 32/3	1	– 2/3	0	4	4/1/3 =12(Minimum Value) →
x_1	15	1	1/3	1	25/3	0	1/3	0	8	8/1/3 = 24
S_3	0	0	– 7/3	– 7	– 154/3	0	– 7/3	1	14	14/–7/3 = – 6
c_j		15	6	9	2	0	0	0		
z_j		15	5	15	125	0	5	0		
$z_j - c_j$		0	– 1	6	123	0	5	0		

↑ Incoming variable

Since $z_2 - c_2 < 0$, the solution is not optimal and therefore, x_2 enters the basis and the basic variable S_1 leaves the basis.

Simplex Table III: Optimal Solution

Basis		x_1	x_2	x_3	x_4	S_1	S_2	S_3	b_i
x_2	6	0	1	9	–32	3	–2	0	12
x_1	15	1	0	–2	19	–1	1	0	4
S_3	0	0	0	14	–126	7	–7	1	42
c_j		15	6	9	2	0	0	0	
z_j		15	6	24	93	3	3	0	
$z_j - c_j$		0	0	15	91	3	3	0	

Since all $z_j - c_j \geq 0$, the solution is optimal and is given by Max Z = 132, $x_1 = 4$, $x_2 = 12$, $x_3 = 0$ $x_4 = 0$

Example 33: Find the optimum integer solution to the following LPP:

Maximise $Z = 3x_1 + 7x_2$

Subject to $3x_1 + 4x_2 \leq 19$

$3x_1 + 6x_2 \leq 21$

x_1, x_2 non-negative integers.

Solution: The Initial simplex table is given by table below:

Simplex Table I: Non-Optimal Solution

B	C_b	X_b	x_1	x_2	S_1	S_2	A_1	Min Ratio $\frac{X_b}{X_2}$
A_1	–M	19	3	4	–1	0	1	$\frac{19}{4} = \frac{19}{4}$
S_1	0	21	3	<u>6</u>	0	1	0	$\frac{21}{6} = \frac{7}{2}$ →
C_j			3	7	0	0	–M	
Z_j			–3M	–4M	M	0	–M	
$C_j - Z_j$			3M + 3	4M + 7↑	–M	0	0	

Entering = x_2, Departing = S_1, Key Element = 6

R_2 (new) = R_2 (old) ÷ 6 = R_2 (old) $.\frac{1}{6}$

R_1 (new) = R_1 (old) – 4R_2 (new)

Simplex Table II: Non-Optimal Solution

B	C_b	X_b	x_1	x_2	S_1	S_2	A_1	Min Ratio $\frac{X_b}{X_1}$
A_1	–M	5	1	0	–1	$-\frac{2}{3}$	1	$\frac{5}{1}=5\rightarrow$
x_2	7	$\frac{7}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{6}$	0	$\frac{7/2}{1/2}=7$
C_j			3	7	0	0	–M	
Z_j			$-M+\frac{7}{2}$	7	M	$\frac{2}{3}M+\frac{7}{6}$	–M	
C_j-Z_j			$M-\frac{1}{2}\uparrow$	0	–M	$-\frac{2}{3}M-\frac{7}{6}$	0	

Entering = x_1, Departing = A_1, Key Element = 1

R_1 (new) = R_1 (old)

R_2 (new) = R_2 (old) $-\frac{1}{2}R_1$ (new)

Simplex Table III: Optimal Solution

B	C_b	X_b	x_1	x_2	S_1	S_2	A_1	Min Ratio
x_1	3	5	1	0	–1	$-\frac{2}{3}$	1	
x_2	7	1	0	1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	
C_j			3	7	0	0	–M	
Z_j			3	7	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	
C_j-Z_j			0	0	$-\frac{1}{2}$	$-\frac{3}{2}$	$-M+\frac{1}{2}$	

Since all $C_j - Z_j \le 0$,

Optimum Solution is arrived with value of variables as:

$x_1 = 5;\ \ x_2 = 1$

Maximise Z = 22

Example 34: Solve the following LPP:

Maximise $z = 10x_1 + 15x_2 + 20x_3$

Subject to constraints, $2x_1 + 4x_2 + 6x_3 \le 24$

$3x_1 + 9x_2 + 6x_3 \le 30$

$x_1, x_2, x_3 \ge 0$

Solution: To convert above problem into standard form, we add a slack variable S_1, and S_2. Now we get the standard form as follows:

Maximise $z = 10x_1 + 15x_2 + 20x_3 + 0S_1 + 0S_2$

Subject to $2x_1 + 4x_2 – 6x_3 + S_1 = 24$

$3x_1 + 9x_2 – 6x_3 + S_2 = 30$

$x_1, x_2, x_3, S_1,$ and $S_2 \ge 0$

The initial simplex table is shown in **table 1.3**.

Table 1.3: Non-Optimal Solution

Basis		x_1	x_2	x_3	S_1	S_2	b_i	b_i/a_{ij}
S_1	0	2	4	6	1	0	24	26/6=4(Min→)
S_2	0	3	9	6	0	1	30	30/6=5
c_j		10	15	20	0	0		
z_j		0	0	0	0	0		
z_j-c_j		–10	–15	–20	0	0		

↑ Incoming Variable (under x_3)

Since all values of $z_j - c_j \le 0$, hence, the initial solution is not optimum. Now S_1 is outgoing variable and x_3 is incoming variable. The key element is 6 as shown in **table 1.3**. The next iteration is shown in **table 1.4**.

Table 1.4: Non-Optimal Solution

Basis		x_1	x_2	x_3	S_1	S_2	b_i	b_i/a_{ij}
x_3	20	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{1}{6}$	0	4	12
S_2	0	1	5	0	–1	1	6	6/1=6(Min→)
c_j		10	15	20	0	0		
z_j		$\frac{20}{3}$	$\frac{40}{3}$	20	$\frac{10}{3}$	0		
z_j-c_j		$-\frac{10}{3}$	$-\frac{5}{3}$	0	$\frac{10}{3}$	0		

↑ Incoming Variable (under x_1)

Since all values of $z_j - c_j \le 0$, hence, the initial solution is again not optimum. Now S_2 is outgoing variable and x_1 is incoming variable. The key element is 1 as shown in **table 1.4**. The next iteration is shown in **table 1.5**.

Table 1.5: Optimal Solution

Basis		x_1	x_2	x_3	S_1	S_2	b_i
x_3	20	0	–1	1	$\frac{1}{2}$	$-\frac{1}{3}$	2
x_1	10	1	5	0	–1	1	6
c_j		10	15	20	0	0	
z_j		10	30	20	0	$\frac{10}{3}$	
z_j-c_j		-0	15	0	0	$\frac{10}{3}$	

In **table 1.5**, since all values of $z_j - c_j \ge 0$, hence we get the optimal solution. The optimal solution is illustrated as below: $x_1 = 6$, $x_2 = 0$, $x_3 = 2$ and $z = 10 \times 6 + 15 \times 0 + 20 \times 2 = 60 + 0 + 40 = 100$

Hence z(optimum) = 100

Example 35: For a company engaged in the manufacture of three products X, Y and Z, the available data are given below. Determine the product mix to maximise the profit.

Operations	Time in Hours Required per Unit of Manufacturing			Total Available Hours per Month
	X	Y	Z	
1	1	2	2	200
2	2	1	1	220
3	3	1	2	180
Profit/Units (₹)	10	15	8	
Minimum sales requirements/month in units	10	20	30	

Solution: Let x, y, z denote the number of units produced per month of products X, Y and Z respectively. Objective is to maximise the monthly profit.

i.e., maximise $Z = 10x + 15y + 8z$

Time constraints for the three operations are.

$x + 2y + 2z \le 200$

$2x + y + z \le 220$

$3x + y + 2z \le 180$

Minimum sales requirements give the following constraints:

$x \geq 10$
$y \geq 20$
$z \geq 30$

Solution of this problem will need introduction of 3 slack, 3 surplus and 3 artificial variables. The problem will involve, therefore, 12 variables and 6 constraints and will require sufficient time for solution by simplex method. The time and effort for solution can be considerably reduced if variables x, y and z having lower bounds of 10, 20 and 30 respectively are substituted as follows:

$x = 10 + x_1$
$y = 20 + x_2$
$z = 30 + x_3$

Where $x_1, x_2, x_3 \geq 0$

Substituting these values in the model, it takes the form:

Maximise $Z = 10(10 + x_1) + 15(20 + x_2) + 8(30 + x_3)$
$= 10x_1 + 15x_2 + 8x_3 + 640$

Subject to, $(10 + x_1) + 2(20 + x_2) + 2(30 + x_3) \leq 200$
$2(10 + x_1) + (20 + x_2) + (30 + x_3) \leq 220$
$3(10 + x_1) + (20 + x_2) + 2(30 + x_3) \leq 200$

where $x_1, x_2, x_3 \geq 0$.

or Maximise $Z = 10x_1 + 15x_2 + 8x_3 + 640$
Subject to $x_1 + 2x_2 + 2x_3 \leq 90$
$2x_1 + x_2 + x_3 \leq 150$
$3x_1 + x_2 + 2x_3 \leq 70$
$x_1, x_2, x_3 \geq 0$

Adding slack variables S_1, S_2 and S_3, we get;

Maximise $Z = 10x_1 + 15x_2 + 8x_3 + 640 + 0S_1 + 0S_2 + 0S_3$

or Maximise

$Z' = Z - 640 = 10x_1 + 15x_2 + 8x_3 + 0S_1 + 0S_2 + 0S_3$

Subject to $x_1 + 2x_2 + 2x_3 + S_1 = 90$
$2x_1 + x_2 + x_3 + S_2 = 150$
$3x_1 + x_2 + 2x_3 + S_3 = 70$
$x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$.

Initial basic feasible solution is obtained by setting $x_1 = x_2 = x_3 = 0$. The solution is $S_1 = 90$, $S_2 = 150$, $S_3 = 70$, $Z' = 0$. This solution and further improved solutions are given in the following tables:

Table I: Non Optimal Solution

Basis		x_1	x_2	x_3	S_1	S_2	S_3	b_i	b_i/a_{ij}
S_1	0	1	(2)	2	1	0	0	90	45→
S_2	0	2	1	1	0	1	0	150	150
S_3	0	3	1	2	0	0	1	70	70
c_j		10	15	8	0	0	0		
z_j		0	0	0	0	0	0	z = 0	
$c_j - z_j$		10	15	8	0	0	0		

↑ Incoming variable

Table II: Non Optimal Solution

Basis		x_1	x_2	x_3	S_1	S_2	S_3	b_i	b_i/a_{ij}
x_2	15	1/2	1	1	1/2	0	0	45	90
S_2	0	3/2	0	0	−1/2	1	0	105	70
S_3	0	(5/2)	0	1	−1/2	0	1	25	10→
c_j		10	15	8	0	0	0		
z_j		15/2	15	15	15/2	0	0	z = 675	
$c_j - z_j$		5/2	0	−7	−15/2	0	0		

↑ Incoming variable

Table III: Optimal Solution

Basis		x_1	x_2	x_3	S_1	S_2	S_3	b_i
x_2	15	0	1	4/5	3/5	0	−1/5	40
S_2	0	0	0	−3/5	−1/5	1	−3/5	90
x_1	10	1	0	2/5	−1/5	0	2/5	10
c_j		10	15	8	0	0	0	
z_j		10	15	16	7	0	1	z = 700
$c_j - z_j$		0	0	−8	−7	0	−1	

∴ Optimal solution is $x_1 = 10$, $x_2 = 40$, $x_3 = 0$, $Z' = 700$.

Substituting these values,

$x = 10 + x_1 = 10 + 10 = 20$ units
$y = 20 + x_2 = 20 + 40 = 60$ units
$z = 30 + x_3 = 30 + 0 = 30$ units
$Z_{max} = Z' + 640 = ₹(700 + 640) = ₹1,340$.

∴ The optimal product mix is to produce 20 units of X, 60 units of Y and 30 units of Z to get the maximum profit of ₹1,340.

Example 36: Solve the following LPP:

Maximise $Z = 1000x_1 + 4000x_2 + 5000x_3$
Subject to $3x_1 + 3x_3 \leq 22$
$x_1 + 2x_2 + 3x_3 \leq 14$
$3x_1 + 2x_2 \leq 14$
and $x_1, x_2 \geq 0$.

Solution: The problem can be expressed in standard form as

Maximise $Z = 1{,}000x_1 + 4{,}000x_2 + 5{,}000x_3 + 0S_1 + 0S_2 + 0S_3$,
Subject to $3x_1 + 3x_3 + S_1 = 22$,
$X_1 + 2x_2 + 3x_3 + S_2 = 14$,
$3x_1 + 2x_2 + S_2 = 14$,
$x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$.

The initial basic feasible solution is

$x_1 = x_2 = x_3 = 0$, $S_1 = 22$, $S_2 = 14$, $S_3 = 14$, $Z = 0$.

Table 1.6 represents this solution:

Table 1.6: Initial Basic Feasible Solution

	C_j	1000	4000	5000	0	0	0		
C_B	Basis	x_1	x_2	x_3	S_1	S_2	S_3	b	Min Ratio
0	s_1	3	0	3	1	0	0	22	$22/3$
0	s_2	1	2	(3)	0	1	0	14	$14/3$ ←
0	s_3	3	2	0	0	0	1	14	∞
	z_j	0	0	0	0	0	0	0	
	$c_j - Z_j$	1000	4000	5000	0	0	0		

↑ (under x_3)

Since $c_j - Z_j$ is positive under some variable columns; initial basic feasible solution is not optimal.

Performing iteration to get optimal solution results in the following table:

Table 1.7: Second Basic Feasible Solution

	C_j	1000	4000	5000	0	0	0		
C_B	Basis	x_1	x_2	x_3	S_1	S_2	S_3	b	Min Ratio
0	s_1	2	−2	0	1	−1	0	8	−4
5000	x_3	$\frac{1}{3}$	$\frac{2}{3}$	1	0	$\frac{1}{3}$	0	$\frac{14}{3}$	7
0	s_3	3	(2)	0	0	0	1	14	7 ←
	Z_j	$\frac{5,000}{3}$	$\frac{10,000}{3}$	5,000	0	$\frac{5,000}{3}$	0	$\frac{70,000}{3}$	
	$c_j - Z_j$	$-\frac{2,000}{3}$	$\frac{2,000}{3}$ ↑	0	0	$-\frac{5,000}{3}$	0		

Further iteration yields the following table:

Table 1.8: Optimal Basic Feasible Solution

	C_j	1,000	4,000	5,000	0	0	0	
C_B	Basis	x_1	x_2	x_3	S_1	S_2	S_3	b
0	s_1	5	0	0	1	−1	1	22
5,000	x_3	$-\frac{2}{3}$	0	1	0	$\frac{1}{3}$	$-\frac{1}{3}$	0
4,000	x_2	$-\frac{3}{2}$	1	0	0	0	$\frac{1}{2}$	7
	$c_j - Z_j$	$-\frac{5,000}{3}$	0	0	0	$-\frac{5,000}{3}$	$-\frac{1,000}{3}$	28,000

Table 1.8 yields the following optimal solution:

$x_1 = 0, x_2 = 7, x_3 = 0; Z_{max} = 28,000.$

1.5.6. Simplex Algorithm (Minimisation Case)

Minimisation case is converted into Maximization case by Min Z = – Max (– Z) and all other steps of maximisation case are followed as they are similar. Constraints may have ≥ and = signs. After ensuring that all $b_i \geq 0$, they are considered in the problem. Artificial variable having no physical meaning have to be considered because the basis matrix cannot be obtained as an identity matrix in the starting simplex table.

This artificial variable technique is used so that the starting basic feasible solution is achieved and the simplex procedure as usual is adopted till the optimal solution is obtained (**figure 1.12**).

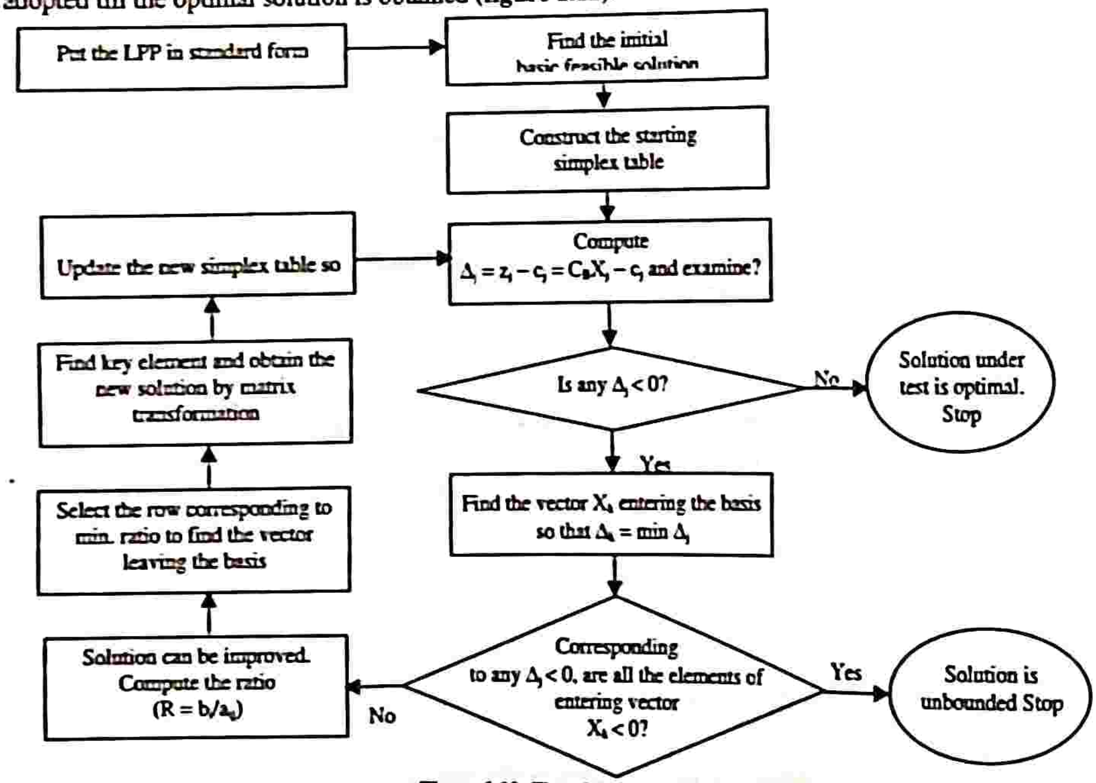


Figure 1.12: Flowchart for Simplex Method

Example 37: Use simplex method to solve the LPP.

Minimise $Z = x_1 - 3x_2 + 2x_3$

Subject to
$$3x_1 - x_2 + 2x_3 \le 7$$
$$-2x_1 + 4x_2 \le 12$$
$$-4x_1 + 3x_2 + 8x_3 \le 10$$
$$x_1, x_2, x_3 \ge 0$$

Solution: Since the given objective function is of minimisation we convert it into maximization using Min Z = – Max (– Z).

Max $Z = -x_1 + 3x_2 - 2x_3$

Subject to Constraints
$$3x_1 - x_2 + 2x_3 \le 7;$$
$$-2x_1 + 4x_2 \le 12;$$
$$-4x_1 + 3x_2 + 8x_3 \le 10$$

We rewrite the inequality of the constraints into an equation by adding slack variables S_1, S_2, S_3 and the standard form of LPP becomes.

Max $Z = -x_1 + 3x_2 - 2x_3 + 0S_1 + 0S_2 + 0S_3$

Subject to Constraints
$$3x_1 - x_2 + 2x_3 + S_1 = 7;$$
$$-2x_1 + 4x_2 + S_2 = 12;$$
$$-4x_1 + 3x_2 + 8x_3 + S_3 = 10$$
$$x_1, x_2, x_3, S_1, S_2, S_3 \ge 0$$

∴ The initial basic feasible solution is given by $S_1 = 7$, $S_2 = 12$, $S_3 = 10$. ($x_1 = x_2 = x_3 = 0$)

Simplex Table I: Non Optimal Solution

Basis		x_1	x_2	x_3	S_1	S_2	S_3	b_i	b_i/a_{ij}
S_1	0	3	–1	2	1	0	0	7	–7
S_2	0	– 2	<u>4</u>	0	0	1	0	12	12/4 = 3(Minimum Value) →
S_3	0	– 4	3	8	0	0	1	10	10/3 = 3.33
c_j		–1	3	– 2	0	0	0		
z_j		0	0	0	0	0	0		
$z_j - c_j$		1	– 3	2	0	0	0		

↑ Incoming variable

Since $z_2 - c_2 < 0$ the solution is not optimum.

Simplex Table II: Non Optimal Solution

Basis		x_1	x_2	x_3	S_1	S_2	S_3	b_i	b_i/a_{ij}
S_1	0	<u>5/2</u>	0	2	1	1/4	0	10	4(Minimum Value) →
x_2	3	–1/2	1	0	0	1/4	0	3	– 3/2
S_3	0	– 5/2	0	8	0	– 3/4	1	1	– 5/2
c_j		– 1	3	– 2	0	0	0		
z_j		– 3/2	3	0	0	3/4	0		
$z_j - c_j$		– 1/2	0	2	0	3/4	0		

↑ Incoming Variable

Since $z_1 - c_1 < 0$, the solution is not optimum. Improve the solution by allowing the variable x_2 to enter into the basis and the variable S_1 to leave the basis.

Simplex Table III: Optimal Solution

Basis		x_1	x_2	x_3	S_1	S_2	S_3	b_i
x_1	–1	1	0	4/5	2/5	1/10	0	4
x_2	3	0	1	2/5	1/5	3/10	0	5
S_3	0	0	0	10	1	–1/2	1	11
c_j		–1	3	– 2	0	0	0	
z_j		–1	3	2/5	1/5	4/5	0	
$z_j - c_j$		0	0	12/5	1/5	4/5	0	

Since all $z_j - c_j \ge 0$, the solution is optimum.

∴ The optimal solution is given by Max Z = 11
$$x_1 = 4, x_2 = 5, x_3 = 0$$

∴ Min $Z = -\text{Max}(-Z) = -11$

∴ Min $Z = -11, x_1 = 4, x_2 = 5, x_3 = 0.$

1.5.7. Advantages of Simplex Method

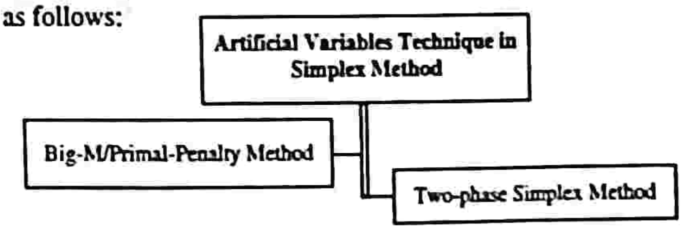
1) Simplex method is used for those problems which have more than two decision variables.
2) Simplex method is a well-known and easy to use method. Through this method, one can easily program the algorithm on a computer. For computation, any function can be quickly included in the program software by just altering the function evaluation. The method is time consuming when done manually but is easy and quick when done with calculators and computers. This ability of the method to program it on machines has made it popular in advanced mathematics.
3) Though the method is simple the identification of errors can be difficult. The simplex method unlike the graphical method deals with an LPP having more than two decision variables.

1.5.8. Disadvantages of Simplex Method

1) Simplex method has its limitations when solving LPP. It is used only in those business situations where a decimal quantity is appropriate. Also, it can be used only when more than two variables are used in a problem. Though this method is very efficient in these situations, but in real-life situations this method does not work because there are hundreds of variables.
2) Simplex method is best adapted in a certain LPP only. This method is used only when the problem is expressed in a standard form with three conditions. Another disadvantage is that the values of the constraints must be non-negative for all variables and must be expressed in = form, with the RHS value to be positive.

1.5.9. Artificial Variables Technique in Simplex Method

For solving the LPP there are two methods (figure below) as follows:



1.5.9.1. Big-M Method/Primal-Penalty Method

The Big-M Method is a method of removing artificial variables from the basis. Artificial variables holding no physical meaning are assigned with variables which are undesirable for the objective function.

To minimise the objective function Z, a huge positive price, also called **penalty** is assigned to each artificial variable. To maximise the objective function Z, a huge negative price, again a penalty is assigned to these artificial variables. For a maximisation problem the penalty will be designated by – M and for a minimisation problem +M, where M > 0.

Steps of Big-M Method

An LPP can be solved by the Big-M Method through the following steps:

Step1: LPP is expressed in a standard form using surplus and artificial variables. The co-efficient of the surplus variable is assigned with a zero and the objective function of the artificial variable is assigned with +M, a huge positive number for Minimisation cases and – M for Maximisation case.

Step 2: Zero value to the original variables is assigned to obtain the initial basic feasible solution.

Step 3: Calculate and then examine the values of $z_j - c_j$ in last row of the simplex table.

i) For optimal solution to the current basic feasible solution, all $z_j - c_j \geq 0$.

ii) We can say that the problem has an unbounded optimal solution if for a column, k, $z_k - c_k$ has the most negative value and that all other entries in this column are positive.

iii) In a condition where one or more $z_j - c_j < 0$, then the basis with the largest negative $z_j - c_j$ value will use a variable. That is,

$z_k - c_k = \text{Min}\ \{z_j - c_j : z_j - c_j < 0\}$

The column to be entered is known as **key or pivot column.**

Step 4: Simplex method of the maximisation case determines the key row and key element. Follow the same method for Big-M method.

For minimisation, convert by applying Min (Z) = – Max (– Z).

Example 38: Solve the following LPP.

Maximise $Z = 2x_1 + 4x_2$

Subject to

$2x_1 + x_2 \leq 18$

$3x_1 + 2x_2 \geq 30$

$x_1 + 2x_2 = 26$

$x_1, x_2 \geq 0$

Solution: After introducing the necessary slack, surplus, and artificial variables, the augmented problem is given here:

Maximise $Z = 2x_1 + 4x_2 + 0S_1 + 0S_2 - MA_1 - MA_2$

Subject to

$2x_1 + x_2 + S_1 = 18$

$3x_1 + 2x_2 - S_2 + A_1 = 30$

$x_1 + 2x_2 + A_2 = 26$

$x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$

The solution is contained in tables 1 through 3:

Simplex Table I: Non Optimal Solution

Basis		x_1	x_2	S_1	S_2	A_1	A_2	b_i	b_i/a_{ij}
S_1	0	2	1	1	0	0	0	18	18
A_1	– M	3	2	0	–1	1	0	30	15
A_2	– M	1	<u>2</u>	0	0	0	1	26	13 →
c_j		2	4	0	0	– M	– M		
z_j		– 4M	– 4M	0	M	– M	– M		
$(z_j - c_j)$		– (4M + 2)	– (4M+4)	0	M	0	0		

↑ Incoming variable

Since $z_1 - c_1 < 0$ and $z_2 - c_2 < 0$. Hence this is not optimum solution for maximization case. So we select the most negative $z_j - c_j$ which is $z_2 - c_2 = -(4M + 4)$

Simplex Table II: Non Optimal Solution

Basis		x_1	x_2	S_1	S_2	A_1	A_2	b_i	b_i/a_{ij}
S_1	0	3/2	0	1	0	0	–1/2	5	10/3
A_1	– M	<u>2</u>	0	0	–1	1	–1	4	2→
x_2	4	1/2	1	0	0	0	1/2	13	26
c_j		2	4	0	0	– M	–M		
z_j		–2M + 2	4	0	M	– M	M + 2		
$(z_j - c_j)$		– 2M	0	0	M	0	2M+2		

↑ Incoming variable

Since $z_1 - c_1 < 0$, the solution is not optimal and therefore, x_1 enters the basis and the basic variable A_1 leaves the basis.

Simplex Table III: Optimal Solution

Basis		x_1	x_2	S_1	S_2	A_1	A_2	b_i
S_1	0	0	0	1	3/4	– 3/4	1/4	2
x_1	2	1	0	0	–1/2	1/2	–1/2	2
x_2	4	0	1	0	1/4	–1/4	3/4	12
c_j		2	4	0	0	– M	– M	
z_j		2	4	0	0	0	2	
$(z_j - c_j)$		0	0	0	0	M	M +2	

Since all $z_j - c_j \geq 0$. Hence this is optimal solution of the given problem.

The optimal solution to the problem is: $x_1 = 2$ and $x_2 = 12$, $S_1 = 2$ and other variables = 0.

The objective function value is $2 \times 2 + 4 \times 12 = 52$.

Example 39: Solve by Big M Method.

Minimise $Z = 60x_1 + 80x_2$

Subject to

$x_2 \geq 200$

$x_1 \leq 400$

$x_1 + x_2 = 500$

$x_1, x_2 \geq 0$

Solution: Converting the problem into the standard form by adding slack, surplus and artificial variables in the set of constraints and assigning appropriate cost to these variables in the objective function, the problem can be re-written as follows:

Min. $Z = 60x_1 + 80x_2 + 0S_1 + 0S_2 + MA_1 + MA_2$

Subject to. $0x_1 + x_2 - S_1 + 0.S_2 + A_1 + 0A_2 = 200$

$x_1 + 0.x_2 + 0.S_1 + S_2 + 0.A_1 + 0.A_2 = 400$

$x_1 + x_2 + 0.S_1 + 0.S_2 + 0.A_1 + A_2 = 500$

Whereas $x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$

From the above system of equations, the initial basic feasible solution can be displayed in the following simplex tables:

Simplex Table I: Non Optimal Solution

C_j →		Contribution per Unit		60	80	0	0	M	M	
↓		Basic Variable	Solution Value	x_1	x_2	S_1	S_2	A_1	A_2	Min. Ratio
M		A_1	200	0	1*	–1	0	1	0	200 →
0		S_2	400	1	0	0	1	0	0	∞
M		A_2	500	1	1	0	0	0	1	500
		Z_j	700M	M	2M	–M	0	M	M	
			$C_j - Z_j$	60 – M	80 – 2M	M	0	0	0	

↑
Incoming variable

Simplex Table II: Non Optimal Solution

C_j →		Contribution per Unit		60	80	0	0	M	
↓		Basic Variable	Solution Value	x_1	x_2	S_1	S_2	A_2	Min. Ratio
80		x_2	200	0	1	–1	0	0	∞
0		S_2	400	1	0	0	1	0	400
M		A_2	300	1*	0	1	0	1	300 →
		Z_j	16000 + 300 M	M	80	M – 80	0	M	
			$C_j - Z_j$	60 – M	0	80 – M	0	0	

↑
Incoming Variable

Simplex Table III: Optimal Solution

C_j →		Contribution per Unit		60	80	0	0
↓		Basic Variable	Solution Value	x_1	x_2	S_1	S_2
80		x_2	200	0	1	–1	0
0		S_2	100	0	0	–1	1
M		x_1	300	1	0	1	0
		Z_1	34,000	60	80	–20	0
			$C_i - Z_i$	0	0	20	0

As all the values in the index row $C_j - Z_j$ are either positive or zero further improvement in the objective function is not possible.

Hence optimal solution is

$x_1 = 300, x_2 = 200,$
$S_1 = 0, S_2 = 100$
and $Z = 34000$

Example 40: ABC printing company is facing a tight financial squeeze and is attempting to cut costs wherever possible. At present it has only one printing contract and luckily, the book is selling well in both the handcover and the paperback editions. It has just received a request to print more copies of this book in either the handcover or the paper back form. The printing cost for the handcover books is ₹600 per 100 books while that for paperback is only ₹500 per 100.

Although the company is attempting to economise, it does not wish to lay-off any employee. Therefore, it feels obliged to run its two printing presses-I and II, at least 80 and 60 hours per week respectively. Press I can produce 100 handcover books in 2 hours or 100 paperback books in 1hour. Press II can produce 100 handcover books in 1 Hour or 100 paperbacks books in 2 hours. Determine how many books of each type should be printed in order to minimise cost (Use Big M method).

Solution: Let x_1, x_2 be the number of batches containing 100 hard cover and paperback books respectively. The LP problem can then be formulated as follows:

Minimise $Z = 600x_1 + 500x_2$
Subject to constraints $2x_1 + x_2 \geq 80$
$x_1 + 2x_2 \geq 60$ and
$x_1, x_2 \geq 0$

By introducing surplus variables S_1, S_2 and artificial variables A_1, A_2 in the inequalities of the constraints, the standard form of the LP problem becomes

Minimise $Z = 600x_1 + 500x_2 + 0S_1 + 0S_2 + MA_1 + MA_2$
Subject to constraints $2x_1 + x_2 - S_1 + A_1 = 80$
$x_1 + 2x_2 - S_2 + A_2 = 60$
and $x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$

the initial basic feasible solution is obtained by setting $x_1 = x_2 = S_1 = S_2 = 0$. We then get:

$A_1 = 80$, $A_2 = 60$ and min $Z = 80M + 60M = 140M$. This initial basic feasible solution is shown in table below:

Simplex Table I: Initial Solution

Basis		x_1	x_2	S_1	S_2	A_1	A_2	b_i	b_i/a_{i1}
A_1	M	2	1	–1	0	1	0	80	80/1
A_2	M	1	2	0	–1	0	1	60	60/2 →
c_i		600	500	0	0	M	M		
z_i		3M	3M	– M	– M	M	M		
$(c_i - z_i)$		600 – 3M	500 – 3M	M	M	0	0		

↑
Incoming variable

As $c_2 - z_2$ value in x_2 column of above table is the largest negative, therefore variable x_2 should be entered to replace basic variable A_2 into the basis. For this, apply following row operations:

R_2(new) → R_2(old) ÷ 2(key element)
R_1(new) → R_1(old) – R_2(new)

to get the new solution as shown in table below:

Simplex Table II: Improved Solution

Basis		x_1	x_2	S_1	S_2	A_1	b_i	b_i/a_{i1}
A_1	M	3/2	0	–1	1/2	1	50	100/3→
x_2	500	1/2	1	0	–1/2	0	30	60
c_i		600	500	0	0	M		
z_i		3M/2+250	500	– M	M/2 – 250	M		
$(c_i - z_i)$		350 – 3M/2	0	M	250– M/2	0		

↑
Incoming variable

As $c_1 - z_1$ value in x_1 column of above table is the largest negative, therefore variable x_1 to replace basic variable A_1 into the basis. For this, apply following row operations:

R_1(new) → R_1(old) × (2/3)(key element)
R_2(new) → R_2(old) – (1/2)R_1(new)

The new table is shown in **table below**:

Simplex Table III: Optimal Solution

Basis		x_1	x_2	S_1	S_2	b_i
x_1	600	1	0	–2/3	1/3	100/3
x_2	500	0	1	1/3	–2/3	40/3
c_j		600	500	0	0	
z_j		600	500	–700/3	–400/3	
$(c_j - z_j)$		0	0	700/3	400/3	

In above table, all $c_j - z_j \geq 0$ and also both artificial variables have been reduced to zero. An optimum solution has been arrived at with $x_1 = 100/3$ batches of hardcover books, $x_2 = 40/3$ batches of paperback books, at a total minimum cost $Z = 600(100/3) + 500(40/3) = (60000/3) + (20000/3) = 80000/3$.

Example 41: An advertising agency wishes to reach two types of audiences – Customers with annual income greater than ₹15,000 (target audience A) and customers with annual income less than ₹15,000 (target audience B). The total advertising budget is ₹2,00,000. One programme of T.V. advertising cost ₹50,000, one programme on radio advertising ₹20,000. For contract reasons, atleast three programmes ought to be on T.V. and the number of radio programmes must be limited to five. Surveys indicate that a single T.V. programme reaches 4,50,000 customers in target audience A and 50,000 in target audience B. One radio programme reaches 20,000 in target audience A and 80,000 in target audience B. Determine the media mix to minimise the total reach. Use simplex method to solve the problem.

Solution: Let us consider that x_1 and x_2 shows the number of program in television and radio respectively. Then,

Total Customers of Television = Customers in Target Audience A and Customers in Target Audience B
= 450000 + 50000
= 500000

Total Customers of Radio = Customers in Target Audience A and Customers in Target Audience B
= 20000 + 80000
= 100000

Thus the linear programming model can be represented as follows:

Maximise $z = 500000\ x_1 + 100000\ x_2 = 5x_1 + x_2$

Subject to the constraints,

$50{,}000x_1 + 20{,}000x_2 \leq 2{,}00{,}000$ or $5x_1 + 2x_2 \leq 20$ (Advertisement Budget)

$x_1 \geq 3$ (Advertisement on T.V.)

$x_2 \leq 5$ (Advertisement on Radio)

and $x_1, x_2 \geq 0$

The above problem can be converted into standard form by adding slack/surplus and/or artificial variables to the LP problem. Now we get the following standard form:

Maximise $z = 5x_1 + x_2 + 0S_1 + 0S_2 + 0S_3 - MA_1$

Subject to the constraints, $5x_1 + 2x_2 + S_1 = 20$

$x_1 - S_2 + A_1 = 3$

$x_2 + S_3 = 5$

and $x_1, x_2, S_1, S_2, S_3, A_1 \geq 0$

By setting the $x_1 = x_2 = S_2 = 0$, we get the initial basic feasible solution. Hence $S_1 = 20$, $A_1 = 3$, $S_3 = 5$ and Max $z = -3M$. The **table 1.9** shows the initial solution:

Table 1.9: Initial Solution

Basis		x_1	x_2	S_1	S_2	S_3	A_1	b_i	b_i/a_{i1}
S_1	0	5	2	1	0	0	0	20	20/5 = 4
A_2	–M	1	0	0	1	0	1	3	3/1=3 →
S_3	0	0	1	0	0	1	0	5	–
c_j		5	1	0	0	0	–M		
z_j		–M	0	0	M	0	–M		
$z_j - c_j$		–M –5	–1	0	M	0	0		

↑ Incoming variable

Since the $z_1 - c_1$ is the largest negative value for x_1 column. Thus x_1 is the incoming variable and A_1 is the outgoing variable and 1 is the corresponding key element. Now let apply the following row operations:

R_2(new) → R_2(old) / 1(key element);

R_1(new) → R_1(old) – $5R_2$(new)

The next iteration is shown in **table 1.10**:

Table 1.10: Improved Solution

Basis		x_1	x_2	S_1	S_2	S_3	b_i	b_i/a_{i1}
S_1	0	0	2	1	5	0	5	5/5 = 1 →
x_1	5	1	0	0	– 1	0	3	–
S_3	0	0	1	0	0	1	5	–
c_j		5	1	0	0	0		
z_j		5	0	0	– 5	0		
$z_j - c_j$		0	– 1	0	– 5	0		

↑ Incoming variable

From **table 1.10**, it is shown that it is not optimal solution as $z_4 - c_4$ is the largest negative value for the column S_2. Thus S_2 is the incoming variable and S_1 is the outgoing variable. Now let apply the following row operations:

R_1 (new) → R_1(old) /5(key element);

R_2 (new) → R_2(old) + R_1(new)

The new solution obtained is shown in **table 1.11**:

Table 1.11: Optimal Solution

Basis		x_1	x_2	S_1	S_2	S_3	b_i
S_2	0	0	2/5	1/5	1	0	1
x_1	5	1	2/5	1/5	0	0	4
S_3	0	0	1	0	0	1	5
c_j		5	1	0	0	0	
z_j		5	2	1	0	0	z = 20
$z_j - c_j$		0	1	1	0	0	

Since all value of $z_j - c_j$ is greater than equal to zero, hence we get the following optimal solution:

$x_1 = 4$, $x_2 = 0$ and Max $z = 20{,}00{,}000$.

1.5.9.2. Two Phase Simplex Method

Linear programming problems with artificial variables can also be solved by another method called the two-phase method. This method is an alternative to the Big-M method as the name suggests, the method divides the procedure into two phases:

1) The first phase of this method is the process of minimising the sum of the artificial variables to get a basic feasible solution to the problem considering the given constraints. This is known as **auxiliary L.P.P.**
2) The second phase starts with the basic feasible solution obtained at the end of Phase I. In this phase, the original objective function is optimised.

Steps of Two Phase Simplex Method

The iterative procedure of the two phase simplex method algorithm is given below:

Step 1: Put the linear programming problem into its standard form. Then, check for starting basic feasible solution of LPP.

1) If the starting basic feasible solution already exists then go to Phase 2 directly.
2) If the basic feasible solution does not exist then go to next step, i.e., Phase I.

Phase I

Step 2: This step involves variables which are required for the basic feasible solution. Artificial variables are added to the left side of each equation to complete the equation that lacks the required starting basic variables. An auxiliary objective function is constructed to minimise the sum of all artificial variables to get a basic feasible solution to the problem. The new objective is,

Minimise $Z = A_1 + A_2 + \ldots A_n$

Maximise $Z^* = -A_1 - A_2 - \ldots - A_n$

Where A_i (i = 1, 2... m) are non-negative artificial variables.

Step 3: The new auxiliary LPP is then solved using simplex algorithm method. At the least interaction following three possible cases may be seen:

1) When Max $Z^* < 0$ and at least one positive value artificial variable is present in the basis, then the original L.P.P. will not have any feasible solution.
2) When Max $Z^* = 0$ and at least one zero value artificial variable is present in the basis, then the original L.P.P. will have feasible solution. In such a case, proceed to Phase 2 to get the basic feasible solution. We may proceed directly to Phase 2 in order to get basic feasible solution or else proceed to Phase 2 after eliminating the artificial basic variable.
3) When Max $Z^* = 0$ and there is no artificial variable in the basis, then a basic feasible solution to the original L.P.P. has been found. Go to Phase 2.

Phase II

Step 4: This phase starts with the optimum basic feasible solution of Phase I. The variables of the objective function are assigned actual coefficient and a zero value to the artificial variables that appear at zero value.

Example 42: Solve the problem given using the two-phase method. The problem is

Minimise $Z = 40x_1 + 24x_2$

Subject to $20x_1 + 50x_2 \geq 4800$

$80x_1 + 50x_2 \geq 7200$

$x_1, x_2 \geq 0$

Solution:

Phase I: First we introduce surplus and artificial variables, and then rewrite the objective function by assigning a zero coefficient to the decision variables and a coefficient 1 to the artificial variables. The problem becomes

Minimisation $Z = 0x_1 + 0x_2 + 0S_1 + 0S_2 + A_1 + A_2$

Subject to $20x_1 + 50x_2 - S_1 + 0S_2 + A_1 + 0A_2 = 4800$

$80x_1 + 50x_2 + 0S_1 - S_2 + 0A_1 + A_2 = 7200$

$x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$

The solution to this problem shall be obtained using simplex method. It is given in table 1 to 5.

Simplex Table I: Non-Optimal Solution

C_B	Basic Variables B	C_j	0	0	0	0	1	1	Minimum Ratio
		Solution Values b(= x_B)	x_1	x_2	S_1	S_2	A_1	A_2	
1	A_1	4800	20	50	-1	0	1	0	240
1	A_2	7200	(80)	50	0	-1	0	1	90 →
		Z_j	100	100	-1	-1	1	1	
		$\Delta_j = Z_j - C_j$	100 ↑	100	-1	-1	0	0	

Since all $z_j - c_j \nleq 0$. Hence this is not optimum solution. So we select most positive $z_j - c_j$

Simplex Table II: Non Optimal Solution

C_B	Basic Variables B	C_j	0	0	0	0	1	1	Minimum Ratio
		Solution Values b(= x_B)	x_1	x_2	S_1	S_2	A_1	A_2	
1	A_1	3000	0	(75/2)	-1	1/4	1	-1/4	80 →
0	x_1	90	1	5/8	0	-1/80	0	1/80	144
		Z_j	0	75/2	-1	1/4	1	-1/4	
		$\Delta_j = Z_j - C_j$	0	75/2 ↑	-1	1/4	0	-5/4	

Simplex Table III: Non Optimal Solution

C_B	Basic Variables B	C_j / Solution Values $b(=x_B)$	x_1	x_2	S_1	S_2	A_1	A_2
		C_j	0	0	0	0	1	1
0	x_2	80	0	1	– 2/75	1/150	2/75	–1/150
0	x_1	40	1	0	1/60	–1/60	–1/60	–1/60
		Z_j	0	0	0	0	0	0
		$\Delta_j = Z_j - C_j$	0	0	0	0	–1	–1

Phase II: The Simplex table III is reproduced here after replacing c_j row by the respective coefficients from the objective function of the original problem, and deleting the columns A_1 and A_2. The problem is then solved using simplex method as:

Simplex Table IV: Optimal Solution

C_B	Basic Variables B	C_j / Solution Values $b(=x_B)$	x_1	x_2	S_1	S_2	Minimum Ratio
		C_j	40	24	0	0	
0	x_2	80	0	1	–2/75	1/150	–3000
0	x_1	40	1	0	(1/60)	–1/60	2400 →
		Z_j	40	24	2/75	–38/75	
		$\Delta_j = Z_j - C_j$	0	0	2/75 ↑	– 38/75	

Since $Z_3 - C_3$ is most positive, the current feasible solution is not optimum (Minimisation Case). Therefore, S_1 enters the basis and the basic variable x_1 leaves the basis.

Simplex Table V: Optimal Solution

C_B	Basic Variables B	C_j / Solution Values $b(=x_B)$	x_1	x_2	S_1	S_2
		C_j	40	24	0	0
24	x_2	144	8/5	1	0	–1/50
0	S_1	2400	60	0	1	–1
		Z_j	194/5	24	0	–12/25
		$\Delta_j = Z_j - C_j$	– 8/5	0	0	–12/25

Since all $Z_j - C_j \leq 0$. (Minimisation case) this is optimal solution. The optimal solution given by Simplex table V is $x_1 = 0$ and $x_2 = 144$.

Example 43: Use two-phase simplex method to solve

Maximise $Z = 5x_1 - 4x_2 + 3x_3$

Subject to
$2x_1 + x_2 - 6x_3 = 20$
$6x_1 + 5x_2 + 10x_3 \leq 76$
$8x_1 - 3x_2 + 6x_3 \leq 50$
$x_1, x_2, x_3 \geq 0$

Solution: Introducing slack variables $S_1, S_2 \geq 0$ and an artificial variable $A_1 \geq 0$ in the constraints of the given LPP, the problem is reformulated in the standard form. Initial basic feasible solution is given by $A_1 = 20$, $S_1 = 76$ and $S_2 = 50$.

Phase I: Assigning a cost-1 to the artificial variable A_1 and Cost 0 to other variables, the objective function of the auxiliary LPP is,

Maximise $Z^* = 0x_1 + 0x_2 + 0x_3 + 0S_1 + 0S_2 - 1A_1$

Subject to
$2x_1 + x_2 - 6x_3 + A_1 = 20$
$6x_1 + 5x_2 + 10x_3 + S_1 = 76$
$8x_1 - 3x_2 + 6x_3 + S_2 = 50$
$x_1, x_2, x_3, S_1, S_2, A_1 \geq 0$

Simplex Table I: Non Optimal Solution

Basis		x_1	x_2	x_3	A_1	S_1	S_2	b_i	b_i/a_{i1}
A_1	–1	2	1	–6	1	0	0	20	20/2 = 10
S_1	0	6	5	10	0	1	0	76	76/6 = 12.66
S_2	0	8	–3	6	0	0	1	50	50/8 = 6.25 →
c_j		0	0	0	–1	0	0		
z_j		–2	–1	6	–1	0	0		
$z_j - c_j$		–2	–1	6	0	0	0		

↑ Incoming variable (x_1)

Since $z_1 - c_1$ and $z_2 - c_2$ is negative therefore this is not optimum solution for maximization case.

Simplex Table II: Non Optimal Solution

Basis		x_1	x_2	x_3	A_1	S_1	S_2	b_i	b_i/a_{i1}
A_1	–1	0	7/4	–15/2	1	0	–1/4	15/2	30/7→
S_1	0	0	29/4	11/2	0	1	–3/4	77/2	154/29
x_1	0	1	–3/8	3/4	0	0	1/8	25/4	
c_j		0	0	0	–1	0	0		
z_j		0	–7/4	15/2	–1	0	1/4		
$z_j - c_j$		0	–7/4	15/2	0	0	1/4		

↑ Incoming variable (x_2)

Since $z_2 - c_2 < 0$ an optimum solution to the auxiliary LPP has not been obtained

Simplex Table III: Non Optimal Solution

Basis		x_1	x_2	x_3	A_1	S_1	S_2	b_i
x_2	0	0	1	–30/7	4/7	0	–1/7	30/7
S_1	0	0	1	256/7	– 29/7	1	2/7	52/7
x_1	0	1	0	–6/7	3/4	0	1/14	55/7
c_j		0	0	0	–1	0	0	
z_j		0	0	0	0	0	0	
$z_j - c_j$		0	0	0	0	1	0	

Since all $z_j - c_j \geq 0$, an optimum solution to the auxiliary LPP has been obtained. Also Max $Z^* = 0$ with no artificial variable in the basis. We go to phase II.

Phase II: Consider the final simplex table of phase I. Consider the actual cost associated with the original variables. Delete the artificial variable A_1 column from the table as it is eliminated in phase I.

Simplex Table I: Optimal Solution

Basis		x_1	x_2	x_3	S_1	S_2	b_i
x_2	−4	0	1	−30/7	0	−1/7	30/7
S_1	0	0	0	256/7	1	2/7	52/7
x_1	5	1	0	−6/7	0	1/14	55/7
c_j		5	−4	3	0	0	
z_j		5	−4	90/7	0	13/14	
$z_j - c_j$		0	0	69/7	0	13/14	

Since all $z_j - c_j \geq 0$ an optimum basic feasible solution has been reached. Hence, an optimum feasible solution to the given LPP is $x_1 = 55/7$, $x_2 = 30/7$, $x_3 = 0$ and Max $Z = 155/7$.

Example 44: Solve the following LPP Two Phase Simplex Method;

Maximise $Z = -4a - 3b - 9c$;

Subject to $2a + 4b + 6c \geq 15$

$6a + b + 6c \geq 12$

$a, b, c \geq 0$

Solution:

Phase I: First we introduce surplus and artificial variables, and then rewrite the objective function by assigning a zero coefficient to the decision variables and a coefficient 1 to the artificial variables. The problem becomes:

Maximize $Z^* = 0a + 0b + 0c + 0S_2 + A_1 + A_2$

Subject to $2a + 4b + 6c - S_1 + 0S_2 + A_1 + 0A_2 = 15$

$6a + b + 6c + 0S_1 - S_2 + 0A_1 + A_2 = 12$

$x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$

Simplex Table I: Non Optimal Solution

Basis		a	b	c	S_1	S_2	A_1	A_2	b_i	Min Ratio $\left(\frac{b_i}{c}\right)$
A_1	−1	2	4	6	−1	0	1	0	15	$\frac{15}{6} = \frac{5}{2}$
A_2	−1	6	1	<u>6</u>	0	−1	0	1	12	$\frac{12}{6} = 2 \rightarrow$
C_j		0	0	0	0	0	−1	−1		
Z_j		−8	−5	−12	1	1	−1	−1		
$C_j - Z_j$		8	5	12	−1	−1	0	0		

↑ Incoming variable

Entering = c, Departing = A_2, Key Element = 6

$R_2\,(\text{new}) = R_2\,(\text{old}) \div 6 = R_2\,(\text{old}) \cdot \frac{1}{6}$

$R_1\,(\text{new}) = R_1\,(\text{old}) - 6R_2\,(\text{new})$

Simplex Table II: Non Optimal Solution

Basis		a	b	c	S_1	S_2	A_1	A_2	b_i	Min Ratio $\left(\frac{b_i}{b}\right)$
A_1	−1	−4	<u>3</u>	0	−1	1	1	−1	3	$\frac{3}{3} = 1 \rightarrow$
c	0	1	$\frac{1}{6}$	1	0	$-\frac{1}{6}$	0	$\frac{1}{6}$	2	$\frac{2}{\frac{1}{6}} = 12$
C_j		0	0	0	0	0	−1	−1		
Z_j		4	−3	0	1	−1	−1	1		
$C_j - Z_j$		−4	3	0	−1	1	0	−2		

↑ Incoming variable

Entering = b, Departing = A_1, Key Element = 3

$R_1\,(\text{new}) = R_1\,(\text{old}) \div 3 = R_1\,(\text{old}) \cdot \frac{1}{3}$

$R_2\,(\text{new}) = R_2\,(\text{old}) - \frac{1}{6} R_1\,(\text{new})$

Simplex Table III: Non Optimal Solution

Basis		a	b	c	S_1	S_2	A_1	A_2	b_i
b	1	$-\frac{4}{3}$	1	0	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	0
c	$\frac{11}{6}$	$\frac{11}{9}$	0	1	$\frac{1}{18}$	$-\frac{2}{9}$	$-\frac{1}{18}$	$\frac{2}{9}$	0
C_j		0	0	0	0	0	−1	−1	
Z_j		0	0	0	0	0	0	0	
$C_j - Z_j$		0	0	0	0	0	−1	−1	

Since all $C_j - Z_j \leq 0$,

Optimum solution is arrived with value of variables as:

$a = 0;\quad b = 1;\quad c = \frac{11}{6}$

Maximise $Z = 0$

Phase II:

Simplex Table I: Optimal Solution

Basis		a	b	c	S_1	S_2	A_1	A_2	b_i	Min Ratio $\left(\frac{b_i}{a}\right)$
b	−3	$-\frac{4}{3}$	1	0	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	1	-
c	−9	$\left(\frac{11}{9}\right)$	0	1	$\frac{1}{18}$	$-\frac{2}{9}$	$-\frac{1}{18}$	$\frac{2}{9}$	$\frac{11}{6}$	$\frac{\frac{11}{6}}{\frac{11}{9}} = \frac{3}{2} \rightarrow$
C_j		−4	−3	−9	0	0	−1	−1		
Z_j		−7	−3	−9	$\frac{1}{2}$	1	$-\frac{1}{2}$	−1		
$C_j - Z_j$		3	0	0	$-\frac{1}{2}$	−1	$-\frac{1}{2}$	0		

↑ Incoming variable

Entering = a, Departing = c, Key Element = $\frac{11}{9}$

$R_2\,(\text{new}) = R_2\,(\text{old}) \div \frac{11}{9} = R_2\,(\text{old}) \cdot \frac{9}{11}$; $R_1\,(\text{new}) = R_1\,(\text{old}) + \frac{4}{3} R_2\,(\text{new})$

Simplex Table II: Optimal Solution

Basis		a	b	c	S_1	S_2	A_1	A_2	b_i
b	–3	0	1	$\frac{12}{11}$	$-\frac{3}{11}$	$\frac{1}{11}$	$\frac{3}{11}$	$-\frac{1}{11}$	3
a	–4	1	0	$\frac{9}{11}$	$\frac{1}{22}$	$-\frac{2}{11}$	$-\frac{1}{22}$	$\frac{2}{11}$	$\frac{3}{2}$
C_i		–4	–3	–9	0	0	–1	–1	
Z_j		–4	–3	$-\frac{72}{11}$	$\frac{7}{11}$	$\frac{5}{11}$	$-\frac{7}{11}$	$-\frac{5}{11}$	
$C_j - Z_j$		0	0	$-\frac{27}{11}$	$-\frac{7}{11}$	$-\frac{5}{11}$	$-\frac{4}{11}$	$-\frac{6}{11}$	

Since all $C_j - Z_j \le 0$,

Optimum Solution is arrived with value of variables as:

$a = \frac{3}{2}$

$b = 3$

$c = 0$

Maximise $Z = -15$

Example 45: Solve the following LPP using the Two-phase method.

Minimise $Z = 10x_1 + 6x_2 + 2x_3$

Subject to $-x_1 + x_2 + x_3 \ge 1$

$3x_1 + x_2 - x_3 \ge 2$

and $x_1, x_2, x_3 \ge 0$

Solution: After applying **Two-Phase Method,** the steps are as follows:

Phase-1

The problem is converted to canonical form by adding slack, surplus and artificial variables as appropriate:

1) As the constraint 1 is of type '≥' we should surplus variable S_1 and add artificial variable A_1.
2) As the constraint 2 is of type '≥' we should subtract surplus variable S_2 and add artificially variable A_2.

After introducing surplus, artificial variables, we have the following equations:

Min $z = A_1 + A_2$

Subject to

$-x_1 + x_2 + x_3 - S_1 + A_1 = 1$

$3x_1 + x_2 - x_3 - S_2 + A_2 = 2$

and $x_1, x_2, x_3, S_1, S_2, A_1, A_2, \ge 0$

Table 1.12: Non-Optimal Solution

B	x_B	x_1	x_2	x_3	S_1	S_2	A_1	A_2	MinRatio $\frac{x_B}{x_1}$
A_1	1	–1	1	1	–1	0	1	0	–
A_2	2	(3)	1	–1	0	–1	0	1	$\frac{2}{3} = \frac{2}{3} \rightarrow$
z = 0	Z_j	2	2	0	–1	–1	1	1	
	C_j	0	0	0	0	0	1	1	
	$C_j - Z_j$	–2	–2	0	1	1	0	0	

↑ Incoming Variable

Negative minimum $C_j - Z_j$ is –2 and its column index is 1. So, the entering variable is x_1.

Minimum ratio is $\frac{2}{3}$ and its row index is 2. So, the leaving basis variable is A_2.

∴ The pivot element is 3.

Entering Variable = x_1,

Departing Variable = A_2, Key Element = 3

$R_2(\text{new}) = R_2(\text{old}) \div 3$

$R_1(\text{new}) = R_1(\text{old}) + R_2(\text{new})$

Table 1.13: Non-Optimal Solution

B	x_B	x_1	x_2	x_3	S_1	S_2	A_1	MinRatio $\frac{x_B}{x_2}$
A_1	$\frac{5}{3}$	0	$\left(\frac{4}{3}\right)$	$\frac{2}{3}$	–1	$-\frac{1}{3}$	1	$\frac{(5/3)}{(4/3)} = \frac{5}{4} \rightarrow$
x_1	$\frac{2}{3}$	1	$\frac{1}{3}$	$-\frac{1}{3}$	0	$-\frac{1}{3}$	0	$\frac{(2/3)}{(1/3)} = 2$
z = 0	Z_j	0	$\frac{4}{3}$	$\frac{2}{3}$	–1	$-\frac{1}{3}$	1	
	C_j	0	0	0	0	0	1	
	$C_j - Z_j$	0	$-\frac{4}{3}$	$-\frac{2}{3}$	1	$\frac{1}{3}$	0	

↑ Incoming Variable

Negative minimum $C_j - Z_j$ is $-\frac{4}{3}$ and its column index is 2. So, the entering variable is x_2.

Minimum ratio is $\frac{5}{4}$ and its row index is 1. So, the leaving basis variable is A_1.

∴ The Pivot element is $\frac{4}{3}$.

Entering Variable = x_2,

Departing Variable = A_1, Key Element = $\frac{4}{3}$

$R_1(\text{new}) = R_1(\text{old}) \times \frac{3}{4}$

$R_2(\text{new}) = R_2(\text{old}) - \frac{3}{4} R_1(\text{new})$

Table 1.14: Non-Optimal Solution

B	x_B	x_1	x_2	x_3	S_1	S_2	MinRatio
x_2	$\frac{5}{4}$	0	1	$\frac{1}{2}$	$-\frac{3}{4}$	$-\frac{1}{4}$	
x_1	$\frac{1}{4}$	1	0	$-\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$	
z = 0	Z_j	0	0	0	0	0	
	C_j	0	0	0	0	0	
	$C_j - Z_j$	0	0	0	0	0	

Since all $C_j - Z_j \ge 0$

Hence, optimal solution is arrived with value of variables as:

$$x_1 = \frac{1}{4}, x_2 = \frac{5}{4}, x_3 = 0$$

Min z = 0

Phase-2

We eliminate the artificial variables and change the objective function for the original,

Table 1.15: Optimal Solution

B	x_B	x_1	x_2	x_3	S_1	S_2	Min Ratio
x_2	$\frac{5}{4}$	0	1	$\frac{1}{2}$	$-\frac{3}{4}$	$-\frac{1}{4}$	
x_1	$\frac{1}{4}$	1	0	$-\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$	
z = 0	Z_j	10	6	–2	–2	–4	
	C_j	10	6	2	0	0	
	$C_j - Z_j$	0	0	4	2	4	

Since all $C_j - Z_j \geq 0$

Hence, optimal solution is arrived with value of variables as:

$$x_1 = \frac{1}{4}, x_2 = \frac{5}{4}, x_3 = 0$$

Min Z = 10

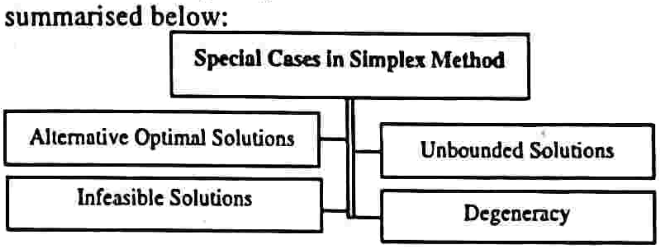
1.5.9.3. Difference between Two Phase and Big-M Method

Table below shows the difference between two-phase and Big-M method:

Two Phase Method	Big-M Method
It is easy to obtain initial basic feasible solution because a zero co-efficient is attached to basic variables.	It is difficult to obtain initial basic feasible solution because a zero co-efficient is not attached to basic variables.
It is easy and less time consuming.	It is tedious because variables (penalties) are involved.
This method does not involve any assumptions at the original system of constraints.	This method uses assumptions for constraints.

1.5.10. Special Cases in Simplex Method

While using simplex method for solving LPP, one may encounter many special situations which are summarised below:



1.5.10.1. Alternative Optimum Solutions

Sometimes an LPP may arrive at an optimum solution but not a unique one. This means that more than one optimum solution exists. In such cases, the following are the characteristics of the final table:

1) An optimum solution.
2) For all basic variables, $(Z_j - C_j)$ values are zero.
3) For non-basic variables, $(Z_j - C_j)$ values may or may not be zero.
4) For atleast one non-basic variable, $(Z_j - C_j)$ value will be zero.
5) More than one optimum solution is possible.
6) For all possible solutions value of objective function Z remains the same.
7) Basis in final table contains basic variable X, *viz.*, x_1, x_2,... and non-basic variables S, *viz.*, S_1, S_2.

Example 46: Maximise $Z = 3x_1 + 2x_2$

Subject to the constraints

$$-x_1 + 2x_2 \leq 4$$
$$3x_1 + 2x_2 \leq 14$$
$$x_1 - x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

Solution: Simplex Table I give the final iteration by using the 'Simplex Algorithm'.

Simplex Table I: Non Optimal Solution

		C_j	3	2	0	0	0
C_B	Basic Variables B	Solution Values b (= x_B)	x_1	x_2	s_1	s_2	s_3
0	s_1	6	0	0	1	$-\frac{1}{5}$	$\frac{8}{5}$
2	x_2	1	0	1	0	$\frac{1}{5}$	$-\frac{3}{5}$
3	x_1	4	1	0	0	$\frac{1}{5}$	$\frac{2}{5}$
	Z_j	14	3	2	0	1	0
Net Evaluation (or Index) Row: $C_j - Z_j$			0	0	0	–1	0

The optimal solution is: $x_1 = 4$, $x_2 = 1$, $s_1 = 6$ and Z = 14. Now, in the **Simplex Table I**, the non-basic variable s_3 has a relative profit of zero. This means that any increase in s_3 will bring no change in the objective function value. In other words, s_3 can be made a basic variable and the resulting basic feasible solution will also have the objective function value equal to 14.

This means, an alternative optimal solution to this problem exists which can be obtained by making s_3 a basic variable as shown below:

Simplex Table II: Multiple Optimal Solution

		C_j	3	2	0	0	0
C_B	Basic Variables	Solution Values	x_1	x_2	s_1	s_2	s_3
0	s_3	$\frac{15}{4}$	0	0	$\frac{5}{8}$	$\frac{3}{8}$	1
2	x_2	$\frac{13}{4}$	0	1	$\frac{3}{8}$	$\frac{1}{8}$	0
3	x_1	$\frac{5}{2}$	1	0	$-\frac{1}{4}$	$\frac{1}{4}$	0
	Z_j	14	3	2	0	1	0
New Evaluation (or Index) row: $C_j - Z_j$			0	0	0	–1	0

Thus alternative optimal solution is:

$$x_1 = \frac{5}{2}, x_2 = \frac{13}{4} \text{ and max } Z = 14.$$

Hence, this problem has no unique solution but has multiple optimal solutions.

1.5.10.2. Unbounded Solution

Unbounded solution occurs when there is no constraint on the solution and the decision variables can be increased infinitely under the same restricted conditions. Such unbounded solution occurs in maximisation problems, where the decision variable can be made infinitely large under the same constraints. This happens in the iteration stage of simplex method when all the entries in minimum ratio column are either infinite or negative.

Example 47: Maximise $Z = 5x_1 + 6x_2 + x_3$

Subject to the constraints

$9x_1 + 3x_2 - 2x_3 \leq 5$
$4x_1 + 2x_2 - x_3 \leq 2$
$x_1 - 4x_2 + x_3 \leq 3$
$x_1, x_2, x_3 \geq 0$

Solution: Introducing slack variables s_1, s_2 and s_3, the problem can be restated in standard form as follows:
Maximise $Z = 5x_1 + 6x_2 + x_3 + 0s_1 + 0s_2 + 0s_3$

Subject to the constraints

$9x_1 + 3x_2 - 2x_3 + s_1 = 5$
$4x_1 + 2x_2 - x_3 + s_2 = 2$
$x_1 - 4x_2 + x_3 + s_3 = 3$
$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$

The initial basic feasible solution is obtained by setting $x_1 = x_2 = x_3 = 0$ so that $s_1 = 5$, $s_2 = 2$ and $s_3 = 3$.

Simplex Table I

		C_j	5	6	1	0	0	0	
C_B	Basic Variables $b(=x_B)$	Solution Values B	x_1	x_2	x_3	s_1	s_2	s_3	Min. Ratio x_B/x_2
0	s_1	5	6	3	–2	1	0	0	5/3
0	s_2	2	4	<u>2</u>	–1	0	1	0	1→
0	s_3	3	1	–4	1	0	0	1	–
	Z_j	0	0	0	0	0	0	0	
New Evaluation (or Index) Row: $C_j - Z_j$			5	6↑	1	0	0	0	

Simplex Table II

		C_j	5	6	1	0	0	0	
C_B	Basic Variables B	Solution Values B $(= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3	Min. Ratio x_B/x_3
0	s_1	2	3	0	$-\frac{1}{3}$	1	$-\frac{3}{2}$	0	–
6	x_2	1	2	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	–
0	s_3	7	9	0	–1	0	2	1	–
	Z_j	6	12	6	–3	0	3	0	
Net Evaluation (or Index) Row: $C_j - Z_j$			–7	0	4	0	–3	0	

From **Simplex Table II**, one observes that the solution is not optimal because a positive value appears in the $C_j - Z_j$ row. According to this X_3 is the incoming variable and the corresponding column is pivot column. Computing the replacement ratio, we notice that all the three ratios are negative. This implies that we can increase x_3 indefinitely without driving one of the basic variables to zero. The given solution is, therefore, unbounded.

1.5.10.3. Infeasible Solution

Now it is very clear that a solution is said to be feasible if it satisfies all the constraints and the non-negativity conditions of a problem. However, in real life situations, it is possible that the constraints are not consistent or there is no feasible solution to the problem. In such cases, a non-feasible solution is obtained also known as infeasible solution and the situation is called **unfeasibility**. Also, a non-feasible situation occurs if an artificial variable appears in the basis of the solution for being optimal.

Example 48: Given the following initial simplex table:

		C_j	15	25	0	0	–M	–M
C_B	Basic Variables B	Solution Values $b(=x_B)$	x_1	x_2	s_1	s_2	A_1	A_2
–M	A_1	20	7	6	–1	0	1	0
0	s_2	30	8	5	0	1	0	0
–M	A_2	18	3	–2	0	0	0	1
Z_j			–10M	–4M	M	0	–M	–M
Net Evaluation (or Index) Row: $C_j - Z_j$			15 + 10M	25 + 4M	–M	0	0	0

Write down the original problem represented by the above table. What are the values x_1, x_2 in this problem?

Solution: The problem represented by the given table is:
Maximise $Z = 15x_1 + 25x_2 + 0s_1 + 0s_2 - MA_1 - MA_2$

Subject to the constraints

$7x_1 + 6x_2 - s_1\ A_1 = 20$ or $7x_1 + 6x_2 \geq 20$
$8x_1 + 5x_2 + s_2 = 30$ or $8x_1 + 5x_2 \leq 30$
$3x_1 - 2x_2 + A_2 = 18$ or $3x_1 - 2x_2 = 18$
$x_1, x_2 \geq 0$

		C_j	15	25	0	0	–M	–M	
C_B	Basic Variables B	Solution Values $b (= x_B)$	x_1	x_2	s_1	s_2	A_1	A_2	Min. Ratio x_B/x_2
–M	A_1	20	<u>7</u>	6	–1	0	1	0	20/7 →
0	s_2	30	8	5	0	1	0	0	30/8
–M	A_2	18	3	–2	0	0	0	1	18/3
Z_j			–10M	–4M	0	0	–M	–M	
Net Evaluation (for Index) Row: $C_j - Z_j$			15 + 10M	25 + 4M	–0	0	0	0	

Simplex Tableau II : Non Optimal Solution

		C_j	15	25	0	0	–M	–M	
C_B	Basic Variables B	Solution Values $b (= x_B)$	x_1	x_2	s_1	s_2	A_1	A_2	Min. Ratio x_B/x_2
15	x_1	$\frac{20}{7}$	1	$\frac{6}{7}$	$-\frac{1}{7}$	0		0	–
0	s_2	$\frac{50}{7}$	0	$\frac{13}{7}$	$\frac{8}{7}$	1		0	25/4 →
–M	A_2	$\frac{66}{7}$	0	$-\frac{32}{7}$	$\frac{3}{7}$	0		1	22
Z_j			15	$\frac{90}{7} + \frac{32M}{7}$	$-\frac{15}{7} - \frac{3M}{7}$	0		–M	
Net Evaluation (or Index) Row: $C_j - Z_j$			0	$\frac{85}{7} - \frac{32}{7}$	$\frac{15}{7} - \frac{3M}{7}$	0		0	

Simplex Tableau III : Infeasible Solution

		C_j	15	25	0	0	$-M$	$-M$
C_B	Basic Variables B	Solution Values b ($= x_B$)	x_1	x_2	s_1	s_2	A_1	A_2
15	x_1	$\frac{15}{4}$	1	$\frac{5}{8}$	0	$\frac{1}{8}$		0
0	s_2	$\frac{25}{4}$	0	$-\frac{13}{8}$	0	$\frac{7}{8}$		0
$-M$	A_2	$\frac{27}{4}$	0	$-\frac{31}{8}$	0	$-\frac{3}{8}$		1
Z_j			15	$\frac{75}{8}+\frac{31M}{8}$	0	$\frac{15}{8}-\frac{3M}{8}$		$-M$
Net Evaluation (or Index) Row: $C_j - Z_j$			0	$\frac{125}{8}-\frac{31M}{8}$	0	$-\frac{15}{89}+\frac{3M}{8}$		0

Now, since there is no positive entry in the net evaluation row, the procedure terminates but sill A_2 exists as a basic variables. Hence the problem has no solution or the problem is infeasible.

1.5.10.4. Degeneracy

Problem of degeneracy is caused when an LPP is solved using simplex method and a situation occurs in which there is a tie between two or more basic variables for 'leaving the bases'. This means that the values of one or more basic variables in the solution column are equal to zero or the minimum ratio to identify the basic variable to 'leave the basis' is not unique.

Degeneracy also occurs during iterations in the simplex method. Therefore, when there is a tie in the minimum ratios for leaving the basis, then the selection is made arbitrary. However, one can reduce the number of iterations for arriving at an optimal solution by adopting the following rules:

1) Detect degeneracy and divide the coefficients of the slack variables by the corresponding positive numbers of the key column in the row starting from left to right.
2) Comparing from left to right, the row that contains the smallest ratio becomes the key row.

In a situation where there is a tie between a slack and an artificial variable to leave the basis, the artificial variable should be given preference for leaving the basis. Also, in such cases, degeneracy need not be resolved.

Example 49: Consider the following system of problem

Max. $Z = 5x_1 + 3x_2$

Subject to $x_1 + x_2 \le 2$

$5x_1 + 2x_2 \le 10$

$-2x_1 - 8x_2 \ge -12$

$x_1, x_2 \ge 0$

Solution: We multiply the constraint three with −1 on both sides to have positive constant at the R.H.S. we get

Max. $Z = 5x_1 + 3x_2$

Subject to $x_1 + x_2 \le 2$

$5x_1 + 2x_2 \le 10$

$2x_1 + 8x_2 \le 12$

We introduce slack variable and assign '0' coefficient and convert the constraints into equation. Resultant objective function and constraint equations are given below:

Max. $Z = 5x_1 + 3x_2 + 0S_1 + 0S_2 + 0S_3$

Subject to $x_1 + x_2 + S_1 + 0S_2 + 0S_3 = 2$

$5x_1 + 2x_2 + 0S_1 + S_2 + 0S_3 = 10$

$2x_1 + 8x_2 + 0S_1 + 0S_2 + S_3 = 12$

$x_1, x_2, S_1, S_2, S_3 \ge 0$

Simplex Table I: Non Optimal Solution

C_j	Contribution per unit		5	3	0	0	0	
↓	Basic Variables	Solution Values	x_1	x_2	s_1	s_2	s_3	Min. Ratio
0	S_1	2	1	1	1	0	0	2
0	S_2	10	5	2	0	1	0	2→
0	S_3	12	2	8	0	0	1	6
		Z_j	0	0	0	0	0	
		$\Delta_j = C_j - Z_j$	5	3	0	0	0	

↑ Incoming Variable

We have resolved degeneracy as below:

	S_1	S_2	S_3
S_1	1/1	0/1	0/1
S_2	0/5	1/5*	0/5

→Least non-negative (+ve value)

Simplex Table II: Non Optimal Solution

C_j →	Contribution per unit		5	3	0	0	0	
↓	Basic Variables	Solution Values	x_1	x_2	s_1	s_2	s_3	Min. Ratio
0	S_1	0	0	3/5	1	−1/5	0	0
5	X_1	2	1	2/5	0	1/5	0	5
0	S_3	8	0	36/5	0	−2/5	1	40/36→
	Z_j	10	5	2	0	1	0	
		$\Delta_j = C_j - Z_j$	0	1	0	−1	0	

↑ Incoming Variable

Simplex Table III: Optimal Solution

C_j	Contribution per unit		5	3	0	0	0
↓	Basic Variables	Solution Values	x_1	x_2	s_1	s_2	s_3
0	S_1	−2/3	0	0	1	−1/6	−1/12
5	X_1	14/9	1	0	0	2/9	−1/18
3	X_2	10/9	0	1	0	−1/18	5/36
		Z_j	5	3	0	17/18	5/36
		$\Delta_j = C_j - Z_j$	0	0	0	−17/18	−5/36

Since all $C_j - Z_j$ element ≤ we have obtained optimum solution

$x_1 = \frac{14}{9}$,

$$\bar{S}_1 = B^{-1}S_1 = \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{1}{3} & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}, \bar{S}_2 = B^{-1}S_2 = \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{1}{3} & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{etc.}$$

, $Z_1 = \frac{100}{9}$ (Maximum Value)

1.6. DUALITY

1.6.1. Introduction

In a general sense the term 'dual' implies two or double. Duality in an LPP means that there are two different ways of analysing each linear programming problem. However, the solutions might be equivalent. Based on the same data, each LPP stated in its original form (primal) gets associated with another LPP (dual). This is known as duality in LPP. Since the dual of the dual is primal, it does not matter which of the two problems is called primal or dual.

The primal (original) LP model has a dual LP problem which is defined directly and systematically from the original. The two problems are so similar that the optimal solution of one automatically provides the optimal solution for the other. The two problems are closely related. Duality in LPP is used for two basic reasons.

1) If the original LPP consists of many constraints and less of variables, it may be difficult to arrive at an optimal solution. So, converting the primal problem into a dual problem can help to reduce the effort and time for arriving at the required decision solution.
2) Duality in LPP is a cost-effective method in a managerial decision-making process.

1.6.2. Primal and Dual Problem

There is a corresponding dual problem associated with every original or primal LP problem. The primal problem is transposed as a dual problem. This implies that:

1) The primal and the dual problem are opposite. In a sense that, if a primal problem is one of minimisation then its dual problem will be one of maximisation.
2) The primal problem with n variables will have a dual problem with n-constraints.
3) The primal problem with m constraints will have a dual problem with m-variables.

To understand duality better, we can say that a standard primal problem is a maximisation problem with ≤ inequalities unlike the standard problem in the LP which is a minimisation problem with the constraints as equalities. The standard primal problem can be explained as below:

Maximise $Z = c_1x_1 + c_1x_2 + ... + c_nx$ .....(1)

Subject to, $a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \le b_1$ .....(2)

$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \le b_2$ .....(3)

... ...

$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \le b_m$ .....(4)

$x_j \ge 0, \quad j = 1, 2, ..., n$ .....(5)

This is a known as a normal maximum problem. The dual of this problem is defined as:

Minimise $W = b_1y_1 + b_2y_2 + ... + b_my_m$ .....(6)

Subject to, $a_{11}y_1 + a_{21}y_2 + ... + a_{m1}y_m \ge c_1$ .....(7)

$a_{12}y_1 + a_{22}y_2 + ... + a_{m2}y_m \ge c_2$ .....(8)

... ...

$a_{1n}y_1 + a_{2n}y_2 + ... + a_{mn}y_m \ge c_n$ .....(9)

$y_i \ge 0, \quad i = 1, 2, ..., m$ .....(10)

1.6.3. Relationship between Primal and Dual

The relation between the standard and its corresponding dual is given by following table:

Table 1.16

If Primal	Then: Dual
1) Objective is to maximise	1) Objective is to minimise
2) Variable x_j	2) Constraint, j
3) Constraint i	3) Variable y_i
4) Variables x_j unrestricted in sign	4) Variable y_i is unrestricted in sign
5) Constraint i is = type	5) Constraint j is = type
6) ≤ Type constraints	6) ≥ Type constraints
7) ≥ 0 Variables	7) ≥ Inequality constraints
8) ≤ Inequality constraints	8) ≥ 0 Variables

Mathematically it is shown below:

Primal	Dual
Max $Z = c_1x_1 + c_2x_2 + ... + c_nx_n$ Subject to the constraints: $a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n \le b_i$ $i = 1, 2, ..., m$ $x_j \ge 0; \quad j = 1, 2, ..., n$	Min $Z^* = b_1w_1 + b_2w_2 + ... + b_mw_m$ Subject to the constraints: $a_{1j}w_1 + a_{2j}w_2 + ... + a_{mj}w_m \ge c_j$ $j = 1, 2, ..., n$ $w_i \ge 0 \quad i = 1, 2, ..., m$
Min $Z = c_1x_1 + c_2x_2 + ... + c_nx_n$ Subject to the constraints: $a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n \ge b_i$ $x_j \ge 0; \quad j = 1, 2, ..., n$ $i = 1, 2, ..., m$	Max $Z^* = b_1w_1 + b_2w_2 + ... + b_mw_m$ Subject to the constraints: $a_{1j}w_1 + a_{2j}w_2 + ... + a_{mj}w_m \le c_j$ $j = 1, ..., n$ w_i unrestricted, $i = 1, 2, ..., m$
Minimise $Z = c_1x_1 + c_2x_2 + ... + c_nx_n$ Subject to the constraints: $a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n = b_i$ $i = 1, 2, ..., m$ $x_j \ge 0; \quad j = 1, 2, ..., n$	Maximise $Z^* = b_1w_1 + b_2w_2 + ... + b_mw_n$ Subject to the constraints: $a_{1j}w_1 + a_{2j}w_2 + ... + a_{mj}w_m \le c_j$ $j = 1, 2, ..., n$ w_i unrestricted, $i = 1, 2, ..., m$
Maximise $Z = c_1x_1 + c_2x_2 + ... + c_nx_n$ Subject to the constraints: $a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n = b_i$ $i = 1, 2, ..., m$ x_j unrestricted, $j = 1, 2, ..., n$	Minimise $Z^* = b_1w_1 + b_2w_2 + ... + b_mw_m$ Subject to the constraints: $a_{1j}w_1 + a_{2j}w_2 + ... + a_{mj}w_m = c_j$ $j = 1, 2, ..., n$ w_j unrestricted, $i = 1, 2, ..., m$

1.6.4. Principles of Duality

The symmetrical form is used to construct the dual from the primal or primal from the dual. The rules for the same are:

1) The maximised objective function of the primal becomes the minimised objective function of the dual and *vice versa*.
2) The inequality signs are reversed in duality. This means that for all maximisation primal problems with ≤ type constraints, there will exist a minimisation dual problem with ≥ type constraints and vice versa. However, this reversal of inequality is true for all constraints except the non- negative conditions.
3) The constraint in the primal corresponds to a variable in the dual and vice versa. This means for a primal problem with m constraints and n variables, there is a dual problem with m variables and n constraints.

4) The right hand side constraints $b_1, b_2, \ldots, b_m$ of the primal become the coefficient of the dual variables $y_1, y_2, \ldots, y_m$ in the dual objective function Z*. Also the coefficients $c_1, c_2, \ldots, c_n$ of the primal variables $x_1, x_2, \ldots, x_n$ in the objective function become the right hand side constraints in the dual.

5) The matrix of coefficients of variables in primal is a transposition of the matrix of the coefficients of variables in dual and *vice versa*.

Example 50: For the LPP given below, write the dual.
Maximise $Z = 40x_1 + 35x_2$
Subject to $2x_1 + 3x_2 \leq 60$; $4x_1 + 3x_2 \leq 96$; $x_1, x_2 \geq 0$

Solution: The dual of above equation is:

Minimise $Z^* = 60y_1 + 96y_2$

Subject to $2y_1 + 4y_2 \geq 40$
$3y_1 + 3y_2 \geq 35$
$y_1, y_2 \geq 0$

Hence, we find that:

1) The primal problem is of the maximization type while the dual is of the minimisation type.

2) The constraint values 60 and 96 of the primal have become the co-efficient of the dual variables y_1 and y_2 in the objective function of the dual in that order; while the coefficients of the variables in the objective function of the primal have become the constraint values in the dual.

3) The first column of the coefficients in the constraints in the primal has become the first row in the constraints in the dual, and the second column has similarly become the second row.

4) The direction of inequalities in the dual is the reverse of that in the primal. Thus, while the inequalities in the primal are of the type ≤, they are of ≥ type in the dual.

We can represent them in matrix form as given in **figure 1.13:**

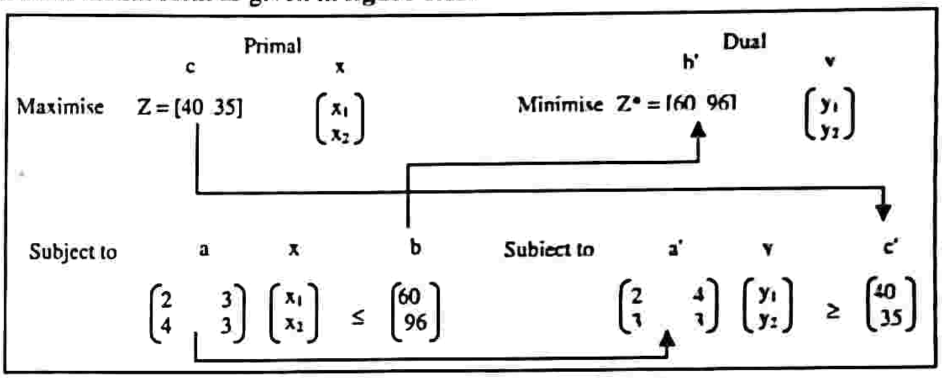


Figure 1.13: Primal and Dual Relationship

Example 51: Formulate the dual of the following linear programming problem:
Maximise $Z = 5x_1 + 3x_2$
subject to the constraints:
$3x_1 + 5x_2 \leq 15$,
$5x_1 + 2x_2 \leq 10$, and $x_1 \geq 0, x_2 \geq 0$

Solution: Standard Primal
Introducing slack variables $s_1 \geq 0$ and $s_2 \geq 0$,
the standard linear programming problem is:

Maximum $Z = 5x_1 + 3x_2 + 0s_1 + 0s_2$

subject to the constraints:

$$3x_1 + 5x_2 + s_1 + 0s_2 = 15,$$
$$5x_1 + 2x_2 + 0s_2 + s_2 = 10,$$
$$x_1, x_2, s_1, s_2 \geq 0,$$

Dual Primal
Let y_1 and y_2 be the dual variables corresponding to the primal constraints. Then the dual problem will be:
Minimise $Z^* = 15y_1 + 10y_2$ subject to the constraints:

$$3y_1 + 5y_2 \geq 5, \quad 5y_1 + 2y_2 \geq 3$$

$$\left.\begin{array}{l} y_1 + 0.y_2 \geq 0 \\ 0.y_1 + y_2 \geq 0 \end{array}\right\} \Rightarrow y_1 \geq 0 \text{ and } y_2 \geq 0$$

y_1 and y_2 unrestricted (redundant).

The dual variables "y_1 and y_2 unrestricted" are dominated by $y_1 \geq 0$ and $y_2 \geq 0$. Eliminating redundancy, the restricted variables are $y_1 \geq 0$ and $y_2 \geq 0$.

Example 52: Write the dual of the following LP problem:
Minimise $Z = 3x_1 - 2x_2 + 4x_3$
Subject to the constraints

$$3x_1 + 5x_2 + 4x_3 \geq 7$$
$$6x_1 + x_2 + 3x_3 \geq 4$$
$$7x_1 - 2x_2 - x_3 \leq 10$$
$$x_1 - 2x_2 + 5x_3 \geq 3$$
$$4x_1 + 7x_2 - 2x_3 \geq 2$$

and $x_1, x_2, x_3 \geq 0$

Solution: Standard Primal
Since the objective function of the given LP problem is of minimisation, the direction of each inequality of ≤ type has to be changed. The standard primal LP problem so obtained is:

Minimise $Z = 3x_1 - 2x_2 + 4x_3$

Subject to the constraints

$$3x_1 + 5x_2 + 4x_3 \geq 7$$
$$6x_1 + x_2 + 3x_3 \geq 4$$
$$-7x_1 + 2x_2 + x_3 \geq -10$$
$$x_1 - 2x_2 + 5x_3 \geq 3$$
$$4x_1 + 7x_2 - 2x_3 \geq 2$$

Dual Primal

If y_1, y_2, y_3, y_4 and y_5 are dual variables corresponding to the five primal constraints in given order, then the dual of this primal LP problem is:

Maximise $Z^* = 7y_1 + 4y_2 - 10y_3 + 3y_4 + 2y_5$

Subject to the constraints

$$3y_1 + 6y_2 - 7y_3 + y_4 + 4y_5 \le 3$$
$$5y_1 + y_2 + 2y_3 - 2y_4 + 7y_5 \le -2$$
$$4y_1 + 3y_2 + y_3 + 5y_4 - 2y_5 \le 4$$

and $y_1, y_2, y_3, y_4, y_5 \ge 0$

Example 53: Write the dual of the following LPP in standard form:

Minimise $Z = 2x_1 + 3x_2 + 4x_3$

Subject to the constraints

$$2x_1 + 3x_2 + 5x_3 \ge 2$$
$$3x_1 + x_2 + 7x_3 = 3$$
$$x_1 + 4x_2 + 6x_3 \le 5$$
$$x_1, x_2 \ge 0, x_3 \text{ unrestricted.}$$

Solution: Standard Primal

Changing the third constraints, the given LPP is

Minimise $Z = 2x_1 + 3x_2 + 4x_3$

Subject to the constraints

$$2x_1 + 3x_2 + 5x_3 \ge 2$$
$$3x_1 + x_2 + 7x_3 = 3$$
$$-x_1 - 4x_2 - 6x_3 \ge -5$$
$$x_1, x_2 \ge 0, x_3 \text{ unrestricted.}$$

Dual Primal

The dual of this problem is:

Maximise $Z^* = 2y_1 + 3y_2 - 5y_3$

Subject to the constraints

$$2y_1 + 3y_2 - y_3 \le 2$$
$$3y_1 + y_2 - 4y_3 = 3$$
$$5y_1 + 7y_2 - 6y_3 \le 4$$
$$y_1, y_2 \ge 0, y_3 \text{ unrestricted.}$$

The standard form of the dual is:

Maximise $Z^* = 2y_1 + 3y_2 - 5(y_3' - y_3'')$

Subject to the constraints

$$2y_1 + 3y_2 - (y_3' - y_3'') \le 2$$
$$3y_1 + y_2 - 4(y_3' - y_3'') = 3$$
$$5y_1 + 7y_2 - 6(y_3' - y_3'') \le 4$$
$$y_1, y_2, y_3', y_3'' \ge 0,$$

Where, y_3, being unrestricted, is equal to $y_3' - y_3''$

Example 54: Write the dual of the following LP problem:

Maximise $z = 5x_1 + 6x_2$

Subject to $4x_1 + 7x_2 = 20$

$$5x_1 + 2x_2 = 10$$
$$6x_1 + 8x_2 = 25$$
$$x_1, x_2 \ge 0$$

Solution: Primal is

Maximise $z_x = 5x_1 + 6x_2$

Subject to

$$4x_1 + 7x_2 = 20$$
$$5x_1 + 2x_2 = 10$$
$$6x_1 + 8x_2 = 25 \text{ and}$$
$$x_1, x_2 \ge 0;$$

Since 1st, 2nd, 3rd constraints in the primal are equalities, the corresponding dual variables y_1, y_2, y_3 will be unrestricted in sign.

Dual is

Minimise $z_y = 20y_1 + 10y_2 + 25y_3$

Subject to

$$4y_1 + 5y_2 + 6y_3 \ge 5$$
$$7y_1 + 2y_2 + 8y_3 \ge 6$$

and y_1, y_2, y_3 unrestricted in sign

Example 55: Convert the L.P.P from Primal into Dual:

Min $Z = 10x_1 + 15x_2$

Subject to the constraints:

$$3x_1 - 4x_2 \le 24$$
$$3x_1 + 5x_2 \ge 20$$
$$\text{Where } x_1, x_2 \ge 0$$

Solution: Standard Primal

Since the objective function of the given LP problem is of minimization, the direction of each inequality of $\le$ type has to be changed. The standard primal LP problem so obtained is:

Min $Z = 10x_1 + 15x_2$

Subject to the constraints

$$-3x_1 + 4x_2 \ge -24$$
$$3x_1 + 5x_2 \ge 20$$
$$\text{Where } x_1, x_2 \ge 0$$

Dual Primal

If y_1, y_2 are dual variables corresponding to the primal constraints in given order, then the dual of this primal LP problem is:

Maximise $Z^* = -24y_1 + 20y_2$

Subject to the constraints $-3y_1 + 3y_2 \le 10$

$4y_1 + 5y_2 \le 15$ and $y_1, y_2 \ge 0,$

Example 56: Obtain the dual problem of the following LPP:

Maximise: $f(x) = 2x_1 + 5x_2 + 6x_3$

Subject to:

$$5x_1 + 6x_2 - x_3 \le 6$$
$$-2x_1 + x_2 + 4x_3 \le 4$$
$$x_1 - 5x_2 + 3x_3 \le 1$$
$$-3x_1 - 3x_2 + 7x_3 \le 6$$
$$x_1, x_2, x_3 \ge 0$$

Also verify that the dual of the dual problem is the primal problem.

Solution: Dual of Dual of a Given Primal is again a Primal

Write down a primal problem with proper variables, constraints and non-negativity restrictions. Then convert this in to its dual and then convert this dual in to its dual which is equal to primal. The dual of the given LPP is

Minimise $Z^* = 6y_1 + 4y_2 + y_3 + 6y_4$

Subject to the constraints

$$5y_1 - 2y_2 + y_3 - 3y_4 \ge 2$$
$$6y_1 + y_2 - 5y_3 - 3y_4 \ge 5$$
$$-y_1 + 4y_2 + 3y_3 + 7y_4 \ge 6$$
$$y_1, y_2, y_3, y_4 \ge 0$$

Taking dual again, we obtain

Maximise

$$Z^* = 2z_1 + 5z_2 + 6z_3$$

Subject to the constraints

$$5z_1 + 6z_2 - z_3 \leq 6$$
$$-2z_1 + z_2 + 4z_3 \leq 4$$
$$z_1 - 5z_2 + 3z_2 \leq 1$$
$$-3z_1 - 3z_2 + 7z_3 \leq 6$$
$$z_1, z_2, z_3 \geq 0.$$

This is equal to primal. Thus **Dual of Dual of a Given Primal is again a Primal.**

1.7. SENSITIVITY ANALYSIS

1.7.1. Introduction

The study of "sensitivity" in respect of a Linear Programming's optimal solution along with parameters discrete changes (variations) is termed as Sensitivity analysis. These changes lead to a varying degree of sensitivity where changes in optimal solution may vary from no change at all to a substantial change for a given LP problem. Thus, sensitivity analysis helps in determination of the range over which there is no change in the optimal solution along with change in LP model parameters, instead of resolving the entire problem with new parameters as a new LP problem. The range for lower and upper parameters within which value may be assumed is easily known from the original optimal solution table that is considered as an initial solution table. The sensitivity studying process for an LP problem optimal solution is generally termed as **post-optimality analysis** because it is done after obtaining the optimal solution for a problem where assumption is made that for model, a given parameter set has been obtained.

1.7.2. Effects on an LP Model by Sensitivity Analysis

During the sensitivity analysis, 5 types of the discrete changes may be investigated in the original LP model. These changes are as follows:

1) c_j the cost or profit association per unit with both non-basic as well as basic decision variables (objective function's co-efficients).
2) b_i resources availability (constraint's right hand side).
3) a_{ij} resources consumption per unit of product (decision variable's co-efficients in constraint's left hand side).
4) New variable addition to the LP problem.
5) New constraint addition to the original LP problem model.

1.7.3. Graphical Approach to Sensitivity Analysis

The graphical method is used in this approach for obtaining the solution for a linear programming problem. **For example,** consider the following problem:

Maximize	$Z = 40x_1 + 35x_2$	Profit
Subject to	$2x_1 + 3x_2 \leq 60$	Raw
Material Constraint	$4x_1 + 3x_2 \leq 96$	Labour
Hours Constraint	$x_1, x_2 \leq 0$	

Here, x_1 gives the product A's number of units and similarly x_2 gives the product B's number of units.

Figure 1.14 shows the graph in which all constraints and Iso-profit lines are depicted as follows:

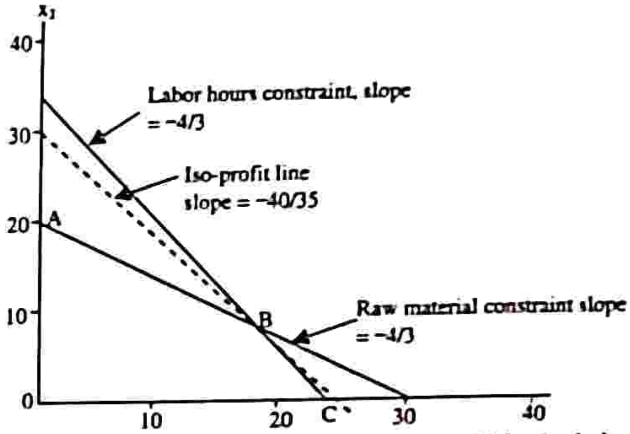
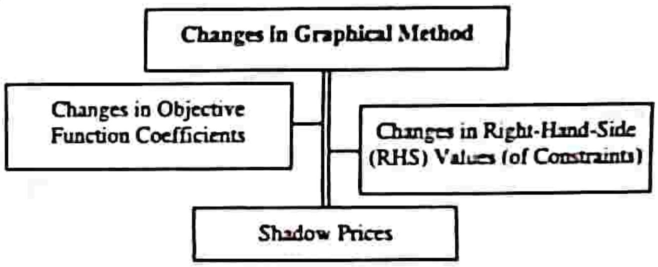


Figure 1.14: Graphic Presentation – Sensitivity Analysis

The point B in figure depicts the optimal solution to the LP in which Z = 1,000, x_2 = 8 and x_1 = 18, where basic variables are x_1 and x_2. Now, in turn one must examine how would there be any change in objective function co-efficient in the problem $(c_j\text{'s})$ or any change in this solution's right-hand-side $(b_i\text{'s})$.

Changes in Graphical Method

Due to the analysis of sensitivity, several changes take place which are as follows:



1.7.3.1. Changes in Objective Function Coefficients

Both products in the above optimal solution are produced as such. Now, if the product A's profits were to sufficiently increase, then the optimal solution is to produce only this product, and similarly if the product A's profits were to sufficiently decrease, then optimal solution for the company will be not to undergo production for this product, and only production of product B will be optimised. Now, we have to determine the profit values for product A such that current product mix will give optimal solution. In other words, if p_1 is the per unit profit for product A, then to keep the current basis optimal what value of P_2 should be taken.

Presently, $P_1 = 40$ where each iso-profit line will fall on linear line $40x_1 + 35x_2 = a$, where a is constant, or we can say $x_2 = (-40/35)x_1$ + constant, and thus slope of this line will be –40/35. Remember, a line possessing equation y = bx + a will have a slope b. By following this, the slope of the line representing labour hour constraints is –4/3 and the line representing the raw material constraints is –2/3.

From the graph, it is observed that any change in P_1 leads to flattening of iso-profit line in comparison to raw material constraint. It results in changing of the optimal solution to a new level (Point A) from the current level (Point B). Along profit P_1, the slope of raw material constraint will be –2/3 and that of iso-profit line will be $-P_1/35$. This follows the rule that $-P_1/35$ must not exceed –2/3 for the solution to be the same and optimal. Thus, there will be flattening of iso-profit lines as compared to the raw material constraint if $P_1 < 70/3$ or $(-P_1/35) > -2/3$, and there will be no optimal solution in current basis.

Likewise, if there is steepening of iso-profit lines comparatively to the labour hour's constraint line, then there will be a change in the optimal solution to a new level given by point C. With this constraint's slope being –4/3, $P_1 > 140/3$ or $-P_1/35 < -4/3$ will be followed and point C will be optimal solution and there will be no optimisation of current basis. Thus, keeping all other parameters unchanged for $70/3 \le P_1 \le 140/3$, there will be continuation of current basis to be optimal and company will continue the production of 8 units of product B and 18 units of product A. Of course, there will be a change in profit even if $70/3 \le P_1 \le 140/3$.

Likewise, determination of the range of profit values can be done on product B, say P_2 for which there would be no change in optimal solution. Here, there will be shift in optimal solution to point A and point C while the conditions be $-40//P_2 > -2/3$ or $P_2 > 60$ and $-40/P_2, > -4/3$ or $P_2 < 30$ respectively. Thus, unaltered optimal solution can be obtained under condition $30 \le P_2 \le 60$ keeping other parameters unchanged.

1.7.3.2. Changes in Right-Hand-Side(RHS) Values (Constraints)

Graphical determination can help to determine whether any change in the right hand side of constraint would lead to change in current basis making it no longer optimal. Let's consider the raw material constraint at first where b_1 is the available quantity which is 60 at present. An increase in b_1 leads to upward shift in the constraint parallel to its actual position currently and a decrease in b_1 will lead to downward shift in the same. Since, both constraints are bind to the current optimal solution, thus due to any change in b_1 as long as the point where there is binding of both these constraints remains feasible. There must be occurrence of the optimal solution at the point of intersection of these constraints.

It is observable from the **figure 1.15** that when $b_1 > 96$, there shall be no longer optimal solution for the current basis. Also, at the point where $b_1 = 96$ and two constraints intersect each other is K, where value of $x_1 = 0$, $x_2 = 32$ and there is full employment of both the resources. Likewise, there is no longer optimal solution with respect to current basis when $b_1 \le 48$. There is intersection of two constraints at point C with value of $b_1 = 48$. There will be binding only with $x_2 < 0$ at an infeasible point while the value of b_1 will be less than 48 and hence no optimal solution will be there on the current basis.

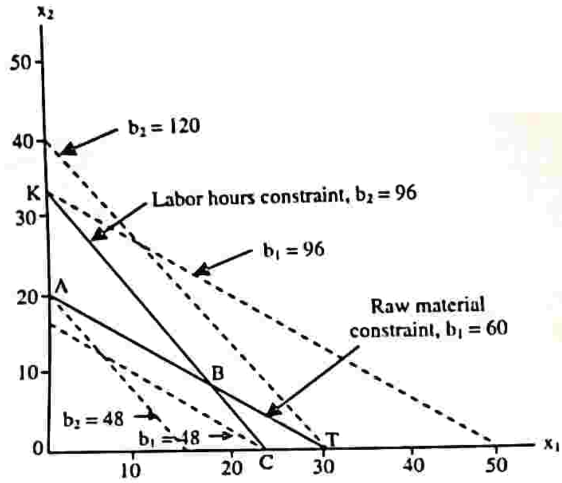


Figure 1.15: Graphic Presentation: RHS Ranging

Thus, it is clearly visible if $48 \le b_1 \le 96$, keeping the all other parameters unchanged, optimal solution can be obtained on current basis. It is noticeable that although the current basis comes out to be optimal at condition $48 \le b_1 \le 96$, still there may be changes in the values of objective function and the decision variables (x_1 and x_2).

For example, there would be change in optimal solution from point B to point BC (another line segment) if the condition is $48 \le b_1 \le 60$, and while the condition is $60 \le b_1 \le 96$, there would be change in optimal solution from point B to the line segment BK's some other point.

The second constraint's range of values for the optimal solution on the current basis can also be determined through this which remains unaffected. Consider b_2 be the value of RHS which is 96 at present. To hold the optimal condition, value of b_2 can be increased up to 120 keeping both the constraints bound to each other where revision of lower limit may get to 60. This means on the current basis, optimality shall be valid for the range $60 \le b_2 \le 120$.

To obtain the optimal solution, a range for current basis may be easily determined that how change in constraint's RHS (right-hand-side) value changes the decision variable's value. Recall that current basis remained optimal for value $-12 \le \Delta \le 36$ on changing the value of b_1 to $60 + \Delta$. Since binding of both the constraints is evaluated due to the current basis; the new decision variable's new values are easy to find by solving the equations, $4ax_1 + 3x_2 = 96$, and $2x_1 + 3x_2 = 60 + \Delta$, which results in the values of x_1 and x_2 as $x_1 = 18 - \Delta_2$, and $x_2 = 8 + (2/3)\Delta$ respectively.

A decrease in output of product A and simultaneous increase in product B's output can be attained by increasing the raw material availability. Suppose, the increase in raw material is to 72kg, then change is represented as $\Delta = 72 - 60 = 12$, for which optimal mix would come out to be $x_1 = 18 - 12/2$ units and $x_2 = 8 + (2/3) \times 12 = 16$ units and the total profit turns out to be 1,040.

Likewise, in the case when the labour availability changes to $96 + \Delta$, then the optimal solution on current basis comes out to be $-36 \leq \Delta \leq 24$. Now again as there is bound of both the constraints on current basis, thus the new optimal solution is obtained by getting solution of the equations simultaneously, which are $4x_1 + 3x_2 = 96 + \Delta$ and $2x_1 + 3x_2 = 60$. The values of x_1 and x_2 comes out to be $x_1 = 18 + \Delta/2$ and $x_2 = 8 - \Delta/3$. Also, the total 90 hours availability will give change of $\Delta = -6$ for which the optimal solution values will be as follows; $x_1 = 18 - 6/2 = 15$ and $x_1 = 8 - (-6/3) = 10$, and $Z = 950$.

1.7.3.3. Shadow Prices

Any change in profit occurring due to unit change in resource's amount is termed as the **shadow price** of a resource. Thus, shadow price deals and states how RHS constraints value changes the optimal Z-value of overall solution. Formally it may be described that the i^{th} constraint's shadow price of a linear problem programming is defined as the amount that helps to improvise the optimal value of the objective function (so that for a maximisation problem, Z-value increases and for a minimisation problem, Z-value decreases) by increasing the RHS constraint's value by unity. However, this is only applicable if the RHS variation of the constraint i leave the optimal solution with current basis.

In **above example**, if the available raw material is $60 + \Delta$ kg on keeping the optimal solution at the current basis, optimal solution comes out to be for values $x_1 = 18 - \Delta/2$ and $x_2 = 8 + (2/3)\Delta$. Also, the value for optimal solution for Z-value is $40 \times (18 - \Delta/2) + 35 (8 + 2\Delta/3)$ or $1,000 + 10\Delta/3$. Thus, an increment in raw material by unit ($\Delta = 1$) shall lead to increase in profit by 10/3 as long as an optimal solution is obtained with the current basis. Hence, raw materials shadow price comes out to be 10/3 per kg.

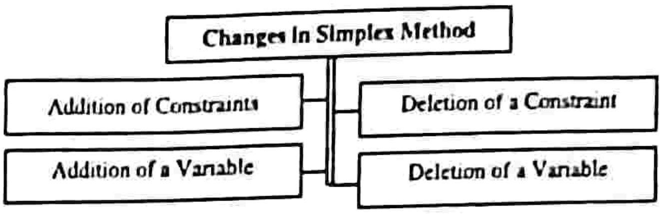
Likewise, if labour hours availability varies as $96 + \Delta$, the variable's optimal values with current basis over the range tends to be optimal at values $x_1 = 18 + \Delta/2$ and $x_2 = 8 - \Delta/3$. The value of Z, objective function with these decision variable's values would come out to be $40 (18 + \Delta/2) + 35 (8 - \Delta/3) = 1,000 + 25\Delta/3$. The current basis that holds within a particular range, for that each additional hour adds a profit of 25/3 whereas reduction by each hour will also reduce the profit with the same rate. Accordingly, labour's shadow price comes out to be 25/3 per hour.

1.7.4. Simplex Method Approach to Sensitivity Analysis

A simplex table is drawn in this approach with the help of simplex method of LPP for obtaining the optimal solution.

Changes in Simplex Method

Various changes can be made in the simplex methods which are as follows:



1.7.4.1. Addition of Variables

In most of the cases, on adding a new variable, there must be re-solving of the problem. However, there is a procedure of net marginal profit that helps in determining whether the new variable addition will possess any impact on the optimal solution or not. The difference between the total marginal resources cost and that of the objective function co-efficient is termed as the Net marginal profit. The total marginal resources cost can be easily calculated with the use of shadow price's current values.

Example 57: Consider the following table which presents an optimal solution to some linear programming problem:

		$c_j \rightarrow$	2	4	1	3	2	0	0	0
C_B	Vectors in basis	X_B	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8
2	Y_1	3	1	0	0	-1	0	0.5	0.2	-1
4	Y_2	1	0	1	0	2	1	-1	0	0.5
1	Y_3	7	0	0	1	-1	-2	5	-0.3	2
	z	17	0	0	0	2	0	2	0.1	2

If the additional constraint $2x_1 + 3x_2 - x_3 + 2x_4 + 4x_5 \leq 5$ were occupy to the system, would there be any change in the optimal solution? Justify your answer.

Solution: The optimal solution from the above table is

$x_1 = 3$, $x_2 = 1$ and $x_3 = 7$ (basic);

$x_4 = x_5 = x_6 = x_7 = x_8 = 0$ (non-basic)

This also satisfies the new additional constraint,

$$2x_1 + 3x_2 - x_3 + 2x_4 + 4x_5 \leq 5$$

Thus the optimum basic feasible solution of the original problem does not change if we introduce the above constraint to it. Hence the additional constraint is redundant and the optimum solution of the given L.P.P. is also optimum for the new L.P.P. If the additional constraint of right-hand side constant is 1 instead of 5, then the optimum solution does not satisfy the additional constraint. In this case the modified optimum simplex table is obtained by inserting the entries corresponding to the new additional constraint. Dual simplex method is then used to obtain the revised optimum solution. It may be observed that the optimum value of the objective function remains the same, while the new optimum basic feasible solution will be:

$x_1 = 3$, $x_2 = 0$, $x_3 = 9$ and $x_5 = 1$.

1.7.4.2. Deletion of Variables

If in the optimal solution, the variable to be deleted is zero; in that case there will be no effect on the optimal solution on deleting the variable. If in the optimal solution the variable to be deleted is not zero, there must be re-solving of the problem to obtain the optimal solution.

A worse objective function value is generally obtained by deletion of a non-zero variable in the original optimal solution. The resulted solution is basically one that is at best no better as compared to the original objective function value.

1.7.4.3. Addition of Constraints

In a linear programming model, on adding a constraint, the first step is to determine whether the current optimal solution will satisfy with this constraint or not. If condition of optimal solution is satisfied, there is no need to resolve the problem. However, if there is violation of the new constraint, there must be resolving of the problem. Due to more constraints in the problem, the optimal objective function value obtained will not be better as compared to the original optimal value which is smaller for maximisation and larger for the maximisation problems.

Example 58: Given the L.P.P.

Maximize $z = 3x_1 + 5x_2$

Subject to:

$$x_1 \le 4,$$
$$3x_1 + 2x_2 \le 18$$
$$\text{and } x_1, x_2 \ge 0.$$

If a new variable x_5 is introduced, with $c_5 = 7$ and $a_5 = [1, 2]$; discuss the effect of adding the new variable and obtain the revised solution if any.

Solution: Introducing slack variables $s_1 \ge 0, s_2 \ge 0$ and then solving by simplex method; the optimum solution is displayed in the table given below:

c_B	y_B	x_B	y_1	y_2	y_3	y_4
0	y_3	4	1	0	1	0
5	y_2	9	3/2	1	0	1/2
		z (=45)	9/2	0	0	5/2

From the above table, we observe that

$$x_B = [4, 9] \text{ and } B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$$

Since a new variable x_5 is introduced, we compute

$$y_5 = B^{-1} a_5 = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} [1, 2] = [1, 1].$$

and $z_5 - c_5 = c_B{}^T y_5 - c_5 = (0, 5)[1, 1] - 7 = -2.$

This indicates that the condition of optimality is violated. Therefore a new (improved) optimum solution can easily be obtained by entering y_5 into the basis:

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
0	y_3	4	1	0	1	0	1*
5	y_2	9	3/2	1	0	1/2	1
		z(=45)	9/2	0	0	5/2	-2

Final Iteration: Optimum solution.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
7	y_5	4	1	0	1	0	1
5	y_2	5	1/2	1	-1	1/2	0
		z(=53)	13/2	0	2	5/2	0

Optimum solution is

$x_1 = 0, x_2 = 5$, and $x_5 = 4$ with maximum $z = 53$.

If we have $c_5 = 3$ instead of 7, then we would have,

$z_5 - c_5 = 2 > 0.$

Thus the optimality of the given problem is not affected by the post-optimal addition of x_5.

1.7.4.4. Deletion of Constraints

There may be two types of constraints to be deleted in a linear programming problem. These are either binding or unbinding (redundant) constraints on the optimal solution. There will be no impact on the optimal solution in case unbinding constraints are deleted, as they only help in enlargement of the feasible region.

Moreover, there will be no impact on the optimal solution if there is no binding because of constraint having a slack or surplus variable of zero value in the basis matrix. There will be post-optimality problem in case a binding constraint is deleted from the LPP. Thus addition of one or two new variables is the simplest way to go along with this kind of problem.

For example, assume a new product **Big Squirts** which requires 5 minutes for the production,'needs 3 pounds of plastic for production and can produce a yielding profit of 10 per dozen. So the model for such a problem is:

Maximise $8X_1 + 5X_2 + 10X_3$

Such that $2X_1 + X_2 + 3X_3 \le 1000$ (Plastic)

$3X_1 + 4X_2 + 5X_3 \le 2400$ (Production time)

$X_1 + X_2 + X_3 \le 700$ (Total units)

$X_1 - X_2 + \le 350$ (Space ray/Zapper mix)

$X_1, X_2, X_3 \ge 0$

The constraint's shadow prices for the above problem comes out to be: ₹3.40, ₹0.40, ₹0, and ₹0, respectively.

Thus, for the production of a dozen Big Squirts the net marginal profit comes out to be:

₹10 – ((₹3.40)(3) + (₹0.40)(5) + (₹0)(1) + (₹0)(0)) = –₹2.20

Thus, production of Big Squirts would not be profitable and thus the optimal solution remains production of 320 dozen Space Rays and 360 dozen Zappers. In case the profit per dozen for the Big Squirts had been 15, there would have been a net margin of ₹2.80. A new optimal solution is indicated via this which includes Big Squirts production yielding an optimal profit that is higher.

Example 59: Consider the optimal Table of a maximisation problem:

Table 1.17

		c_j	3	5	0	0	7
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	x_5
7	x_5	4	1	0	1	0	1
5	x_2	5	1/2	1	–1	1/2	0
	$z_j - c_j$	53	13/2	0	2	5/2	0

Find the change in the optimal solution, when the basic variable x_2 is deleted.

Solution: Since this problem is of maximisation type we assign a cost – M to the variable x_2. The modified simplex table becomes:

Table 1.18

		c_j	3	–M	0	0	7
C_B	Y_B	X_B	x_1	x_2	x_3	$x_4\downarrow$	x_5
7	x_5	4	1	0	1	0	1
–M	$\bar{x}_2$	5	1/2	1	–1	[1/2]	0
			$\frac{-M+8}{2}$	0	M + 7	–M/2	0

First Iteration: Introduce x_4 and drop x_2.

Table 1.19

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	x_5
7	x_5	4	1	0	1	0	1
0	x_4	10	1	2	–2	1	0
		28	4	M	7	0	0

Since all $z_j - c_j \geq 0$, the current solution is optimum. The optimal solution to the new problem is:

$x_1 = 0, x_2 = 0$ and $x_5 = 4$ with maximum of $z = 28$.

1.8. EXERCISE

1.8.1. Short Answer Type Questions

1) What is quantitative technique?
2) What are the characteristics of quantitative technique?
3) Discuss the quantitative approaches to decision making.
4) What is the role of QT in decision making?
5) Discuss the quantitative approach to problem solving.
6) Classify the models of quantitative techniques.
7) What are the general methods of solving quantitative models?
8) Explain the quantitative analysis process in detail.
9) What are the advantages and limitation of quantitative techniques?
10) What is the scope of quantitative techniques?
11) What is Graphical Method? Give the procedure of graphical method.
12) Explain the areas where linear programming can be applied.
13) What is Simplex method?
14) What do you mean by Linear Programming? Describe the limitations of L.P.
15) What are the components of linear programming?
16) What is the meaning of duality? What is the principle of duality?
17) Explain dual simplex method.
18) What is sensitivity analysis?
19) Give the simplex method approach to sensitivity analysis.

1.8.2. Long Answer Type Questions

1) Solve the L.P. problem:
   Minimise: $Z = x_1 - 3x_2 + 2x_3$
   Subject to the constraints:
   $3x_1 - x_2 + 3x_3 \leq 7$
   $-2x_1 + 4x_2 \leq 12$
   $-4x_1 + 3x_2 + 8x_3 \leq 10$
   and $x_1, x_2, x_3 \geq 0$

   [**Ans:** Max $Z = 11, x_1 = 4, x_2 = 5, x_3 = 0$, Min $Z = -11, x_1 = 4, x_2 = 5, x_3 = 0$]

2) A company makes two kinds of belts. Belt A is of high quality and belt B is of lower quality. The respective profits are ₹8 and ₹6 per belt. Each belt of type A requires twice as much time as belt of type B and if all belts were of type B, the company could make 1,000 belts per day. The supply of leather is sufficient for only 800 belts (both A and B combined). Belt A requires a fancy buckle and only 400 such buckles are available per day. Three are only 700 buckles a day available for type B. Determine the number of belts to be produced for each type so as to maximise profit. Formulate and solve the problem graphically.

   [**Ans:** x(A type belt) = 200, y(B type belt) = 600 max z = 5200]

3) Use simplex method to solve the problem:
   Minimise: $15/2x_1 - 3x_2$
   Subject to the constraints:
   $3x_1 - x_2 - x_3 \geq 3$
   $x_1 - x_2 + x_3 \geq 2$
   and $x_1, x_2, x_3 \geq 0$

   [**Ans:** $x_1 = 5/4, x_2 = 0, x_3 = 3/5$, Minimise $Z = 0$]

4) Solve the L.P.P.
   Max $Z = 3X_1 + 2X_2$
   Subject to $4X_1 + 3X_2 \leq 12$
   $4X_1 + X_2 \leq 8$
   $4X_1 - X_2 \leq 8$
   $X_1, X_2 \geq 0$

   [**Ans:** $X_1 = 3/2, X_2 = 2$ and, Max $Z = 17/2$]

5) Using simplex method solves the L.P.P.
   Maximize $Z = X_1 + X_2 + 3X_3$
   Subject to $3X_1 + 2X_2 + X_3 \leq 3$
   $2X_1 + X_2 + 2X_3 \leq 2$
   $X_1, X_2 X_3, \geq 0$

   [**Ans:** $X_1 = 0, X_2 = 0, X_3 = 1$, Max $Z = 3$]

6) Solve graphically the following LPP:
   Maximise $Z = 8x_1 + 16x_2$
   Subject to $x_1 + x_2 \leq 200$
   $x_2 \leq 125$
   $3x_1 + 6x_2 \leq 900$
   $x_1, x_2 \geq 0$

   [**Ans:** $x_1 = 100$ $x_2 = 100$ and Max $Z = 2400$ or $x_1 = 50$ $x_2 = 125$ and Max $Z = 2400$]

7) Solve graphically:
   Maximise $Z = 10x_1 + 15x_2$
   Subject to $2x_1 + x_2 \leq 26$
   $2x_1 + 4x_2 \leq 56$
   $x_1 - x_2 \geq -5$
   $x_1, x_2 \geq 0$

   [**Ans:** $x_1 = 8$ and $x_2 = 10$ and Z is maximum (it is equal to 230)]

8) A retired person wants to invest upto an amount of ₹30,000 in fixed income securities. His broker recommends investing in two bonds – Bond A yielding 7% and Bond B yielding 10%. After some consideration, he decides to invest at most ₹12,000 in Bond B and atleast ₹6,000 in Bond A. He also wants the amount invested in Bond A to be atleast equal to the amount invested in Bond B. What should the broker recommend if the investor wants to maximise his return on investment? Solve graphically.

[Ans: (invest ₹18,000 in Bond A and ₹12,000 in Bond B. It would yield a return of ₹2,460]

9) Solve the LPP by simplex method.

Maximize: $z = 3x_1 + 5x_2 + 4x_2$

Subject to $2x_1 + 3x_2 \leq 8$

$2x_2 + 5x_3 \leq 10$

$3x_1 + 2x_2 + 4x_3 \leq 15$

and $x_1, x_2, x_3 \geq 0$

[Ans: $x_1 = 89/41$, $x_2 = 50/41$, $x_3 = 62/41$; max $z = 765/41$]

10) Use the simplex method to solve the following LP problem.

Maximize $Z = 3x_1 + 5x_2 + 4x_3$

Subject to $2x_1 + 3x_2 \leq 8$

$2x_2 + 5x_3 \leq 10$

$3x_1 + 2x_2 + 4x_3 \leq 15$

and $x_1, x_2, x_3 \leq 0$

[Ans: $x_1 = 89/41$, $x_2 = 50/41$, $x_3 = 62/41$; Max $Z = \frac{765}{41}$]

11) Solve the following problem using simplex method

Maximize $Z = 21x_1 + 15x_2$

Subject to $-x_1 - 2x_2 \geq -6$

$4x_1 + 3x_2 \leq 12$

$x_1, x_2 \geq 0$

[Ans: $x_1 = 3$, $x_2 = 0$, Max $Z = 63$]

12) Mahendra is in jewellery business and makes rings and bracelets of silver and gold. Each ring takes 3 units of silver and 1 unit of gold and each bracelet takes 1 unit of silver and 2 units of gold. Mahendra has 9 units of silver and 8 units of gold. Each ring brings ₹40 as profit and each bracelet ₹50 as profit. His objective is to maximise profit. Develop a linear programming model and obtain the optimum solution and then answer the following:

i) What happens if Mahendra has 1 unit less silver?
ii) What happens if he has 1 unit more gold?
iii) What happens if the contribution of rings increases by ₹10 per ring?

[Ans: Optimum solution: 2 rings and 3 bracelets; maximum profit is ₹230.

i) 1.6 rings and 3.2 bracelets; maximum profit is ₹224.
ii) 1.8 rings and 3.6 bracelets; maximum profit is ₹252.
iii) Solution remains unchanged but maximum profit increases to ₹260.]

13) Write down dual of the following LPP and solve it:

Maximise $Z = 8x_1 + 4x_2$

Subject to the constraints:

$4x_1 + 2x_2 \leq 30$

$2x_1 + 4x_2 \leq 24$

$x_1, x_2 \geq 0$

[Ans: Minimise $Z^* = 30w_1 + 24w_2$

subject to the constraints:

$4w_1 + 2w_2 \geq 8$

$2w_1 + 4w_2 \geq 4$

$w_1 \geq 0$ and

$w_2 \geq 0$

Optimum solution: $x_1 = 6$ and $x_2 = 3$; maximum $z = 60$]

Unit 2

Linear Programming Extensions

2.1. TRANSPORTATION MODELS

2.1.1. Introduction

Transportation model is defined as the study of optimal transportation and resource allocation.

Transportation model is defined as the distribution of goods from many points of supply to a number of points of demand, where the points of supply are known as origins or sources and points of demand are known as destinations or sinks. It is also involves in determination of the minimum cost to allocate to a product from several supply sources to several destinations.

The transportation problem is to transport different amounts of a single homogenous product, which are initially kept at various origins, to different destinations with the objective of minimising the total transportation cost.

For example, let us consider a cold drink manufacturing company has four plants which are situated in four different cities. Shops which are located in four different cities consume the total production of these four plants. Here in this example, supply and demand of cold drinks lead to a transportation problem as we need to find a transportation schedule that minimises the total cost of transporting cold drinks from different plant locations to various shops.

Transportation model is a special class of the linear programming problem (LPP).

2.1.2. Mathematical Model of Transportation Problem

Given m sources and n destinations, the supply at source i is a_i and the demand at destination j is b_j. The cost of shipping one unit of goods from source i to destination j is c_{ij}. The goal is to minimise the total transportation cost while satisfying all the supply and demand restrictions.

Let,

a_i = Quantity of commodity available at origin i,

b_j = Quantity of commodity required at destination j,

c_{ij} = Cost of transporting one unit of commodity from source/origin to destination j

x_{ij} = Quantity transported from origin i to destination j.

Then the problem is to determine the transportation schedule so as to minimise the total transportation cost satisfying supply and demand constraints.

Mathematically, the problem may be stated as a linear programming problem as follows:

Minimise Total Cost $z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$

Subject to $\sum_{j=1}^{n} x_{ij} = a_i \quad \text{for } i = 1, 2, ..., m$

$\sum_{i=1}^{m} x_{ij} = b_j \quad \text{for } j = 1, 2, ..., n$

And $x_{ij} \geq 0 \quad \text{for all } i = 1, 2, ..., m \text{ and } j = 1, 2, ..., n$

For ease in presentation and solution, a transportation problem is generally presented as shown in transportation **table 2.1.**

Table 2.1: Transportation Table

Origin (I)	Destination (J)				Supply(a_i)
	1	2		n	
O_1	c_{11} x_{11}	c_{12} x_{12}	...	c_{1n} x_{1n}	a_1
O_2	c_{21} x_{21}	c_{22} x_{22}	...	c_{2n} x_{2n}	a_2
...	...	...	...	...	...
O_m	c_{m1} x_{m1}	c_{m2} x_{m2}	...	c_{mn} x_{mn}	a_m
Demand, b_j	b_1	b_2	...	b_n	$\Sigma a_i = \Sigma b_j$

Let $O_1, O_2, \ldots, O_m$ be m plant having $a_1, a_2, \ldots, a_m$ product respectively is. Let $d_1, d_2, \ldots, d_n$ be n destinations each of which have the $b_1, b_2, \ldots, b_n$ requirements respectively.

The objective function minimises the total cost of transportation (z) between various sources and destinations. The constraint i in the first set of constraints ensures that the total units transported from the source I is less than or equal to its supply.

The constraint j in the second set of constraints ensures that the total units transported to the destination j is greater than or equal to its demand.

Total Supply = Total Demand

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} x_{bj}$$ (Also called **rim condition**)

That is, the total capacity (or supply) must be equal to the total requirement (or demand).

Example 1: There are three fertilizer plants A, B, and C of a Tata Chemicals in Lucknow. The production of fertilizer at every plant is as follows:

Plant A: 7 million tonne **Plant B:** 6 million tonne **Plant C:** 12 million tonne

The four distribution centers of company demands the following quantity from these three plants:

Distribution Centre X: 8 million tonne **Distribution Centre Y:** 8 million tonne

Distribution Centre Z: 4 million tonne **Distribution Centre W:** 5 million tonne

For each one million tonne quantity transported by every plant to each distribution centre, the costs are in hundreds of rupees as shown in table below:

Plant	Distribution Centre			
	X	Y	Z	W
A	4	5	13	9
B	3	2	8	3
C	7	10	17	11

Solution: The standard form for this transportation problem is illustrated below:

Plant	Distribution Centre				
	X	Y	Z	W	Supply
A	$x_{11} = 4$	$x_{12} = 5$	$x_{13} = 13$	$x_{14} = 9$	7
B	$x_{21} = 3$	$x_{22} = 2$	$x_{23} = 8$	$x_{24} = 3$	6
C	$x_{31} = 7$	$x_{32} = 10$	$x_{33} = 17$	$x_{34} = 11$	12
Demand	8	8	4	5	25 / 25

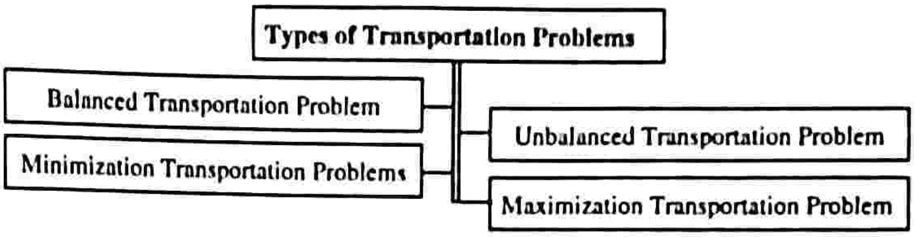
Each cell of table $x_{11}, x_{12}, \ldots, x_{34}$ shows the transportation cost. The right column and bottom row of table illustrates the supply and demand respectively.

2.1.3. Assumptions in Transportation Problem

1) The total requirement at different destinations is equal to the available quantity of the item at various sources.
2) The items can be transported easily from all sources to destinations.
3) The cost of the product for unit transportation from all sources to destinations is definitely and exactly known.
4) The cost of transportation on a given route is directly proportional to the number of units transported on that route.
5) To minimise the total transportation cost for the organisation as a whole and not for individual supply and distribution centres is the main objective.
6) The demand of a destination can be satisfied by the use of more than one source

2.1.4. Types of Transportation Problems

There are mainly four types of transportation problems:



1) **Balanced Transportation Problem:** If the sum of the supplies of all the sources is equal to the sum of the demands of all the destinations, then such type of problem is known as balanced transportation problem. Mathematically, it is represented by the following relationship:

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

2) **Unbalanced Transportation Problem:** If the sum of the supplies of all the sources is not equal to the sum of the demands of all the destinations, then this kind of problem is known as unbalanced transportation problem. Mathematically, it is represented by the following relationship:

$$\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$$

If $\sum_{i=1}^{m} a_i \geq \sum_{j=1}^{n} b_j$, then include a dummy destination to absorb the excess supply.

If $\sum_{i=1}^{m} a_i \leq \sum_{j=1}^{n} b_j$, then include a dummy source to absorb the excess demand.

3) **Minimisation Transportation Problems:** Generally, transportation model is used for solving the cost minimisation problems. It is also used for solving the problems in which objective is to maximise total value or benefit.
4) **Maximisation Transportation Problem:** Generally, transportation model is used for solving the cost minimisation problems. It is also used for solving the problems in which objective is to maximise total value or benefit. In this case the unit profit or pay-off p_{ij} related with each route,(i, j) is given instead of unit cost c_{ij}.

Then the objective function in terms of total profit or pay-off is defined as follows:

$$\text{Maximise} Z = \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} x_{ij}$$

2.1.5. Some Basic Solutions for Transportation Problem

1) **Feasible Solution:** A feasible solution to a transportation problem is referred to as a set of non-negative values x_{ij}, i = 1, 2,.., m; j = 1, 2, ...,n that satisfies the constraints.
2) **Basic Feasible Solution:** Basic feasible solution for a transportation problem is a feasible solution that has not more than (m + n –1) non-negative allocation.
3) **Optimal Solution:** An optimal solution is a feasible solution (not necessarily basic) which minimises the total transportation cost or maximises total revenue.
4) **Non-Degenerate Basic Feasible Solution:** Non-degenerate basic feasible solution of a transportation problem is a feasible solution that has exactly (m + n – 1) allocation in independent positions.
5) **Degenerate Basic Feasible Solution:** A basic feasible solution which consists of less than (m + n – 1) non-negative allocation is known as degenerate basic feasible solution.

2.1.6. Transportation Algorithm

The solution algorithm to a transportation problem may be summarised into the following steps:

Step 1) Formulate the Problem and Setup in the Matrix Form: The transportation problem is formulated in the similar manner as the LP problem is formulated. Here the total transportation cost is the objective function and supply and demand available at each source and destination is taken as constraints.

Step 2) Obtain an Initial Basic Feasible Solution: The following three methods are used to obtain an initial solution:

1) North-West Corner Method
2) Least Cost Method
3) Vogel's Approximation Method

The following conditions must be satisfied by the intial solution which is obtained by any of the three methods:

1) The solution must satisfy all the supply and demand constraints which is also called rim condition.
2) The number of postive allocation must be equal to $(m + n - 1)$, where m represents the number of rows and n represents the number of columns.

Any solution which satisfies the above conditions is termed as non-degenerate basis feasible solution and the solution which does not satisfy the above conditions is called degenerate solution.

Step 3) Test the Initial Solution for Optimality: The Modified Distribution (MODI) method is used for testing the optimality of the solution obtained in step 2. Stop the procedure if the current solution is optimal. If solution is not normal then determine a new improved solution.

Step 4) Updating the Solution: Repeat step 3 until an optimal solution is reached.

2.1.7. Phases of Solution of Transportation Problems

There are two phases in which all transportation problems can be solved:

Phase I: Initial Basic Feasible Solution: Generally, a transportation problem consist of 'm' and 'n' number of origins and destinations respectively. A feasible solution exists when the following condition satisfies:

$$\sum a_i = \sum b$$

i.e., sum of the origin capacities must be equal to the sum of destination requirements.

In the transportation problem matrix, a feasible solution satisfies $m + n - 1$ number of allocations.
Also, the number of allocations which is equal to $m + n - 1$ is same as the order of matrix $(m \times n)$ or number of basic cells.
Following are some important methods which are used to determine the initial basic feasible solution:

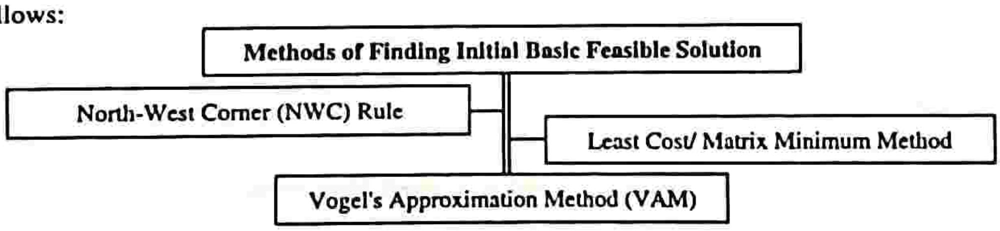
1) North-West Method,
2) Matrix Minimum/Least Cost Method, and
3) Vogel's Approximation Method.

Phase II: Optimum Basic Feasible Solution: An optimal solution is defined as a basic feasible solution that minimises the total transportation cost. The following two methods are used to obtain an optimal solution:

1) Stepping Stone Method
2) Modified Distribution (MODI) Method

2.1.8. Initial Basic Feasible Solution(IBFS)

The methods which are used in solving a transportation problem to determine an initial basic feasible solution can be summarised as follows:



2.1.8.1. North-West Corner(NWC) Rule

In North-West Corner method, the initial assignment is made in the left upper (north-west) corner of the transportation table.

The method can be summarised as follows:

Step 1: In the transportation table, the first allocation is done in the cell placed at the upper left-hand i.e., north-west corner of the table. The amount allocated in the first cell should be the maximum possible value i.e., $x_{11} = \min(a_1, b_1)$. The cell (1, 1) in the transportation table is assigned this value x_{11}.

Step 2: The allocation in step 1 is of such a magnitude that either the capacity of the origin of the first row is satisfied or the requirement of the destination of the first column is satisfied or both. The following three conditions are evaluated on the basis of first allocation:

1) If $b_1 > a_1$, move vertically downwards to the second row in the cell (2, 1) and make the second allocation of x_{21} = min $(a_2, b_1 - a_1)$.
2) If $b_1 < a_1$, move horizontally right-side to the second column in the cell (1, 2) and make the second allocation of x_{12} = min $(a_1 - b_1, b_2)$.
3) If $b_1 = a_1$, then there is a tie for the second allocation. The second allocation can be done in cell (2, 2) in the similar way as was done in point (1) and (2) mentioned above.

Step 3: In the transportation table, begin from the new north-west corner and repeat steps 1 and 2 until all the requirements are satisfied.

Advantages of North West Corner Method

1) This method offers step by step solution and hence proves to be very effective.
2) In this method, optimum solution is obtained in a very simple manner.

Disadvantages of North West Corner Method

1) This method does not consider cost factor which is sought to be minimised.
2) To obtain optimal solution, North West Corner rule takes more time.

Example 2: Find the initial basic feasible solution of following transportation problem by applying NWC method:

	P	Q	R	Supply
A	4	9	6	7
B	5	5	3	10
C	7	6	9	9
D	3	8	4	16
Demand	9	13	20	42

Solution: To compute initial basic feasible of transportation problem using NWC method, the steps are as follows:

Step 1: First select the cell located at the North-west corner of above table, i.e., cell (A, P).

Step 2: Now allocate the possible minimum value taken either from supply or demand, i.e., minimum (7, 9) =7. Hence allocate minimum value 7 at the cell (A, P).

Step 3: As the whole supply (7) of A is finished, hence shift to other cell (B, P).

Step 4: Total remaining demand at the P is now 2 units. So min (9, 2) =2.
Allocate this unit (2) at the cell (B, P)

Step 5: Since the demand is exhausted at the cell (B, P), we move to another cell (B, Q). Here the total remaining supply is now 8 units. So min(8, 13) = 8
Allocate this value (8) at the cell (B, Q).

Step 6: Total supply at the B is now finished, hence shift to another cell (C, Q). The remaining demand at Q is 5 units. Select the min(5, 9) = 5
Allocate this unit (5) at the cell (C, Q).

Step 7: Since the total demand at Q is now exhausted, hence the remaining supply at the C is now 4 units. Move to another cell at (C, R). Choose min (4, 20) = 4 units.
Allocate this unit (4) at the cell (C, R).

Step 8: Now remaining demand is now 16 units and total supply is also 16 units. Hence allocate this at the cell (D, R).

	P	Q	R	Supply
A	4 (7)	9	6	7
B	5 (2)	5 (8)	3	10
C	7	6 (5)	9 (4)	9
D	3	8	4 (16)	16
Demand	9	13	20	42

Since the total number of allocation (6) equal to m(row) + n (column) – 1= 4 + 3 – 1 = 6, hence the solution is feasible.
Total Transportation Cost = $4 \times 7 + 5 \times 2 + 5 \times 8 + 6 \times 5 + 9 \times 4 + 4 \times 16 = 28 + 10 + 40 + 30 + 36 + 64 = 208$

Example 3: With the help of North-West Corner method, solve the following transportation problem:

	W_1	W_2	W_3	W_4	Supply
P_1	21	40	30	15	10
P_2	60	25	50	55	8
P_3	45	10	60	10	17
Demand	6	7	8	14	35

Solution: The allocations of units by using north-west corner method are shown below:

	W_1	W_2	W_3	W_4	Supply
P_1	21 (6)	40 (4)	30	15	10
P_2	60	25 (3)	50 (5)	55	8
P_3	45	10	60 (3)	10 (14)	17
Demand	6	7	8	14	35

Since total number of allocations (6) is equal to m(row) + n(column) − 1= 3 + 4 − 1 = 6, hence the above solution is feasible.

$$\text{Transportation Cost (TC)} = 21 \times 6 + 40 \times 4 + 25 \times 3 + 50 \times 5 + 60 \times 3 + 10 \times 14$$
$$= 126 + 160 + 75 + 250 + 180 + 140 = 931$$

Example 4: Applying the North-West Corner method, solve the following transportation problem:

		To				
		X	Y	Z	W	Supply
From	D_1	4	5	5	6	19
	D_2	3	7	2	5	15
	D_3	5	4	4	3	11
	Demand	9	13	16	7	45

Solution: Using the North-West Corner method, the following allocations of units can be done:

		To				
		X	Y	Z	W	Supply
From	D_1	4 (9)	5 (10)	5	6	19/10/0
	D_2	3	7 (3)	2 (12)	5	15/12/0
	D_3	5	4	4 (4)	3 (7)	11/7/0
	Demand	9/0	13/3/0	16/4/0	7/0	45

$$\text{Total Minimum Transportation Cost (TC)} = 4 \times 9 + 5 \times 10 + 7 \times 3 + 2 \times 12 + 4 \times 4 + 3 \times 7$$
$$= 36 + 50 + 21 + 24 + 16 + 21 = 168$$

Example 5: Solve the following transportation problem using North-West Corner method:

	W_1	W_2	W_3	Supply
P_1	7	6	9	20
P_2	5	7	3	28
P_3	4	5	8	17
Demand	21	25	19	65

Solution: Applying NWC Method, we have

	W_1	W_2	W_3	Supply
P_1	7 (20)	6	9	20/0
P_2	5 (1)	7 (25)	3 (2)	28/27/2/0
P_3	4	5	8 (17)	17/0
Demand	21/1/0	25/0	19/17/0	65

Total Minimum Transportation Cost $= 7 \times 20 + 5 \times 1 + 7 \times 25 + 3 \times 2 + 8 \times 17 = 462$

2.1.8.2. Least Cost / Matrix Minimum Method

In least cost method, the initial assignment is made in the cell having the smallest unit cost in the transportation table.

The following steps are involved in the least cost method:

Step 1: Select the cell (O_i, D_j) from the transportation table which has the minimum unit cost and allocate maximum possible to this cell. That row or column is eliminated whose demand and supply requirements are satisfied. If there is any tie in the smallest unit cost for two or more cells then one selects that cell in which maximum allocation is possible.

Step 2: For all remaining rows and columns, adjust the supply and demand. Repeat the process among the remaining rows and columns with the smallest unit cost.

Step 3: The process will continue until the supply at various origins and demand at various destinations are fulfilled.

Advantages of Least Cost Method

1) In this method transportation cost is considered and hence it provides accurate solution
2) In this method, it is easy and simple to calculate optimum solution.

Disadvantages of Least Cost Method

1) This method does not provide the solution which is closer to optimal solution.
2) This method does not follow any systematic rule in the situation when there is a tie in the minimum cost and personal observation is used for the selection of allocation cell.

Example 6: Applying the matrix minima method, solve the following transportation problem:

	A_1	A_2	A_3	A_4	Supply
X_1	8	12	8	6	4500
X_2	7	15	7	13	6000
X_3	5	9	10	11	7000
Demand	6000	4500	3000	4000	17500

Solution: Following allocation will appear while applying the matrix minima method:

From ↓ / To →	A_1	A_2	A_3	A_4	Supply
X_1	8	12 (500)	8	6 (4000)	4500/500/0
X_2	7	15 (3000)	7 (3000)	13	6000/3000/0
X_3	5 (6000)	9 (1000)	10	11	7000/1000/0
Demand	6000/0	4500/3500/3000/0	3000/0	4000/0	17500

Total Transportation Cost = $12 \times 500 + 6 \times 4000 + 15 \times 3000 + 7 \times 3000 + 5 \times 6000 + 9 \times 1000$

$= 6000 + 24000 + 45000 + 21000 + 30000 + 9000 = 135000$

Example 7: Find the initial basic feasible solution of the following transportation problem using least cost method:

From / To	W_1	W_2	W_3	W_4	Supply
F_1	4	6	3	9	12
F_2	5	2	5	2	13
F_3	7	1	4	8	10
Demand	8	9	10	8	35

Solution: To find out the minimum transportation cost, first select the cell having minimum cost and then allocate the maximum possible units in that cell. In case there are two cells have minimum cost (tie) then select that cell in which the maximum allocation of units is possible.

From / To	W_1	W_2	W_3	W_4	Supply
F_1	4	6	3	9	12
F_2	5	2	5	2	13
F_3	7	1 (9)	4	8	10/1
Demand	8	9/0	10	8	35

The minimum transportation cost (1) occurs in the cell (F_3, W_2) as shown in **table above.** Hence allocation at this cell is min (9, 10) = 9 units. Now the demand at the W_2 is fully exhausted, so cross-out this column. The remaining supply at the F_3 is now 1 unit. Until the total demand and supply not consumed fully, repeat this step for other cells also. The final allocation table will be as follows:

↓From / To →	W_1	W_2	W_3	W_4	Supply
F_1	4 (2)	6	3 (10)	9	12/2/0
F_2	5 (5)	2	5	2 (8)	13/5/0
F_3	7 (1)	1 (9)	4	8	10/1/0
Demand	8/6/1/0	9/0	10/0	8/0	35

Since allocations (6) are equal to m (row) + n (column) – 1 = 3 + 4 – 1 = 6, hence the solution is feasible.

Total Transportation cost $= 4 \times 2 + 3 \times 10 + 5 \times 5 + 2 \times 8 + 7 \times 1 + 1 \times 9 = 8 + 30 + 25 + 16 + 7 + 9 = 95$

Example 8: Using matrix minima method, find out the initial basic feasible solution of following transportation problem. Also calculate the total transportation cost:

Factory	Warehouses				Supply
	A	B	C	D	
X	21	16	25	13	21
Y	17	18	14	22	27
Z	32	27	12	41	19
Demand	14	15	18	20	67

Solution: Using the matrix minima method, the final allocation table will be as follows:

Factory	A	B	C	D	Supply
X	21	16 (1)	25	13 (20)	21
Y	17 (14)	18 (13)	14	22	27
Z	32	27 (1)	12 (18)	41	19
Demand	14	15	18	20	67

Total Transportation Cost(TC) $= 16 \times 1 + 13 \times 20 + 17 \times 14 + 18 \times 13 + 27 \times 1 + 12 \times 18$

$= 16 + 260 + 238 + 234 + 27 + 216 = 991$

Example 9: Determine the initial basic feasible solution using LCM.

		Destination				
		D_1	D_2	D_3	D_4	Supply
Supply	S_1	21	16	15	3	11
	S_2	17	18	14	23	13
	S_3	32	27	18	41	19
	Demand	6	10	12	15	

Solution: Using the matrix minima method, the final allocation table will be as follows:

	D_1	D_2	D_3	D_4	Supply
S_1	21	16	15	3 (11)	11
S_2	17 (1)	18	14 (12)	23	13
S_3	32 (5)	27 (10)	18	41 (4)	19
Demand	0	0	0	0	

The total transportation cost $= 3 \times 11 + 17 \times 1 + 14 \times 12 + 32 \times 5 + 27 \times 10 + 41 \times 4 = 812$

2.1.8.3. Vogel's Approximation Method (VAM)

The vogel's approximation method is preferred for obtaining the initial basic feasible solution because the obtained solution is either optimal or very close to optimal solution. Hence, it takes less time to reach the final solution.

Various steps involved in Vogel's approximation method are as follows:

Step 1: Identify the smallest and next-to-smallest cost in each row of the transportation table. For each row calculate the differences between them which are known as '**Penalties**'. Write the penalties of each row and column alongside the transportation table against the respective rows and columns.

Step 2: Among all the rows and columns identify that row and column which has the largest penalty. Select the cell with least cost in this identified row or column and allocate the feasible number of units to this cell. That row or column is eliminated whose demand and supply requirements are satisfied. If there is any tie in the largest penalties for two or more rows then one selects either of them.

Step 3: For the reduced transportation table repeat the step 1 to calculate the column and row penalties and then go to step 2.

Repeat the process until all the requirements are satisfied.

Advantages of Vogel's Approximation Method

1) It is a very systematic method.
2) The time required to solve the transportation problem is comparatively less.
3) This method involves less number of computations.

Disadvantages of Vogel's Approximation Method

1) The solution which is provided by this method is nearest to the optimal solution.
2) This method proves to be tedious when the given matrix is large.

Example 10: Determine the initial basic feasible solutions of following transportation problem so as the products are transported at a lowest cost. The **table below** shows the cost structure of transportation problem:

To → / From ↓	F_1	F_2	F_3	F_4	Supply
O_1	6	4	1	5	14
O_2	8	9	2	7	16
O_3	2	6	3	4	15
Demand	10	12	15	8	45

Solution: The steps of Vogel's' approximation methods are described below:

To→ / From ↓	F_1	F_2 (12)	F_3	F_4 (2)	Supply	Row Penalty I	II	III	IV	V	VI
O_1	6	4	1 (15)	5 (1)	14/12	3	1	1	1	1	4
O_2	8 (10)	9	2	7 (5)	16/1	5↑	1	2↑	–	–	–
O_3	2	6	3	4	15/5	1	2	2	2↑	–	–
Demand	10	12	15	8/7/2	45						

Column Penalty	F_1	F_2	F_3	F_4
I	4	2	1	1
II	4↑	2	–	1
III	–	2	–	1
IV	–	2	–	1
V	–	4	–	5↑
VI	–	4↑	–	–

Step 1: Initially calculate the penalties of each row and column by subtracting the two smallest costs.

Step 2: Now detect the high penalty in each row and column.

Step 3: At the lowest cost cell of row or column that has high penalty, the maximum possible allocation can be done.

Step 4: If the allocation is exhausted to any row or column, then that row or column is eliminated. Next the penalty is calculated again for other rows or columns.

Step 5: Repeat the above procedure until the whole demand and supply is not fully consumed.

Total Minimum Transportation cost = 4 × 12 + 5 × 2 + 2 × 15 + 7 × 1 + 2 × 10 + 4 × 5 = 48 + 10 + 30 + 7 + 20 + 20 = 135

Example 11: A company works at six places. It has three cement plants located at the places X, Y and Z and these are producing 50, 40 and 60 units daily. Company has also three distribution centres located at the place P, Q and R with daily demands of 20, 95 and 35 units. The following table shows per unit cost of shipping to distribution centres from manufacturing plants. Find out which route the company wants to follow in order to minimise the total transportation cost.

Distribution Centres

Plant		P	Q	R
	X	6	5	1
	Y	3	8	7
	Z	5	5	2

Solution: After implementing the Vogel's approximation method (VAM), the following allocation table will be shown:

	P	Q	R	Supply	Row Penalty I	II	III	IV	V
X	6	5 (15)	1 (35)	50	4↑	1	5	5↑	–
Y	3 (20)	8 (20)	7	40	4	5↑	8↑	–	–
Z	5	5 (60)	2	60	3	0	5	5	5
Demand	20	95	35	150					

Column Penalty		P	Q	R
	I	2	0	1
	II	2	0	–
	III	–	0	–
	IV	–	0	–
	V	–	5↑	–

Total Minimum transportation Cost = 5 × 15 + 1 × 35 + 3 × 20 + 8 × 20 + 5 × 60 = 75 + 35 + 60 + 160 + 300 = 630

Example 12: Find the basic feasible solution of the following transportation problem.

	1	2	3	4	Supply
1	2	3	11	7	6
2	1	0	6	1	1
3	5	8	15	9	10
Demand	7	5	3	2	

Ans: Applying VAM Method, we have following allocations:

	D_1	D_2	D_3	D_4	Supply	Row Penalties		
P_1	2 (1)	3 (5)	11	7	6	1	1	5↑
P_2	1	0	6	1 (1)	1	1	–	–
P_3	5 (6)	8	15 (3)	9 (1)	10	3	3	4
Demand	7	5	3	2	17			
Column Penalties	1	3	5	6↑				
	3	5↑	4	2				
	3	–	4	2				

Transportation Cost = $2 \times 1 + 5 \times 6 + 3 \times 5 + 15 \times 3 + 1 \times 1 + 9 \times 1 = 2 + 30 + 15 + 45 + 1 + 9 = 102$

Example 13: Consider the following transportation problem:

		Destination				Availability
		1	2	3	4	
	1	21	16	25	13	11
Source	2	17	18	14	23	13
	3	32	27	18	41	19
Requirement		6	10	12	15	43

Determine an initial basic feasible solution using Vogel's approximation method.

Solution: For Vogel's approximation method, the steps are given below:

Step 1: First compute the penalties by taking difference between the two lowest cost cell in each row and in each column.

Step 2: Identify the largest penalty among row and column penalty.

Step 3: Allocate feasible number of units in the lowest cost cell of row or column having the largest penalty.

Step 4: Once the allocation is done fully to a row or column, ignore that row or column for further consideration by eliminating that particular row or column.

Step 5: Calculate the penalties again and repeat the procedure until the supply and demand exhaust.

The allocations are as below:

	D_1	D_2	D_3	D_4	Supply	Row Penalties I	II	III	IV	V
S_1	21	16	25	(11) 13	11	3	–	–	–	–
S_2	(6) 17	(3) 18	14	(4) 23	13	3	3	3	4	18
S_3	32	(7) 27	(12) 18	41	19	9	9	9	9↑	27↑
Demand	6	10	12	15	43					
Column Penalties I	4	2	4	10↑						
II	15	9	4	18↑						
III	15↑	9	4	–						
IV	–	9	4	–						
V	–	9	–	–						

$\because m + n - 1 = 3 + 4 - 1 = 6$

$\therefore$ The solution is feasible.

Total Minimum Transportation Cost (TC) $= 13 \times 11 + 17 \times 6 + 18 \times 3 + 23 \times 4 + 27 \times 7 + 18 \times 12$
$= 143 + 102 + 54 + 92 + 189 + 216 = 796$

Example 14: For the given transportation problem, obtain the initial basic feasible solution by VAM.

							Availability
	9	12	9	6	9	10	5
	7	3	7	7	5	5	6
	6	5	9	11	3	11	2
	6	8	11	2	2	10	9
Demand	4	4	6	2	4	2	

Solution: Applying VAM Method, we have following allocations:

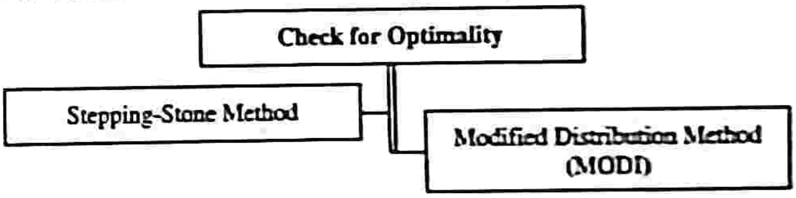
							Availability	Row Penalty I	II	III	IV	V	VI	VII	VIII
	9	12	9 (5)	6	9	10	5	3	3	0	0	0	0	9↑	-
	7	3 (4)	7	7	5	5 (2)	6	2	2	2	4↑	-	-	-	-
	6 (1)	5	9 (1)	11	3	11	2	2	2	2	1	3	3	9	9↑
	6 (3)	8	11	2 (2)	2 (4)	10	9	0	0	4↑	2	5↑	-	-	-
Demand	4	4	6	2	4	2	22								

Column Penalty						
I	0	2	2	4	1	5↑
II	0	2	2	4↑	1	-
III	0	2	2	-	1	-
IV	0	2	2	-	-	-
V	0	-	0	-	-	-
VI	3↑	-	0	-	-	-
VII	-	-	0	-	-	-
VIII	-	-	9	-	-	-

Transportation Cost $= 9 \times 5 + 3 \times 4 + 5 \times 2 + 6 \times 1 + 9 \times 1 + 6 \times 3 + 2 \times 2 + 2 \times 4 = 45 + 12 + 10 + 6 + 9 + 18 + 4 + 8 = 112$

2.1.9. Check for Optimality

Once the initial basic feasible solution is obtained, the next step in this method is to test whether the obtained solution is optimal or not. There are two methods which are used for optimality test of a basic feasible solution. It is shown in **figure below:**



2.1.9.1. Optimum Solution By Stepping-Stone Method

In this method, the optimality test is performed by calculating the opportunity cost for each empty cell. This method involves the procedure of determining the potential, if one exists, to improve each of the non-basic variables with respect to the objective function. Next step is to find out what would be the effect on the total cost if one unit is assigned to each such cell considered above. With the help of this information, we get to know whether the solution is optimal or not. If the solution is not optimal then one improves the solution.

Steps of Stepping Stone Method

Step 1) Find an initial basic feasible solution using any one of the following:
- i) North West Corner Rule
- ii) Matrix Minimum Method
- iii) Vogel Approximation Method

Step 2) Let 'm' is the number of rows and 'n' is the number of columns. It is noted that the number of occupied cells is exactly equal to m + n – 1.

Step 3) Choose an unoccupied cell. Start at this cell and trace a closed path from it, until finally you return to the same unoccupied cell.

Step 4) Plus (+) and minus (–) signs are assigned on each corner cell of the closed path at alternate basis. Assignment starts with the plus sign at unoccupied cell to be evaluated.

Step 5) Unit transportation costs are added which are associated with each of the cell traced in the closed path. This provides net changes in terms of cost.

Step 6) Steps 3 to 5 are repeated until all unoccupied cells are evaluated.

Step 7) If all the net changes computed are less than zero then the current solution requires to be improved and the total transportation cost will be reduced. Then move to step 8.

Step 8) Choose the unoccupied cell which contains the most negative net cost change and find the maximum number of units that can be allocated to this cell. The number of units that can be shipped to the entering cell is indicated by the smallest value with a negative position on the closed path.

This number is added to the unoccupied cells which are lying on the marked path with a plus sign. This number is subtracted to the cells which are lying on the marked path with a minus sign. Finally the solution obtained is optimal if all the values of unoccupied cells are greater than or equal to zero.

Formation of Loops in TP

An ordered set of four or more cells is said to form a loop if it has the following conditions:

Condition 1: Any two adjacent cells of the set lie either in the same row or in the same column; and

Condition 2: No three or more adjacent cells lie in the same row or in the same column.

The first cell of the set will follow the last one in the set. One get a closed path satisfying the above conditions (1) and (2) if he/she joins the cells of loop by horizontal and vertical line segments.

Properties of Loop

1) Every loop has an even number of cells.
2) A feasible solution to a transportation problem is basic if and only if, the corresponding cells in the transportation table do not form a loop.

For example, consider two sets X = {(1, 1), (4, 1), (4, 4), (2, 4), (2, 3), (1, 3)} and
X' = {(3, 1), (3, 4), (2, 4), (2, 3), (2, 2), (1, 2), (1, 1)}

Where, (i, j) denotes the (i, j)[th] cell of the transportation table. Then it is shown that the set X makes a loop whereas the set X' does not make a loop, because cells (2, 4), (2, 3) and (2, 2) occur in the same row.

Figure below shows loop and non-loop conditions:

Loop

(1, 1)	(1, 3)		
		(2, 3)	(2, 4)
(4, 1)			(4, 4)

Non-Loop

(1, 1)	(1, 2)		
	(2, 2)	(2, 3)	(2, 4)
	(3, 1)		(3, 4)

Example 15: Consider the following table that shows manufacturing plants and warehouses of a company and the shipping costs to transport the available quantity (supply) from manufacturing plants A, B, C to requirements (demand) at every warehouses X, Y, Z and W.

Warehouse					
Plant	**X**	**Y**	**Z**	**W**	**Supply**
A	3	2	5	2	15
B	2	1	4	4	24
C	2	3	4	3	21
Demand	13	12	16	19	60

Solution: After implementing the stepping stone method, we have the following steps:

Step 1: Find-Out the Initial Feasible Solution: The initial step of stepping-stone method is to find the initial feasible solution that satisfies the rim requirements. There are various different methods that can be used for this, such as:

1) North-West Corner (NWC) Method,
2) Least Cost Method (LCM) and
3) Vogel's Approximation Method (VAM).

Applying VAM Method

After implementing the VAM, we have the following table:

	Warehouse					Row Penalty					
Plant	X	Y	Z	W	Supply	I	II	III	IV	V	VI
A	3	2	5	2 (15)	15	0	1	0	–	–	–
B	2 (12)	1 (12)	4	4	24	1↑	2	–	–	–	–
C	2 (1)	3	4 (16)	3 (4)	21	1	1	1↑	1	1	2↑
Demand	13	12	16	19	60						

Column Penalty	X	Y	Z	W
I	0	1	0	1
II	–	–	0	1↑
III	1	–	1	1
IV	2	–	4↑	3
V	2	–	–	3↑
VI	2	–	–	–

Transportation Costs = $15 \times 2 + 12 \times 2 + 1 \times 12 + 2 \times 1 + 4 \times 16 + 3 \times 4 = 30 + 24 + 12 + 2 + 64 + 12 =$ ₹144

Step 2: Optimality Test: The transportation cost is now ₹144. There are two methods that can be used for optimality test. These are as below:

1) Stepping Stone Method and
2) MODI Method

Applying Stepping-Stone Method

Now apply the stepping stone method on the above example that has initial basic feasible solution obtained by VAM. From the above table, it is known that there are six cells that are empty. These cells are: (A, X), (A, Y), (A, Z), (B, Z), (B, W) and (C, Y).

Let us consider the non-occupied cell (A, X) for improvement. One unit shipping cost of an item on this path result an increase of ₹3 and this will deduct one unit from the cell (A, W) and hence there is decrease of ₹2 in the cost. In order to maintain the feasibility, deduct one unit from the cell (C, X) and introduce one unit in the cell (C, W).

The result of this process is that there is an increase of ₹2 in the cost. This is shown as below:

Increment in cost after increasing one unit in cell (A, X) $= +3$

Decrement in cost after decreasing one unit in cell (A, W) $= -2$

Increment in cost after increasing one unit in cell (C, W) $= +3$

Decrement in cost after decreasing one unit in cell (C, X) $= \underline{-2}$

Total effect on the cost $= +2$

Hence, + sign indicates that there is an increase of cost of ₹2 per unit by taking the route AX. This also shows that there is a decrease of cost in this route is ₹2. Consider similar process for the empty cell (A, Z), then the cells (A, W), (C, Z) and (C, W) will form closed loop as shown in **figure 2.1**.

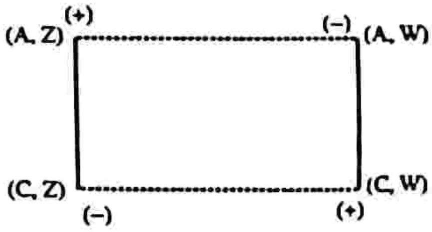


Figure 2.1: Closed Loop

In above case, one unit is moved from cell (A, W) to cell (A, Z) then the adjustments can be done in the cells (C, W) and (C, Z) in such a manner that demand constraints is not violated.

One unit transfer in this case will provide total effect on the cost $= 5 - 4 + 3 - 2 =$ ₹2, which shows an increase in the cost. This also shows per unit opportunity cost for cell (A, Z).

Table 2.2 shows the opportunity costs for different empty cells of the solution to be calculated:

Table 2.2: Opportunity Cost

Cell	Loop Formed by the Cells	Per Unit Change in Cost	Opportunity Cost
(A, X)	(A, X) – (C, X) + (C, W) – (A, W)	3 – 2 + 3 – 2 = 2	– 2
(A, Y)	(A, Y) – (B, Y) + (B, Z) – (C, Z) + (C, W) – (A, W)	2 – 1 + 4 – 4 + 3 – 2 = 2	– 2
(A, Z)	(A, Z) – (C, Z) + (C, W) – (A, W)	5 – 4 + 3 – 2 = 2	– 2
(B, X)	(B, X) – (C, X) + (C, Z) – (B, Z)	2 – 2 + 4 – 4 = 0	0
(B, W)	(B, W) – (B, Z) + (C, Z) – (C, W)	4 – 4 + 4 – 3 = 1	– 1
(C, Y)	(C, Y) – (C, Z) + (B, Z) – (B, Y)	3 – 4 + 4 – 1 = 2	– 2

One can find the opportunity cost by multiplying per unit alteration in cost by –1. From **table 2.2**, it is shown that the opportunity cost for cell (B, X) is not less than zero. Hence solution is not optimal. Now consider another loop for the cells(C, X) and (B, X). The new solution is shown below:

Initial Solution Warehouse

Plant	X	Y	Z	W
A	3	2	5	2 (15)
B	2 (12)	1 (12)	4	4
C	2 (1)	3	4 (16)	3 (4)

Optimality can be checked for this in order to calculate opportunity costs for all empty cells. In this case, opportunity costs for the cell (B, Z) will only be non-negative. The cell (B, X) will be empty this cell at the former iteration. Hence, the optimal solution will be obtained with the minimum transportation cost of ₹144. If some cells have zero opportunity costs, while others are negative then this shows that solution is optimal. In such case the alternative optimal solution is also possible.

Note: In case there are various cells that have negative cost, then choose that cell which shows the most negative cost.

2.1.9.2. Optimum Solution by Modified Distribution Method (MODI) or UV Method

MODI method proves to be an efficient method to test whether the determined solution of a transportation problem is optimal or not when compared to stepping-stone method.

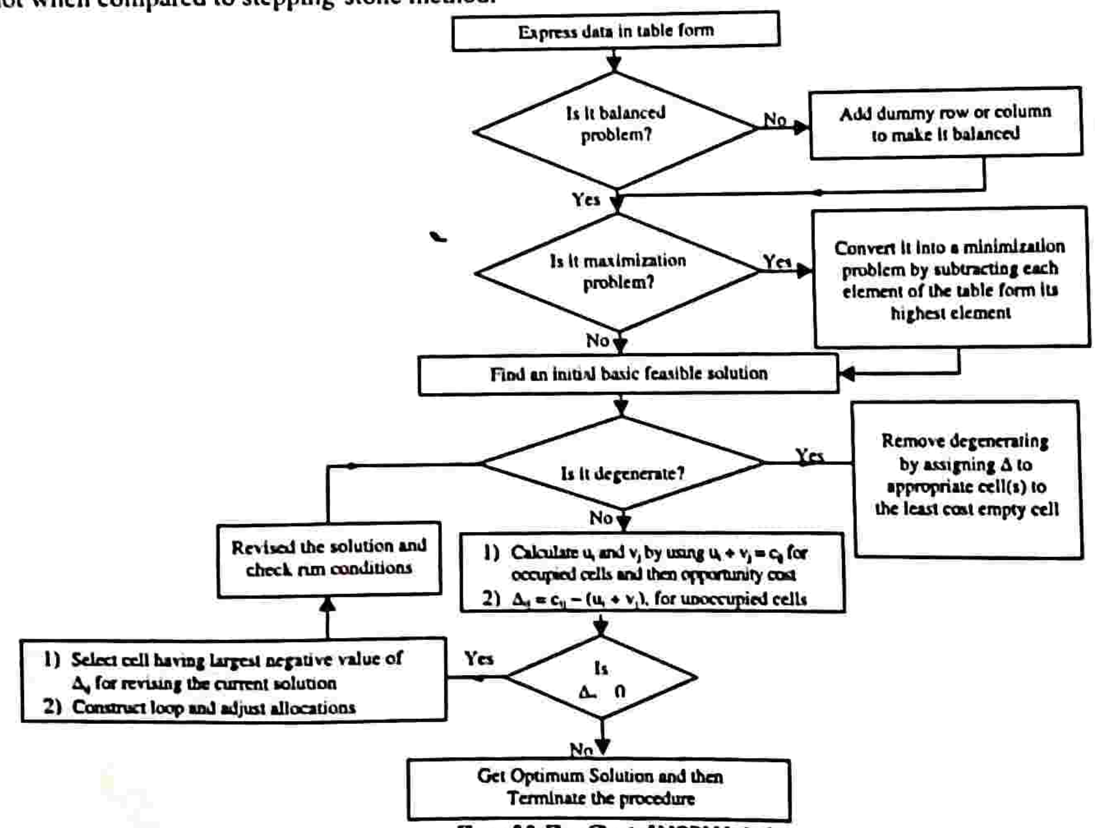


Figure 2.2: Flow Chart of MODI Method

In stepping-stone method, a closed loop is drawn and the opportunity cost of each empty cell is evaluated. This proves to be a complex exercise and takes long calculation time if a large number of sources and destinations are involved. The number of steps required in the evaluation of the empty cells in MODI method is less and it avoids the extensive scanning as done in stepping-stone method. We can determine the opportunity cost of each of the empty cells with a straightforward computational scheme in MODI method **(figure 2.2)**.

Steps of MODI Method

Step 1: Add a column on the right hand side and a row in the bottom of the transportation table titled u_i and v_j respectively.

Step 2: In this step following sub steps are performed:

i) For the rows/columns which have maximum number of allocations, select the value of u_i and v_j equal to zero. Normally, the first row has allocated the value 0(zero) i.e. $u_1 = 0$.

ii) In the first row, consider every occupied cell individually and allocate the column value v_j When the occupied cell is in the j^{th} column of the row, which is such that the sum of the row and the column values is equal to the unit cost value in the occupied cell, consider these values and pick other occupied cells one by one and find the appropriate values of u_i's, taking in each case $u_i + v_j = c_{ij}$ Thus, if u_i is the row value of the i^{th} row, v_j is the column value of the j^{th} column and c_{ij} is the unit cost of the cell in the i^{th} row and j^{th} column, then the following equation gives the row and column values:

$$u_i + v_j = c_{ij}$$

Step 3: After finding all values of u_i and v_j, calculate for each unoccupied cell $\Delta_{ij} = c_{ij} - (u_i + v_j)$, where Δ_{ij}'s represent the opportunity costs of various cells. After calculating the opportunity costs follow the same process which is performed in the stepping stone method. The solution is called an optimal solution if all the empty cells have positive opportunity. If values of Δ_{ij}'s are negative then the given solution is not an optimal solution and if one or more Δ_{ij}'s values are negative i.e., $\Delta_{ij} < 0$, then choose the cell which has the largest opportunity cost value and form a closed loop. According to the procedure of this method transfer the units along the route. Test the result solution for optimality condition and improve if required. The process is repeated until an optimal solution is obtained.

Hence the condition for the solution of being optimal solution is:

$$c_{ij} - (u_i + v_j) \geq 0$$

Example 16: Find the optimum solution of the following transportation problem using Least Cost/Matrix Minima Method and MODI method, where cells shows the transportation costs in rupees.

	W_1	W_2	W_3	W_4	Supply
O_1	6	4	1	5	14
O_2	8	9	2	7	16
O_3	4	3	6	2	5
Demand	6	10	15	4	

Solution: After implementing the least cost method, we got the final allocation table as below:

	W_1	W_2	W_3	W_4	Supply
O_1	6	4	1 (14)	5	14
O_2	8 (6)	9 (9)	2 (1)	7	16
O_3	4	3 (1)	6	2 (4)	5
Demand	6	10	15	4	35

Total Transportation Cost = 1 × 14 + 8 × 6 + 9 × 9 + 2 × 1 + 3 × 1 + 2 × 4 = 14 + 48 + 81 + 2 + 3 + 8 = ₹156

Optimality Test Using MODI Method: Since the number of allocations (6) is equal to $(m + n - 1) = 3 + 4 - 1 = 6$, hence the solution shows non-degeneracy. Now using the MODI method, find the u_i and v_j, $c_{ij} = u_i + v_j$ for all non-empty cells (i, j). Substituting, $u_1 = 0$, we get the following:

$c_{13} = u_1 + v_3 \Rightarrow 1 = 0 + v_3 \Rightarrow v_3 = 1 - 0 = 1,$

$c_{21} = u_2 + v_1 \Rightarrow 8 = 1 + v_1 \Rightarrow v_1 = 8 - 1 = 7,$

$c_{32} = u_3 + v_2 \Rightarrow 3 = u_3 + 8 \Rightarrow u_3 = 3 - 8 = -5,$

$c_{23} = u_2 + v_3 \Rightarrow 2 = u_2 + 1 \Rightarrow u_2 = 2 - 1 = 1$

$c_{22} = u_2 + v_2 \Rightarrow 9 = 1 + v_2 \Rightarrow v_2 = 9 - 1 = 8$

$c_{34} = u_3 + v_4 \Rightarrow 2 = -5 + v_4 \Rightarrow v_4 = 2 + 5 = 7$

We have following allocation table:

	W_1	W_2	W_3	W_4	Supply	u_i
O_1	6	4	1 (14)	5	14	$u_1 = 0$
O_2	8 (6)	9 (9)	2 (1)	7	16	$u_2 = 1$
O_3	4	3 (1)	6	2 (4)	5	$u_3 = -5$
Demand	6	10	15	4	35	
v_j	$v_1 = 7$	$v_2 = 8$	$v_3 = 1$	$v_4 = 7$		

Find d_{ij}, $d_{ij} = c_{ij} - (u_i + v_j)$, for all empty cells (i, j), we have the following:

$d_{11} = 6 - (0 + 7) = -1$

$d_{12} = 4 - (0 + 8) = -4$

$d_{14} = 5 - (0 + 7) = -2$

$d_{24} = 7 - (1 + 7) = -1$

$d_{31} = 4 - (-5 + 7) = 2$

$d_{32} = 6 - (-5 + 1) = 10$

Since value of $d_{12} = -4$ is most negative. Therefore loop starts from cell (O_1, W_1). We have the following table:

	W_1	W_2	W_3	W_4	Supply
O_1	6	4 (+) (Δ)	1 (–) ($14 - \Delta$)	5	14
O_2	8 (6)	9 (–) ($9 - \Delta$)	2 ($1 + \Delta$) (+)	7	16
O_3	4	3 (1)	6	2 (4)	5
Demand	6	10	15	4	35

$\therefore 9 - \Delta = 0 \Rightarrow \Delta = 9$

We got the following table:

	W_1	W_2	W_3	W_4	Supply	u_i
O_1	6	4 (9)	1 (5)	5	14	$u_1 = 0$
O_2	8 (6)	9	2 (10)	7	16	$u_2 = 1$
O_3	4	3 (1)	6	2 (4)	5	$u_3 = -1$
Demand	6	10	15	4	35	
v_j	$v_1 = 7$	$v_2 = 4$	$v_3 = 1$	$v_4 = 3$		

Again find d_{ij}, $d_{ij} = c_{ij} - (u_i + v_j)$, for all empty cells (i, j), we have the following:

$d_{11} = 6 - (0 + 7) = -1,$ $d_{14} = 5 - (0 + 3) = 2$

$d_{22} = 9 - (1 + 4) = 4,$ $d_{24} = 7 - (1 + 3) = 3$

$d_{31} = 4 - (-1 + 7) = -2,$ $d_{33} = 6 - (1 - 1) = 6$

Since $d_{31} = -2$ is most negative. We have the following loop:

	W_1	W_2	W_3	W_4	Supply
O_1	6	4 (+) (9 + Δ)	1 (−) (5 − Δ)	5	14
O_2	8 (−) (6 − Δ)	9	2 (+) (10 + Δ)	7	16
O_3	4 (+) (Δ)	3 (−) (1 − Δ)	6	2 (4)	5
Demand	6	10	15	4	35

∵ $1 - \Delta = 0$

∴ $\Delta = 1$

Again we have following table:

	W_1	W_2	W_3	W_4	Supply	u_i
O_1	6	4 (10)	1 (4)	5	14	$u_1 = 0$
O_2	8 (5)	9	2 (11)	7	16	$u_2 = 1$
O_3	4 (1)	3	6	2 (4)	5	$u_3 = -3$
Demand	6	10	15	4	35	
v_j	$v_1 = 7$	$v_2 = 4$	$v_3 = 1$	$v_4 = 5$		

Again find d_{ij}, $d_{ij} = c_{ij} - (u_i + v_j)$, for all empty cells (i, j), we have the following:

$d_{11} = 6 - (0 + 7) = -1$ $d_{14} = 5 - (0 + 5) = 0$

$d_{22} = 9 - (1 + 4) = 4$ $d_{24} = 7 - (5 + 1) = 1$

$d_{32} = 3 - (4 - 3) = 2$ $d_{33} = 6 - (-3 + 1) = 8$

We have the following loop:

	W_1	W_2	W_3	W_4	Supply
O_1	(Δ) 6 (+)	4 (10)	1 (−) (4 − Δ)	5	14
O_2	8 (−) (5 − Δ)	9	2 (+) (11 + Δ)	7	16
O_3	4 (1)	3	6	2 (4)	5
Demand	6	10	15	4	35

∵ $4 - \Delta = 0 \Rightarrow \Delta = 4$. After this we have following allocation table:

	D_1	D_2	D_3	D_4	Supply	u_i
1	6 (4)	4 (10)	1	5	14	$u_1 = 0$
2	8 (1)	9	2 (15)	7	16	$u_2 = 2$
3	4 (1)	3	6	2 (4)	5	$u_3 = -2$
Demand	6	10	15	4	35	
v_j	$v_1 = 6$	$v_2 = 4$	$v_3 = 0$	$v_4 = 4$		

Again find d_{ij}, $d_{ij} = c_{ij} - (u_i + v_j)$, for all empty cells (i, j), we have the following:

$d_{13} = 1 - (0 + 0) = 1$,

$d_{14} = 5 - (0 + 4) = 1$

$d_{22} = 9 - (2 + 4) = 3$,

$d_{24} = 7 - (4 + 2) = 1$

$d_{32} = 3 - (-2 + 4) = 1$,

$d_{33} = 6 - (-2 + 0) = 8$

Since all values of d_{ij} is positive (≥ 0). Therefore this solution is optimum.

Hence, Total Transportation Cost $= 6 \times 4 + 8 \times 1 + 4 \times 1 + 4 \times 10 + 2 \times 15 + 2 \times 4$
$= 24 + 8 + 4 + 40 + 30 + 8 =$ ₹114

Example 17: Obtain an optimal solution to the following transportation problem by U-V method. Use VAM to get the starting BFS.

From \ To	I	II	III	IV	Supply
A	19	30	50	10	7
B	70	30	40	60	9
C	40	8	70	20	18
Demand	5	8	7	14	

Solution: Finding Basic Feasible Solution (BFS) Using Vogel's Approximation Method (VAM)

Total number of Supply Constraints: 3

Total number of Demand Constraints: 4

The problem is now as follows:

Table 2.3

	I	II	III	IV	Supply	Row Penalty
A	19	30	50	10	7	9
B	70	30	40	60	9	10
C	40	8	70	20	18	12
Demand	5	8	7	14	34	
Column Penalty	21	22	10	10		

The value 22, which is the maximum penalty, is present in the column II. In this column, the minimum value c_{ij} is $c_{32} = 8$. The maximum allocation in this cell is 8. Hence, the demand of II is satisfied by this maximum allocation as well as the supply of C from reduces 18 to 10 (18 – 8 = 10).

Table 2.4

	I	II	III	IV	Supply	Row Penalty	
A	19	30	50	10	7	9	9
B	70	30	40	60	9	10	20
C	40	8 (8)	70	20	10	12	20
Demand	5	0	7	14			
Column Penalty	21	22	10	10			
	21	–	10	10			

Now the maximum penalty 21 is present in the column I. In this column, the minimum value c_{ij} is c_{11} = 19. The maximum allocation in this cell is 5. Hence, the demand of I is satisfied by this maximum allocation as well as the supply of C from 7 to 2 (7 – 5 = 2) is also adjusted.

Table 2.5

	I	II	III	IV	Supply	Row Penalty		
A	19 (5)	30	50	10	2	9	9	40
B	70	30	40	60	9	10	20	20
C	40	8 (8)	70	20	10	12	20	50
Demand	0	0	7	14				
Column Penalty	21	22	10	10				
	21	–	10	10				
	–	–	10	10				

The maximum penalty 50 is present in the row C. In this row the minimum value c_{ij} is c_{34} = 20. The maximum allocation in this cell is 10. Hence, the demand of column IV is satisfied by this maximum allocation as well as the supply of C from 14 to 4 (14 – 10 = 4) is also adjusted.

Table 2.6

	I	II	III	IV	Supply	Row Penalty			
A	19 (5)	30	50	10	2	9	9	40	40
B	70	30	40	60	9	10	20	20	20
C	40	8 (8)	70	20 (10)	0	12	20	50	–
Demand	0	0	7	4					
Column Penalty	21	22	10	10					
	21	–	10	10					
	–	–	10	10					
	–	–	10	50					

The maximum penalty 50 is present in the column IV. In this column the minimum value c_{ij} is c_{14} = 10. The maximum allocation in this cell is 2. Hence, the demand of column IV is satisfied by this maximum allocation as well as the supply of A from 4 to 2 (4 – 2 = 2) is also adjusted.

Table 2.7

	I	II	III	IV	Supply	Row Penalty				
A	19 (5)	30	50	10 (2)	0	9	9	40	40	–
B	70	30	40	60	9	10	20	20	20	20
C	40	8 (8)	70	20 (10)	0	12	20	50	–	–
Demand	0	0	7	2						

Column Penalty	I	II	III	IV
	21	22	10	10
	21	–	10	10
	–	–	10	10
	–	–	10	50
	–	–	40	60

The value 60 which is the maximum penalty now is present in the column IV. In this column, the minimum value c_{ij} is c_{24} = 60. The maximum allocation in this cell is 2. Hence, the demand of column IV is satisfied by this maximum allocation as well as the supply of row B from 9 to 7 (9 – 2 = 7) is also adjusted.

Table 2.8

	I	II	III	IV	Supply	Row Penalty					
A	19 (5)	30	50	10 (2)	0	9	9	40	40	–	–
B	70	30	40	60 (2)	7	10	20	20	20	20	40
C	40	8 (8)	70	20 (10)	0	12	20	50	–	–	–
Demand	0	0	7	0							

Column Penalty	I	II	III	IV
	21	22	10	10
	21	–	10	10
	–	–	10	10
	–	–	10	50
	–	–	40	60
	–	–	40	–

The maximum penalty now is 40 which is present in the row B. In this row the minimum c_{ij} is c_{23} = 40. The maximum allocation in this cell is 7. Hence, the demand of column III is satisfied by this maximum allocation as well as the supply of row B from 7 to 0 (7 – 7 = 0) is also adjusted.

Final allocation table is now as follows:

Table 2.9

	I	II	III	IV	Supply	Row Penalty					
A	19 (5)	30	50	10 (2)	7	9	9	40	40	–	–
B	70	30	40 (7)	60 (2)	9	10	20	20	20	20	40
C	40	8 (8)	70	20 (10)	18	12	20	50	–	–	–
Demand	5	8	7	14							

Column Penalty	I	II	III	IV
	21	22	10	10
	21	–	10	10
	–	–	10	10
	–	–	10	50
	–	–	40	60
	–	–	40	–

The minimum transportation cost = $19 \times 5 + 10 \times 2 + 40 \times 7 + 60 \times 2 + 8 \times 8 + 20 \times 10 = 779$

Here, the number of allocated cells = 6 is equal to $m + n - 1 = 3 + 4 - 1 = 6$

∴ This solution is non-degenerate.

Optimality Test Using MODI (U-V) Method

Allocation table is as follows:

Table 2.10

	I	II	III	IV	Supply
A	19 (5)	30	50	10 (2)	7
B	70	30	40 (7)	60 (2)	9
C	40	8 (8)	70	20 (10)	18
Demand	5	8	7	14	

Step-1 of Optimality Test

1) Find u_i and v_j for all occupied cells(i, j), where $c_{ij} = u_i + v_j$

Substituting, $v_4 = 0$, we get

A-IV: $c_{14} = u_1 + v_4 \Rightarrow 10 = u_1 \Rightarrow u_1 = 10$

A-I: $c_{11} = u_1 + v_1 \Rightarrow 19 = 10 + v_1 \Rightarrow v_1 = 19 - 10 = 9$

B-IV: $c_{24} = u_2 + v_4 \Rightarrow 60 = u_2 \Rightarrow u_2 = 60$

B-III: $c_{23} = u_2 + v_3 \Rightarrow 40 = 60 + v_3 \Rightarrow v_3 = 40 - 60 = -20$

C-IV: $c_{34} = u_3 + v_4 \Rightarrow 20 = u_3 \Rightarrow u_3 = 20$

C-II: $c_{32} = u_3 + v_2 \Rightarrow 8 = 20 + v_2 \Rightarrow v_2 = 8 - 20 = -12$

Table 2.11

	I	II	III	IV	Supply	u_i
A	19 (5)	30	50	10 (2)	7	$u_1 = 10$
B	70	30	40 (7)	60 (2)	9	$u_2 = 60$
C	40	8 (8)	70	20 (10)	18	$u_3 = 20$
Demand	5	8	7	14		
v_j	$v_1 = 9$	$v_2 = -12$	$v_3 = -20$	$v_4 = 0$		

2) Find d_{ij} for all unoccupied cells(i, j), where $d_{ij} = c_{ij} - (u_i + v_j)$

A-II: $d_{12} = c_{12} - (u_1 + v_2) = 30 - (10 - 12) = 32$

A-III: $d_{13} = c_{13} - (u_1 + v_3) = 50 - (10 - 20) = 60$

B-I: $d_{21} = c_{21} - (u_2 + v_1) = 70 - (60 + 9) = 1$

B-II: $d_{22} = c_{22} - (u_2 + v_2) = 30 - (60 - 12) = -18$

C-I: $d_{31} = c_{31} - (u_3 + v_1) = 40 - (20 + 9) = 11$

C-III: $d_{33} = c_{33} - (u_3 + v_3) = 70 - (20 - 20) = 70$

Table 2.12

	I	II	III	IV	Supply	u_i
A	19 ⑤	30(32)	50(60)	10 ②	7	$u_1 = 10$
B	70 (1)	30(−18)	40 ⑦	60 ②	9	$u_2 = 60$
C	40(11)	8 ⑧	70(70)	20 ⑩	18	$u_3 = 20$
Demand	5	8	7	14		
v_j	$v_1 = 9$	$v_2 = -12$	$v_3 = -20$	$v_4 = 0$		

3) Now from all the opportunity cost (i.e. d_{ij}), select the minimum negative value. Here it is $d_{22} = -18$. Now draw a closed path from B-II.

Closed path is B-II → B-IV → C-IV → C-II

Following table illustrates the closed path and plus/minus sign allocation:

Table 2.13

	I	II	III	IV	Supply	u_i
A	19 ⑤	30(32)	50(60)	10 ②	7	$u_1 = 10$
B	70 (1)	+ 30(−18)	40 ⑦	− 60 ②	9	$u_2 = 60$
C	40(11)	− 8 ⑧	70(70)	+ 20 ⑩	18	$u_3 = 20$
Demand	5	8	7	14		
v_j	$v_1 = 9$	$v_2 = -12$	$v_3 = -20$	$v_4 = 0$		

4) In all the negative positions on the closed path, the minimum allocated value is 2. Subtract this minimum value from all negative (–) positions and add this minimum value to all positive (+) positions.

Table 2.14

	I	II	III	IV	Supply
A	19 ⑤	30	50	10 ②	7
B	70	30 ②	40 ⑦	60	9
C	40	8 ⑥	70	20 ⑫	18
Demand	5	8	7	14	

5) Till the optimal solution is not found, repeat the sub-steps from 1 to 4.

Step-2 of Optimality Test

1) Find u_i and v_j for all occupied cells(i, j), where $c_{ij} = u_i + v_j$

Substituting, $u_1 = 0$, we get

A-I: $c_{11} = u_1 + v_1 \Rightarrow 19 = 0 + v_1 \Rightarrow v_1 = 19$

A-IV: $c_{14} = u_1 + v_4 \Rightarrow 10 = 0 + v_4 \Rightarrow v_4 = 10$

C-IV: $c_{34} = u_3 + v_4 \Rightarrow 20 = u_3 + 10 \Rightarrow u_3 = 20 - 10 = 10$

C-II: $c_{32} = u_3 + v_2 \Rightarrow 8 = 10 + v_2 \Rightarrow v_2 = 8 - 10 = -2$

B-II: $c_{22} = u_2 + v_2 \Rightarrow 30 = u_2 - 2 \Rightarrow u_2 = 30 + 2 = 32$

B-III: $c_{23} = u_2 + v_3 \Rightarrow 40 = 32 + v_3 \Rightarrow v_3 = 40 - 32 = 8$

Table 2.15

	I	II	III	IV	Supply	u_i
A	19 (5)	30	50	10 (2)	7	$u_1 = 0$
B	70	30 (2)	40 (7)	60	9	$u_2 = 32$
C	40	8 (6)	70	20 (12)	18	$u_3 = 10$
Demand	5	8	7	14		
v_j	$v_1 = 19$	$v_2 = -2$	$v_3 = 8$	$v_4 = 10$		

2) Find d_{ij}, for all unoccupied cells(i, j), where $d_{ij} = c_{ij} - (u_i + v_j)$

A-II: $d_{12} = c_{12} - (u_1 + v_2) = 30 - (0 - 2) = 32$

A-III: $d_{13} = c_{13} - (u_1 + v_3) = 50 - (0 + 8) = 42$

B-I: $d_{21} = c_{21} - (u_2 + v_1) = 70 - (32 + 19) = 19$

B-IV: $d_{24} = c_{24} - (u_2 + v_4) = 60 - (32 + 10) = 18$

C-I: $d_{31} = c_{31} - (u_3 + v_1) = 40 - (10 + 19) = 11$

C-III: $d_{33} = c_{33} - (u_3 + v_3) = 70 - (10 + 8) = 52$

Table 2.16

	I	II	III	IV	Supply	u_i
A	19 (5)	30(32)	50(42)	10 (2)	7	$u_1 = 0$
B	70 (19)	30 (2)	40 (7)	60(18)	9	$u_2 = 32$
C	40(11)	8 (6)	70(52)	20 (12)	18	$u_3 = 10$
Demand	5	8	7	14		
v_j	$v_1 = 19$	$v_2 = -2$	$v_3 = 8$	$v_4 = 10$		

Since all $d_{ij} \geq 0$, so final optimal solution is arrived.

Table 2.17

	I	II	III	IV	Supply
A	19 (5)	30	50	10 (2)	7
B	70	30 (2)	40 (7)	60	9
C	40	8 (6)	70	20 (12)	18
Demand	5	8	7	14	

The minimum total transportation cost = $19 \times 5 + 10 \times 2 + 30 \times 2 + 40 \times 7 + 8 \times 6 + 20 \times 12 = 743$

Example 18: The following table represents the factory capacities, store requirements ar unit cost (in rupee) of shipping from each factory to each store. Find the optional transportation plan so as to minimise the transportation cost.

Stores

		S_1	S_2	S_3	S_4	S_5	S_6	S_7	Factory Capacity
	F_1	5	6	4	3	7	5	4	7000
Factory	F_2	9	4	3	4	3	2	1	4000
	F_3	8	4	2	5	4	8	3	10000
	Store Demand	1500	2000	4500	4000	2500	3500	3000	

Solution: Initial feasible solution using VAM is as follows:

	S_1	S_2	S_3	S_4	S_5	S_6	S_7	Supply	Row Penalty
F_1	(1500) 5	6	4	(4000) 3	7	5	(1500) 4	7000	1 1 1 1 1 1 1 1 3
F_2	9	4	3	4	3	(3500) 2	(500) 1	4000	1 1 2 - - - - - -
F_3	8	(2000) 4	(4500) 2	5	(2500) 4	8	(1000) 3	10000	1 1 1 1 1 1 2 - -
Demand	1500	2000	4500	4000	2500	3500	3000		

Column Penalty

	S_1	S_2	S_3	S_4	S_5	S_6	S_7
	3	0	1	1	1	3	2
	-	0	1	1	1	3	2
	-	0	1	1	1	-	2
	-	2	2	2	3	-	1
Column Penalty	-	2	2	2	-	-	1
	-	-	2	2	-	-	1
	-	-	-	2	-	-	1
	-	-	-	3	-	-	4
	-	-	-	3	-	-	-

The minimum total transportation cost $= 5 \times 1500 + 3 \times 4000 + 4 \times 1500 + 2 \times 3500 + 1 \times 500 + 4 \times 2000 + 2 \times 4500 + 4 \times 2500 + 3 \times 1000 = 63000$

Here the number of allocated cell = 9 is equal to $m + n - 1 = 3 + 7 - 1 = 9$

$\therefore$ This solution is non-degenerate

Optimality test Using MODI Method

Allocation table is:

	S_1	S_2	S_3	S_4	S_5	S_6	S_7	Supply
F_1	(1500) 5	6	4	(4000) 3	7	5	(1500) 4	7000
F_2	9	4	3	4	3	(3500) 2	(500) 1	4000
F_3	8	(2000) 4	(4500) 2	5	(2500) 4	8	(1000) 3	10000
Demand	1500	2000	4500	4000	2500	3500	3000	

We first find u_i and v_j for all occupied cells(i, j), where $c_{ij} = u_i + v_j$

Substituting, $u_3 = 0$, we get;

$c_{32} = u_3 + v_2 \Rightarrow v_2 = 4 - 0 \Rightarrow v_2 = 4$ $c_{33} = u_3 + v_3 \Rightarrow v_3 = 2 - 0 \Rightarrow v_3 = 2$

$c_{35} = u_3 + v_5 \Rightarrow v_5 = 4 - 0 \Rightarrow v_5 = 4$ $c_{37} = u_3 + v_7 \Rightarrow v_7 = 3 - 0 \Rightarrow v_7 = 3$

$c_{17} = u_1 + v_7 \Rightarrow u_1 = 4 - 3 \Rightarrow u_1 = 1$ $c_{11} = u_1 + v_1 \Rightarrow v_1 = 5 - 1 \Rightarrow v_1 = 4$

$c_{14} = u_1 + v_4 \Rightarrow v_4 = 3 - 1 \Rightarrow v_4 = 2$ $c_{27} = u_2 + v_7 \Rightarrow u_2 = 1 - 3 \Rightarrow u_2 = -2$

$c_{26} = u_2 + v_6 \Rightarrow v_6 = 2 + 2 \Rightarrow v_6 = 4$

	S_1	S_2	S_3	S_4	S_5	S_6	S_7	Supply	u_i
F_1	(1500) 5	6	4	(4000) 3	7	5	(1500) 4	7000	1
F_2	9	4	3	4	3	(3500) 2	(500) 1	4000	–2
F_3	8	(2000) 4	(4500) 2	5	(2500) 4	8	(1000) 3	10000	0
Demand	1500	2000	4500	4000	2500	3500	3000		
v_j	4	4	2	2	4	4	3		

Now we find d_{ij} for all unoccupied cells (i, j), where $d_{ij} = c_{ij} - (u_i + v_j)$

$d_{12} = c_{12} - (u_1 + v_2) = 6 - (1 + 4) = 1$ $\quad$ $d_{13} = c_{13} - (u_1 + v_3) = 4 - (1 + 2) = 1$

$d_{15} = c_{15} - (u_1 + v_5) = 7 - (1 + 4) = 2$ $\quad$ $d_{16} = c_{16} - (u_1 + v_6) = 5 - (1 + 4) = 0$

$d_{21} = c_{21} - (u_2 + v_1) = 9 - (-2 + 4) = 7$ $\quad$ $d_{22} = c_{22} - (u_2 + v_2) = 4 - (-2 + 4) = 2$

$d_{23} = c_{23} - (u_2 + v_3) = 3 - (-2 + 2) = 3$ $\quad$ $d_{24} = c_{24} - (u_2 + v_4) = 4 - (-2 + 2) = 4$

$d_{25} = c_{25} - (u_2 + v_5) = 3 - (-2 + 4) = 1$ $\quad$ $d_{31} = c_{31} - (u_3 + v_1) = 8 - (0 + 4) = 4$

$d_{34} = c_{34} - (u_3 + v_4) = 5 - (0 + 2) = 3$ $\quad$ $d_{36} = c_{36} - (u_3 + v_6) = 8 - (0 + 4) = 4$

	S_1	S_2	S_3	S_4	S_5	S_6	S_7	Supply	u_i
F_1	(1500) 5	6 (1)	4 (1)	(4000) 3	7 (2)	5 (0)	(1500) 4	7000	1
F_2	9 (7)	4 (2)	3 (3)	4 (4)	3 (1)	(3500) 2	(500) 1	4000	–2
F_3	8 (4)	(2000) 4	(4500) 2	5 (3)	(2500) 4	8 (4)	(1000) 3	10000	0
Demand	1500	2000	4500	4000	2500	3500	3000		
v_j	4	4	2	2	4	4	3		

Since all $d_{ij} \geq 0$ so final optimal solution is arrived.

	S_1	S_2	S_3	S_4	S_5	S_6	S_7	Supply
F_1	(1500) 5	6	4	(4000) 3	7	5	(1500) 4	7000
F_2	9	4	3	4	3	(3500) 2	(500) 1	4000
F_3	8	(2000) 4	(4500) 2	5	(2500) 4	8	(1000) 3	10000
Demand	1500	2000	4500	4000	2500	3500	3000	

The minimum total transportation cost = $5 \times 1500 + 3 \times 4000 + 4 \times 1500 + 2 \times 3500 + 1 \times 500 + 4 \times 2000 + 2 \times 4500 + 4 \times 2500 + 3 \times 1000 = 63000$

Note: Alternate solution is available with unoccupied cell $F_1 S_6$: d_{16} = [0] but with the same optimal value.

Example 19: Find an optimal solution to following transportation problem:

Origin	Destination				Supply
	A	B	C	D	
X	2	2	2	1	30

Y	10	8	5	4	70
Z	7	6	6	8	50
Demand	40	30	40	40	

Solution: Applying the Vogel's method we have the following table:

	A	B	C	D	Supply	Row Penalty
X	2 (30)	2	2	1	30	1 – – – – –
Y	10	8	5 (30)	4 (40)	70	1 1 3 – – –
Z	7 (10)	6 (30)	6 (10)	8	50	0 0 0 0 0 6
Demand	40	30	40	40		
Column Penalty	5	4	3	3		
	3	2	1	4		
	3	2	1	–		
	7	6	6	–		
	–	6	6	–		
	–	–	6↑	–		

The minimum transportation cost = $2 \times 30 + 5 \times 30 + 4 \times 40 + 7 \times 10 + 6 \times 30 + 6 \times 10 = 680$

Here, the numbers of allocated cells = 6 is equal to $m - n - 1 = 3 + 4 - 1 - 6$

∴ This solution is non-degenerate

Optimality Test Using MODI Method

Step of Optimality Test

1) Find u_i and v_j for all occupied cells (i, j), where $C_{ij} = u_i + v_j$ substituting $u_3 = 0$, we get

   ZA: $C_{31} = u_3 + v_1 \Rightarrow 7 = 0 + v_1 \Rightarrow v_1 = 7$

   XA: $C_{11} = u_1 + v_1 \Rightarrow 2 = u_1 + 7 \Rightarrow u_1 = 2 - 7 = -5$

   ZB: $C_{32} = u_3 + v_2 \Rightarrow 6 = 0 + v_2 \Rightarrow v_2 = 6$

   ZC: $C_{33} = u_3 + v_3 \Rightarrow 6 = 0 + v_3 \Rightarrow v_3 = 6$

   YC: $C_{23} = u_2 + v_3 \Rightarrow 5 = u_2 + 6 \Rightarrow u_2 = 5 - 6 = -1$

   YD: $C_{24} = u_2 + v_4 \Rightarrow 4 = -1 + v_4 \Rightarrow v_4 = 4 + 1 = 5$

	A	B	C	D	Supply	u_i
X	2 (30)	2	2	1	30	$u_1 = -5$
Y	10	8	5 (30)	4 (40)	70	$u_2 = -1$
Z	7 (10)	6 (30)	6 (10)	8	50	$u_3 = 0$
Demand	40	30	40	40		
v_j	$v_1 = 7$	$v_2 = 6$	$v_3 = 6$	$v_4 = 5$		

2) Find d_{ij} for all unoccupied cells (i, j), where $d_{ij} = C_{ij} - (u_i + v_j)$

   XB: $d_{12} = C_{12} - (u_1 + v_2) = 2 - (-5 + 6) = 1$

   XC: $d_{13} = C_{13} - (u_1 + v_3) = 2 - (-5 + 6) = 1$

   XD: $d_{14} = C_{14} - (u_1 + v_4) = 1 - (-5 + 5) = 1$

   YA: $d_{12} = C_{21} - (u_2 + v_1) = 10 - (-1 + 7) = 4$

   YB: $d_{22} = C_{22} - (u_2 + v_2) = 8 - (-1 + 6) = 3$

   ZD: $d_{34} = C_{34} - (u_3 + v_4) = 8 - (0 + 5) = 3$

	A	B	C	D	Supply	u_i
X	2 [30]	2(1)	2(1)	1(1)	30	$u_1 = -5$
Y	10(4)	8(3)	5 [30]	4 [40]	70	$u_2 = -1$
Z	7 [10]	6 [30]	6 [10]	8(3)	50	$u_3 = 0$
Demand	40	30	40	40		
v_j	$v_1 = 7$	$v_2 = 6$	$v_3 = 6$	$v_4 = 5$		

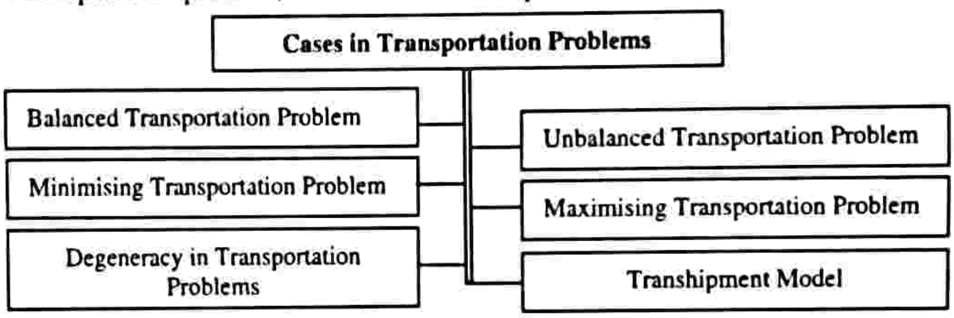
Since all $d_{ij} \geq 0$ so final optimal solution is arrived.

	A	B	C	D	Supply
X	2 [30]	2	2	1	30
Y	10	8	5 [30]	4 [40]	70
Z	7 [10]	6 [30]	6 [10]	8	50
Demand	40	30	40	40	

The minimum transportation cost = $2 \times 30 + 5 \times 30 + 4 \times 40 + 7 \times 10 + 6 \times 30 + 6 \times 10 = 680$

2.1.10. Cases in Transportation Problems

While solving a transportation problem, different varieties of problems arise; some of them are as follows:



2.1.10.1. Balanced Transportation Problems

If the sum of the supplies of all the sources is equal to the sum of the demands of all the destinations, then such type of problem is known as balanced transportation problem. Mathematically, it is represented by the following relationship:

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

Example 20: Solve the following transportation problem:

From ↓ / To→	W_1	W_2	W_3	W_4	W_5	Supply
O_1	20	28	32	55	70	50
O_2	48	33	40	44	25	100
O_3	35	55	22	43	48	150
Demand	100	70	50	40	40	300

Solution: Applying the matrix minima method, we obtain the following allocation table:

To → / From ↓	W_1	W_2	W_3	W_4	W_5	Supply
O_1	20 (50)	28	32	55	70	50/0
O_2	48	33 (60)	40	44	25 (40)	100/60/0
O_3	35 (50)	55 (10)	22 (50)	43 (40)	48	150/100/50/0
Demand	100/50/0	70/10/0	50/0	40/0	40/0	300

Here, the number of allocations (7) is equal to m (row) + n (column) – 1 = 3 + 5 – 1 = 7.
∴ the solution is feasible.

Total Transportation Cost = 20 × 50 + 33 × 60 + 25 × 40 + 35 × 50 + 55 × 10 + 22 × 50 + 43 × 40
= 1000 + 1980 + 1000 + 1750 + 550 + 1100 + 1720 = ₹9100

Example 21: Solve the following transportation problem to minimise the total transportation cost for shifting the goods from factories (A, B and C) to warehouses (P, Q and R) where unit transportation cost, availability and demand, at factories and warehouses respectively are given in the following matrix:

		Warehouse			
		P	Q	R	Availability
Factory	A	1	2	0	30
	B	2	3	4	35
	C	1	5	6	35
	Demand	30	40	30	

Find the allocation so that the total transportation cost is minimum.

Solution: Applying VAM Method,

	P	Q	R	Supply	Row Penalties
A	1	2	0 (30)	30	1 1
B	2	3 (30)	4	35	1 1
C	1 (30)	5 (5)	6	35	4 4
Demand	30	40	30	100	
Column Penalties	0	1	4		
	1	2	–		

Transportation Cost = 1 × 30 + 3 × 30 + 5 × 5 + 0 × 30 = 30 + 90 + 25 = 145

Optimality Test (MODI Method)
Since the number of basic cells = (m + n – 1) = 4 + 3 – 1 = 6
No. of allocations = 4

As the number of allocated cells (4) is less than number of basic cells (6), hence the solution is degenerate. We now introduce small amount Δ in the cell (1, 1) to eliminate the degeneracy. When we input small amount Δ in the cell (1, 1), then number of occupied cells has becomes (m + n – 1).

Now let us calculate u_i and v_j for all allocated cells by using $u_i + v_j = c_{ij}$.

	P	Q	R	Availability	u_i
A	1 (Δ)	2	0 (30)	30	$u_1 = 0$
B	2	3 (30)	4	35	$u_2 = -2$
C	1 (30)	5 (5)	6	35	$u_3 = 0$
Demand	30	40	30	100	
v_j	$v_1 = 1$	$v_2 = 5$	$v_3 = 0$		

Now we calculate $d_{ij} = c_{ij} - (u_i + v_j)$ for all non-occupied cells. Thus we have,

$d_{12} = c_{12} - (u_1 + v_2) = 2 - (0 + 1) = 1$

$d_{21} = c_{21} - (u_2 + v_1) = 2 - (-2 + 1) = 3$

$d_{23} = c_{23} - (u_2 + v_3) = 4 - (-2 + 0) = 6$

$d_{33} = c_{33} - (u_3 + v_3) = 6 - (0 + 0) = 6$

Since all values of d_{ij} are positive hence it is an optimal solution.

∴ Transportation Cost = 145

2.1.10.2. Unbalanced Transportation Problems

If the sum of the supplies of all the sources is not equal to the sum of the demands of all the destinations, then this kind of problem is known as unbalanced transportation problem. Mathematically, it is represented by the following relationship:

$$\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$$

Example 22: A company has 3 factories A, B and C which supply to 4 warehouses at P, Q, R and S. The monthly production capacity (tonnes) A, B and C are 120, 80 and 200 respectively. The monthly requirement (tonnes) for the warehouses P, Q, R and S are 60, 50, 140 and 50 respectively.

The transportation cost (₹ per tonne) Matrix is given below:

Factories

Warehouse		A	B	C
	P	4	3	7
	Q	5	8	4
	R	2	4	7
	S	5	8	4

Use Vogel's method to determine transportation distribution of product to warehouses to minimise the transportation cost.

Solution: After applying the Vogel's method we have following allocations:

	A	B	C	Capacity	Penalty I	II	III	IV	V	VI
P	4	3 (60)	7	60	1	1	4↑	–	–	–
Q	5	8	4 (50)	50	1	1	4	4↑	–	–
R	2 (120)	4 (20)	7	140	2	2	3	3	3	4
S	5	8	4 (50)	50	1	1	4	4	4↑	–
Dummy	0	0	0 (100)	100	0	–	–	–	–	–
Demand	120	80	200	400						
Penalty I	2	3	4↑							
II	2↑	1	0							
III	–	1	0							
IV	–	4	0							
V	–	4	3							
VI	–	4↑	–							

Total Minimum Transportation Cost (TC) $= 3\times 60 + 4\times 50 + 2\times 120 + 4\times 20 + 4\times 50 + 0\times 100$
$= 180 + 200 + 240 + 80 + 200 + 0 = ₹900$

Example 23: Applying North-West Corner Method, solve the following transportation problem:

From ↓ / To →	A	B	C	D	Supply
O_1	6	8	7	12	500
O_2	10	13	9	11	400
O_3	8	10	12	14	900
Demand	700	500	400	300	1900 \ 1800

Solution: Here the Total demand = 1900 and Total Supply = 1800. It is not a balanced problem (i.e., unbalanced) because the total demand is not equal to total supply. For making this problem balanced, we add a dummy row with value 0 and with supply 100 units.

From ↓ / To →	A	B	C	D	Supply
O_1	6 (500)	8	7	12	500
O_2	10 (200)	13 (200)	9	11	400
O_3	8	10 (300)	12 (400)	14 (200)	900
O_4 (Dummy)	0	0	0	0 (100)	100
Demand	700	500	400	300	1900 \ 1900

Total Transportation Cost = 6 × 500 + 10 × 200 + 13 × 200 + 10 × 300 + 12 × 400 + 14 × 200 + 0 × 100
= 3000 + 2000 + 2600 + 3000 + 4800 + 2800+ 0 = 18200

2.1.10.3. Minimising Transportation Problems

Generally, transportation model is used for solving the cost minimisation problems. It is also used for solving the problems in which objective is to maximise total value or benefit.

Example 24: Determine optimal solution of the following transportation problem in order to minimise product's transportation cost. The structure of per unit cost is shown in **table below:**

	X	Y	Z	Supply
P	16	19	12	15
Q	22	11	19	16
R	14	18	9	11
Demand	10	17	15	42

Solution: After implementing the VAM method, we have the following allocation table:

Row Penalty

	X	Y	Z		I	II	III	IV	V
P	(10) 16	(1) 19	(4) 12	15	4	4	4	4	12
Q	22	(16) 11	19	16	8↑	-	-	-	-
R	14	18	(11) 9	11	5	5↑	-	-	-
	10	17	15	42					

Column Penalty

	X	Y	Z
I	2	7	3
II	2	1	3
III	16	19↑	12
IV	16↑	-	12
V	-	-	12↑

The allocation will be as follows:

Allocate 10 to cell (P, X) = 160,

Allocate 1 to cell (P, Y) = 19

Allocate 4 to cell (P, Z) = 48,

Allocate 16 to cell (Q, Y) = 176

Allocate 11 to cell (R, Z) = 99

Total Cost = 160 + 19 + 48 + 176 + 99 = ₹502

Example 25: Solve the following transportation problem by VAM.

					Supply
	11	13	17	14	250
	16	18	14	10	300
	21	24	13	10	400
Demand	200	225	275	250	950

Solution: Applying VAM Method, we have the following table:

					Supply	Row Penalties
	11 (200)	13 (50)	17	14	250	2 1 4 – – –
	16	18 (175)	14	10 (125)	300	4 4 4 4↑ – –
	21	24	13 (275)	10 (125)	400	3 3 3 3 3 10↑
Demand	200	225	275	250	950	

Column Penalties				
	5↑	5	1	0
	–	5↑	1	0
	–	6↑	1	0
	–	–	1	0
	–	–	13↑	10
	–	–	–	10

Transportation Cost = 11 × 200 + 13 × 50 + 18 × 175 + 10 × 125 +13 × 275 + 10 × 125
= 2200 + 650 + 3150 + 1250 + 3575 + 1250
= ₹12075

Optimality Test (MODI Method)

Since the number of basic cells = 4 + 3 – 1 = 6

No. of allocations = 6

∴The solution is non-degenerate.

Now we calculate u_i and v_j using $u_i + v_j = c_{ij}$ for all allocated cells.

					Supply	u_i
	11 (200)	13 (50)	17	14	250	$u_1 = 0$
	16	18 (175)	14	10 (125)	300	$u_2 = 5$
	21	24	13 (275)	10 (125)	400	$u_3 = 5$
Demand	200	225	275	250	950	
v_j	$v_1 = 11$	$v_2 = 13$	$v_3 = 8$	$v_4 = 5$		

Now we find net evaluations $d_{ij} = c_{ij} - (u_i + v_j)$ for all non-occupied cells. Then we have,

$d_{13} = c_{13} - (u_1 + v_3) = 17 - (0 + 8) = 9$, $d_{14} = c_{14} - (u_1 + v_4) = 14 - (0 + 5) = 9$

$d_{21} = c_{21} - (u_2 + v_1) = 16 - (5 + 11) = 0$, $d_{23} = c_{23} - (u_2 + v_3) = 14 - (5 + 8) = 1$

$d_{31} = c_{31} - (u_3 + v_1) = 21 - (5 + 11) = 5$, $d_{32} = c_{32} - (u_3 + v_2) = 24 - (5 + 13) = 6$

Since all $d_{ij} \geq 0$, so final optimal solution is as follows:

					Supply
	11 (200)	13 (50)	17	14	250
	16	18 (175)	14	10 (125)	300
	21	24	13 (275)	10 (125)	400
Demand	200	225	275	250	950

Optimal Transportation Cost $= 11 \times 200 + 13 \times 50 + 18 \times 175 + 13 \times 275 + 10 \times 125 + 10 \times 125$
$= 2200 + 650 + 3150 + 3575 + 1250 + 1250$
$=$ ₹12075

2.1.10.4. Maximising Transportation Problems

Generally, transportation model is used for solving the cost minimisation problems. It is also used for solving the problems in which objective is to maximise total value or benefit. In this case the unit profit or pay-off p_{ij} related with each route, (i, j) is given instead of unit cost c_{ij}.

Then the objective function in terms of total profit or pay-off is defined as follows:

$$\text{Maximise} Z = \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} x_{ij}$$

For solving the maximisation transportation problem, the algorithm is same as that for the minimisation problem, but few adjustments are required in Vogel's Approximation Method (VAM) and MODI optimality tests to find the initial solution because in this problem profits are given instead of costs. In each row or column, by using VAM method to find initial solution, penalties are computed as difference between the largest and next largest pay-off. In this case row and column differences represent pay-offs. Allocations are made in the cells with the largest pay-off corresponding to the highest row or column difference. In VAM method, penalties are calculated in each row or column as difference between the largest and next largest pay-off.

The criterion of optimality for the maximisation problem is just opposite of the rule for minimisation problem. For maximisation case the solution is called optimal when all opportunity costs d_{ij} for the unoccupied cells are zero or negative.

Example 26: Consider the profit matrix as shown below:

	A	B	C	D	E	Supply
X	19	21	16	15	15	150
Y	9	13	11	19	11	200
Z	18	19	20	24	14	125
Demand	80	100	75	45	125	

Maximise the profit.

Solution: As it is maximisation problem, we have to transform this problem into minimisation problem. For this, we have to subtract every element of matrix from the largest element (24) of the matrix as follows:

	A	B	C	D	E	Supply
X	5	3	8	9	9	150
Y	15	11	13	5	13	200
Z	6	5	4	0	10	125
Demand	80	100	75	45	125	

Here, total demand = 425 and total supply = 475. Since total demand is not equal to total supply, so it is an unbalanced problem. For balancing this problem, we have to add dummy column with demand equal to 50.

After applying Vogel's approximation method, the initial solution of above problem will be as follows:

	A	B	C	D	E	Dummy	Supply
X	5 (50)	3 (100)	8	9	9	0	150
Y	15 (25)	11	13	5	13 (125)	0 (50)	200
Z	6 (5)	5	4 (75)	0 (45)	10	0	125
Demand	80	100	75	45	125	50	475

Since total allocation (8) is equal to (m + n − 1 = 3 + 6 − 1 = 8), so this problem is not degeneracy. For testing the optimality of above initial solution, apply the MODI method as follows:

	A	B	C	D	E	Dummy	Supply	u_i
X	5 (50)	3 (100)	8	9	9	0	150	$u_1 = 5$
Y	15 (25) (−)	11	13	5 (+)	13 (125)	0 (50)	200	$u_2 = 15$
Z	6 (5) (+)	5	4 (75)	0 (45) (−)	10	0	125	$u_3 = 6$
Demand	80	100	75	45	125	50	475	
v_j	$v_1 = 0$	$v_2 = -2$	$v_3 = -2$	$v_4 = -6$	$v_5 = -2$	$v_6 = -15$		

As the all opportunity costs are not zero or greater than zero, hence we have to drop the cell (Y, A) and add the cell (Y, D). Hence the revised table is as below:

	A	B	C	D	E	Dummy	Supply	u_i
X	5 (50)	3 (100)	8	9	9	0	150	$u_1 = -6$
Y	15	11	13	5 (25)	13 (125)	0 (50)	200	$u_2 = 0$
Z	6 (30)	5	4 (75)	0 (20)	10	0	125	$u_3 = -5$
Demand	80	100	75	45	125	50	475	
v_j	$v_1 = 11$	$v_2 = 9$	$v_3 = 9$	$v_4 = 5$	$v_5 = 13$	$v_6 = 0$		

As all the opportunity costs $d_{ij} \geq 0$, hence this solution is optimal.

Total Profit = 50 × 19 + 100 × 21 + 25 × 19 + 125 × 11 + 50 × 0 + 30 × 18 + 75 × 20 + 20 × 24
= 950 + 2100 + 475 + 1375 + 0 + 540 + 1500 + 480 = ₹7420

Example 27: A company manufacturing air-coolers has two plants located at X and Y places with a capacity of 200 units and 100 units per week respectively. The company supplies the air-coolers to its four showrooms situated at A B, C and D which have a maximum demand of 75, 100, 100 and 30 units respectively. Due to the differences in raw material cost and transportation costs, the profit per unit in rupees differs which is shown in the table below. Plan the production programme so as to maximise the profit. The company may have its production capacity at both plants partly or wholly unused.

	A	B	C	D
X	90	90	100	110
Y	50	70	130	85

Solution: Since the total demand (i.e., 305 units) is greater than the total capacity, (i.e., 300 units). Therefore, the problem is unbalanced. Therefore, we add a dummy plant with its weekly capacity of (305 – 300) 5 units and transportation cost from this plant to all demanding cities is zero. Now, the modified table is shown below:

	A	B	C	D	Supply
X	90	90	100	110	200
Y	50	70	130	85	100
S_{dummy}	0	0	0	0	5
Demand	75	100	100	30	

Problem is Maximization, so convert it to minimization by subtracting all the elements from max element (130):

	A	B	C	D	Supply
X	40	40	30	20	200
Y	80	60	0	45	100
Dummy	130	130	130	130	5
Demand	75	100	100	30	

The initial solution is obtained by using Vogel's Approximation Method as given in table below:

Initial Solution Obtained by Vogel's Approximation Method

To / From	A	B	C	D	Capacity	Row Penalty				
X	40 (70)	40	30 (100)	20	200 (30)	10	20	0	40	40
Y	80	60	0	45 (100)	100	45	–	–	–	–
Dummy	130 (5)	130	130	130	5	0	0	0	130	–
Demand	75	100	100	30	305					
	40	20	30	25						
Column	90	90	–	110						
Penalty	90	90	–							
	90	–	–							
	40	–	–							

The allocations in the original problem are as follows:

To / From	A	B	C	D	Supply
X	90 (70)	90 (100)	100	110 (30)	200
Y	50	70	130 (100)	85	100
Dummy	0 (5)	0	0	0	5
Demand	75	100	100	30	305

The maximum profit $= 90 \times 70 + 90 \times 100 + 110 \times 30 + 130 \times 100 + 0 \times 5 =$ ₹31600

Here, the number of allocated cells = 5, which is one less than to $m + n - 1 = 3 + 4 - 1 = 6$

This solution is degenerate

To resolve degeneracy, we make use of an artificial quantity (d).

The quantity Δ is assigned to that unoccupied cell, which has the minimum transportation cost.

The quantity Δ is assigned to XC, which has the minimum transportation cost = 30.

Using MODI's Method, we notice that this initial solution is also optimal solution. The optimal solution is shown in **table below:**

To / From	A	B	C	D	Supply
X	40 (70)	40 (100)	30 (Δ)	20 (30)	200
Y	80	60	0 (100)	45	100
Dummy	130 (5)	130	130	130	5
Demand	75	100	100	30	305

Iteration-1 of Optimality Test

1) Find u_i and v_j for all occupied cells (i, j), where $c_{ij} = u_i + v_j$.
   Substituting, $u_2 = 0$, we get
   $c_{11} = u_1 + v_1 \Rightarrow v_1 = c_{11} - u_1 \Rightarrow v_1 = 40 - 0 \Rightarrow v_1 = 40$
   $c_{31} = u_3 + v_1 \Rightarrow u_3 = c_{31} - v_1 \Rightarrow u_3 = 130 - 40 \Rightarrow u_3 = 90$
   $c_{12} = u_1 + v_2 \Rightarrow v_2 = c_{12} - u_1 \Rightarrow v_2 = 40 - 0 \Rightarrow v_2 = 40$
   $c_{13} = u_1 + v_3 \Rightarrow v_3 = c_{13} - u_1 \Rightarrow v_3 = 30 - 0 \Rightarrow v_3 = 30$
   $c_{23} = u_2 + v_3 \Rightarrow u_2 = c_{23} - v_3 \Rightarrow u_2 = 0 - 30 \Rightarrow u_2 = -30$
   $c_{14} = u_1 + v_4 \Rightarrow v_4 = c_{14} - u_1 \Rightarrow v_4 = 20 - 0 \Rightarrow v_4 = 20$

To / From	A	B	C	D	Supply	u_i
X	40 (70)	40 (100)	30 (Δ)	20 (30)	200	0
Y	80	60	0 (100)	45	100	-30
Dummy	130 (5)	130	130	130	5	90
Demand	75	100	100	30	305	
v_j	40	40	30	20		

2) Find d_{ij} for all unoccupied cells (i, j), where $d_{ij} = c_{ij} - (u_i + v_j)$
   $d_{21} = c_{21} - (u_2 + v_1) = 80 - (-30 + 40) = 70$
   $d_{22} = c_{22} - (u_2 + v_2) = 60 - (-30 + 40) = 50$
   $d_{24} = c_{24} - (u_2 + v_4) = 45 - (-30 + 20) = 55$
   $d_{32} = c_{32} - (u_3 + v_2) = 130 - (90 + 40) = 0$
   $d_{33} = c_{33} - (u_3 + v_3) = 130 - (90 + 30) = 10$
   $d_{34} = c_{34} - (u_3 + v_4) = 130 - (90 + 20) = 20$

To / From	A	B	C	D	Supply	u_i
X	40 [70]	40 [100]	30 [Δ]	20 [30]	200	0
Y	80 (70)	60 (50)	0 [100]	45 (55)	100	–30
Dummy	130 [5]	130 (0)	130 (10)	130 (20)	5	90
Demand	75	100	100	30	305	
v_j	40	40	30	20		

Since all $d_{ij} \geq 0$, So final optimal solution is arrived.

To / From	A	B	C	D	Supply
X	40 [70]	40 [100]	30 [Δ]	20 [30]	200
Y	80	60	0 [100]	45	100
Dummy	130 [5]	130	130	130	5
Demand	75	100	100	30	305

Allocation in the original problem is shown in **table below:**

To / From	A	B	C	D	Supply
X	90 [70]	90 [100]	100 [Δ]	110 [30]	200
Y	50	70	130 [100]	85	100
Dummy	0 [5]	0	0	0	5
Demand	75	100	100	30	305

The maximum profit $= 90 \times 70 + 90 \times 100 + 110 \times 30 + 130 \times 100 + 0 \times 5 =$ ₹31600

2.1.10.5. Degeneracy in Transportation Problems

In transportation problem degeneracy arises if it satisfies the following condition:

Number of occupied cells < (Number of columns + number of rows – 1)

Let the transportation problem consists of 'm' number of origins and 'n' numbers of destinations. Then its basic feasible solution is known as degenerate if:

Number of occupied cells < (m + n – 1)

Generally, degeneracy arises during the calculation of the initial basic feasible solution or during the testing of the optimal solution:

1) **Resolution of Degeneracy in the Initial Stage:** For resolving degeneracy, allocate a very small amount (almost close to zero) to one or more of the unoccupied cells which results in the number of occupied cells equal to (m + n – 1). For resolving degeneracy, one or more unoccupied cells are allocated by a small quantity 'Δ' (close to zero) and after performing this activity the occupied cells will become (m + n – 1). In a minimisation transportation problem, 'Δ' is allocated to that unoccupied cell which has lowest transportation cost. While in a maximisation problem, 'Δ' is allocated to that cell which has highest payoff value. In some cases, Δ must be added in one of those cells which uniquely identify the values of u_i and v_j.

2) **Resolution of Degeneracy during Solution Stage:** This is another method used to resolve degeneracy which arises during optimality test. In this method 'Δ' is assigned to one or more cells which are recently vacated. Then the new solution consists of a number of occupied cells equal to (m + n – 1). Once the optimal solution is reached, it may be removed.

Example 28: A pharmaceutical company transports its medical equipment from source A, B, C to destinations X, Y, Z and W. The demand, supply and time of shipment are shown in the following table. Show how company will make transportation plan so that total cost for shipment is minimum?

Source		Destinations X	Y	Z	W	Supply
Source	A	10	22	0	22	8
	B	15	20	12	8	13
	C	21	12	10	15	11
	Demand	5	11	8	8	

Solution: Applying the VAM method, we have the following table:

	X	Y	Z	W	Supply	Row Penalty I	II	III
A	10	22	0 (8)	22	8	10	–	–
B	15 (5)	20	12	8 (8)	13	4	7	7↑
C	21	12 (11)	10	15	11	2	3	6
Demand	5	11	8	8	32			
Home Penalty I	5	8	10↑	7				
II	6	8↑	–	7				
III	6	–	–	7				

Total Transportation Costs = 8 × 0 + 5 × 15 + 8 × 8 + 11 × 12 = 0 + 75 + 64 + 132 = ₹271

Test for Optimality using MODI Method

Since the number of allocations (4) is not equal to (m + n – 1) = 4 + 3 – 1 = 6, hence solution is degenerate. For eliminating degeneracy, we have to add small amount Δ_1 and Δ_2 in the cell (A, X) and cell (C, Z) respectively. Hence now total numbers of allocated cells has become equal to (m + n – 1). Let us calculate the value of u_i and v_j for all occupied cells using the $u_i + v_j = c_{ij}$, we have following table:

	X	Y	Z	W	Supply	u_i
A	10 (Δ_1)	22	0 (8)	22	8	$u_1 = 0$
B	15 (5)	20	12	8 (8)	13	$u_2 = 5$
C	21	12 (11)	10 (Δ_2)	15	11	$u_3 = 10$
Demand	5	11	8	8	32	
v_j	$v_1 = 10$	$v_2 = 2$	$v_3 = 0$	$v_4 = 3$		

Now we have to calculate $d_{ij} = c_{ij} - (u_i + v_j)$ for all non-allocated cells. We get the following values:

$d_{12} = c_{12} - (u_1 + v_2) = 22 - (0 + 2) = 20,$ $\quad d_{14} = c_{14} - (u_1 + v_4) = 22 - (0 + 3) = 19$

$d_{22} = c_{22} - (u_2 + v_2) = 20 - (5 + 2) = 13,$ $\quad d_{23} = c_{23} - (u_2 + v_3) = 12 - (5 + 0) = 7$

$d_{31} = c_{31} - (u_3 + v_1) = 21 - (10 + 10) = 1,$ $\quad d_{34} = c_{34} - (u_3 + v_4) = 15 - (10 + 3) = 2$

Since all values of $d_{ij} \geq 0$, hence it is an optimal solution.

∴ Total Transportation Cost = ₹271

Example 29: A company has three plants with capacities of 60, 70 and 80 units respectively to meet the demands of three warehouses with respective requirement of 50, 80 and 80 units. Given the following per unit costs transportation, find the optimum transportation plan.

	Warehouse	A	B	C
Plant	X	8	7	3
	Y	3	8	9
	Z	11	3	5

Solution: The problem is balanced as, total demand = total supply. The initial solution is determined by VAM.

	A	B	C	Supply	Row Penalty			
X	8	7	3 (60)	60	4	4	3	3
Y	3 (50)	8	9 (20)	70	5	1	9	–
Z	11	3 (80)	5 (Δ)	80	2	2	–	–
Demand	50	80	80	210				
Column Penalty	5	4	2					
	–	4	2					
	–	–	6					
	–	–	3					

It is observed that in the initial solution the number of occupied cells are 4, which is less than $(m + n - 1) = (3 + 3 - 1) = 5$. Hence the initial solution is not feasible and there is the problem of degeneracy.

Thus to make the solution feasible we introduce a small number Δ in one of the independent empty cells of initial solution with least cost. Evidently this is the cell (Z, C).

Optimal solution can be obtained by MODI method.

Determine all u_i and v_j values, for each unoccupied cell $\Delta_{ij} = u_i + v_j - c_{ij}$.

Plant	A	B	C	Supply	u_i
X	8 (–11)	7 (– 6)	3 (60)	60	0
Y	3 (50)	8 (– 1)	9 (20)	70	6
Z	11 (–12)	3 (80)	5 (Δ)	80	2
Demand	50	80	80	210	
v_j	–3	1	3		

Now since all $\Delta_{ij} < 0$, the current feasible solution is optimum.

Hence, the transportation cost of the optimum schedule is

$= 3 \times 60 + 50 \times 3 + 20 \times 9 + 3 \times 80 + 5 \times \Delta = 750 + 5\Delta \quad = 750$ since $\Delta \rightarrow 0$

2.1.10.6. Transhipment Models

It is a generalisation of transportation problem. In this case there is the possibility that a shipment from source to destination and vice versa as well as source to source and destination to destination. In some cases there are the chances of any economy as a result. It can be formulated like a transportation problem in addition a large number of sources and destinations.

For example, consider an industry with two factories F_1 and F_2 as well as three warehouses W_1, W_2 and W_3. It can be considered as a simple transportation problem if the transportation is only from factories to warehouses. On the other hand if the transportation is done from one factory to another factory or warehouse and from warehouse to any factory or another warehouse then it is considered as the **transhipment problem.** Like this, transportation of material comprising two altered transportation modes- roads and railways or the between stations that are linked by broad gauge and metre gauge lines. This transportation will certainly involve transhipment.

To drive the transhipment, disregard the distinction between sources to destinations. So that, the transportation problem with s sources and d destinations can be converted into transhipment problem with s + d sources and s + d destinations. If total flow into a node is equal to total flow out from node, node represents a **pure transient node.**

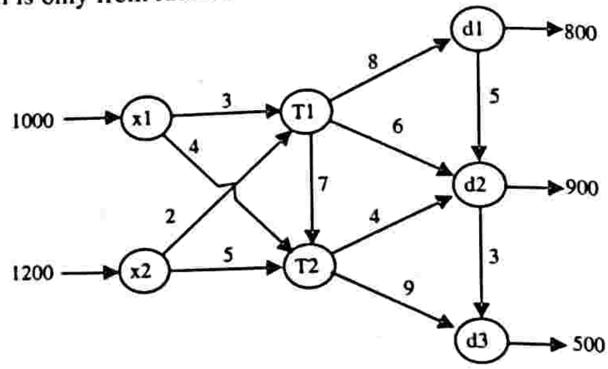


Figure 2.3: Transshipment Network between Plants and Dealers

Example 30: Two vehicle manufacturing plants, x1 and x2, are linked to three dealers, d1, d2, and d3, by way of two transit centres, T1 and T2. This network is shown in **figure 2.3**. The supply amounts at plants x1 and x2 are 1000 and 1200 cars, and the demand amounts at dealers dl, d2, and d3, are 800, 900, and 500 vehicle. The shipping costs per car (in ₹00) between pairs of nodes are shown by links of the network. Transhipment occurs in the network because the total supply amount of 2200 (= 1000 + 1200) vehicle at nodes x1 and x2 could pass via any node of the network before ultimately reaching their destinations at nodes d1, d2, and d3. Every node with input and output arcs is considered as a source and designation and this node is work as transhipment node. The other remaining nodes are either pure supply nodes (x1 and x2) or pure demand nodes (d3).

This transhipment problem can be changed into standard transportation problem having six sources(x1, x2, T1, T2, dl, and d2) and five destinations (T1, T2, d1, d2, and d3). The total amounts of supply and demand at the different nodes are computed as follows:

Supply at a pure supply node = Original supply
Demand at a **pure demand node** = Original demand
Supply at a transhipment node = Original supply + Buffer amount
Demand at a **transhipment node** = Original demand + Buffer amount

According to the requirement the buffer amount must be greater to allow all of the original supply (or demand) units to transfer via any of the transhipment nodes. Let us consider that B be the needed buffer amount then we have:
B = Total supply (or demand) = 1000 + 1200 (or 800 + 900 + 500) = 2200 cars

With the help of buffer B and the unit shipping costs; one can construct the equivalent regular transportation model as shown in **table 2.18.**

Table 2.18: Transshipment Model

	T1	T2	d1	d2	d3	
x1	3	4	M	M	M	1000
x2	2	5	M	M	M	1200
T1	0	7	8	6	M	B
T2	M	0	M	4	9	B
d1	M	M	0	5	M	B
d2	M	M	M	0	3	B
	B	B	800 + B	900 + B	500	

The solution of the above transportation model is illustrated in **figure 2.4**. The effect of transhipment: Dealer d2 receives 1400 vehicle, keeps 900 vehicles to satisfy its demand, and sends the remaining 500 vehicle to dealer D3.

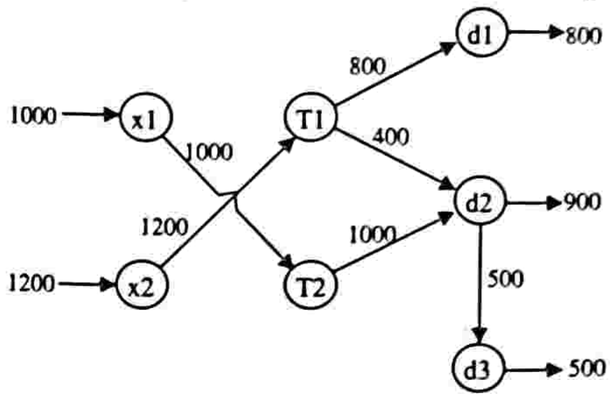


Figure 2.4: Solution of the Transshipment Model

Example 31: The requirements and capacity at the factories in terms of unit transportation costs for the company XYZ Ltd. are shown in **table 2.19**:

Table 2.19

Warehouses

		W_1	W_2	W_3	Supply
Factory	F_1	2	6	30	150
	F_2	6	10	50	300
	Demand	150	150	150	450

Factory to factory, unit transportation cost is shown in following **table 2.20**:

Table 2.20

Factory

		F_1	F_2
Factory	F_1	0	130
	F_2	2	0

Warehouse to warehouse, unit transportation costs are shown in following **table 2.21**:

Table 2.21

Warehouse

		W_1	W_2	W_3
Warehouse	W_1	0	46	2
	W_2	2	0	6
	W_3	130	6	0

Warehouse to factory, unit transportation costs are shown in following **table 2.22**:

Table 2.22

	Factory	
Warehouse	F_1	F_2
W_1	6	30
W_2	50	6
W_3	90	110

Solution:

Step 1: First we have to convert the above transshipment problem as a simple transportation problem:

One can get a formulation of transhipment problem by adding a 450(buffer stock). This stock shows the total capacity and total requirement in the original transportation problem to each row and column of the transhipment problem using the **table 2.19, table 2.20, table 2.21** and **table 2.22**. Thereafter the final transportation problem consists of m + n = 5 origins and m + n = 5 destinations as shown in table below:

Table 2.23: Transshipment Table

	F_1	F_2	W_1	W_2	W_3	Supply
F_1	0	130	2	6	30	150 + 450 = 600
F_2	2	0	6	10	50	300 + 450 = 750
W_1	6	30	0	46	2	450
W_2	50	6	2	0	6	450
W_3	90	110	130	6	0	450
Demand	450	450	150 + 450 = 600	150 + 450 = 600	150 + 450 = 600	2,700

Step 2: We get the following allocations after solving the transportation problem:

	F_1	F_2	W_1	W_2	W_3	Supply
F_1	(150) 0	130	(300) 2	(150) 6	30	600
F_2	(300) 2	(450) 0	6	10	50	750
W_1	6	30	(300) 0	46	(150) 2	450
W_2	50	6	2	0	6	450
W_3	90	110	130	(450) 6	0	450
Demand	450	450	600	600	(450) 600	2,700

Thus we have $x_{11} = 150$, $x_{13} = 300$, $x_{14} = 150$, $x_{21} = 300$
$x_{22} = 450$, $x_{33} = 300$, $x_{35} = 150$, $x_{44} = 450$,
$x_{55} = 450$

The details of the above transhipment problem are shown below:

1) The $x_{21} = 300$ units are transported from the factory F_2 to factory F_1. An increment of availability to 450 units consisting of originally 150 units from F_1.
2) From factory F_1 the $x_{13} = 300$ is transported to W_1 and $x_{14} = 150$ is transport to W_2.
3) From 300 units available at W_1 transport $x_{35} = 150$ units to W_3.

Total Transshipment Cost = $2 \times 300 + 6 \times 150 + 2 \times 300 + 2 \times 150 = 2400$

As allocations are transported from factories to warehouses as per **table 2.22**, hence the minimal transportation cost allocations are $x_{13} = 150$, $x_{21} = 150$, $x_{22} = 150$ with a minimal cost of ₹ 6,900. Thus, transshipment reduces the cost of cargo movement in such case.

Example 32: Consider the following transhipment problem with two sources and three destinations. The unit cost of transportation between different possible nodes is given in the following table. Find the optimal shipping plan such that the total cost is minimised.

		Destination					
		S_1	S_2	D_1	D_2	D_3	Supply
	S_1	0	3	12	4	12	800
	S_2	5	0	3	6	10	700
	D_1	8	10	0	4	20	–
Source	D_2	20	12	5	0	15	–
	D_3	8	10	30	8	0	–
	Demand	–	–	500	400	600	

Solution: For this trans-shipment, total supply = total demand = 800 + 700 = 500 + 400 + 600 = 1500

Adding 1500 units to each supply/demand point, we get the following table. Initial basic feasible solution (IBFS) obtained by the vogel's approximation method is also shown.

	S_1	S_2	D_1	D_2	D_3	Supply
S_1	0	3	12	4	12	800 + 1500 = 2300
S_2	5	0	3	6	10	700 + 1500 = 2200
D_1	8	10	0	4	20	1500
D_2	20	12	5	0	15	1500
D_3	8	10	30	8	0	1500
Demand	1500	1500	500 + 1500 = 2000	400 + 1500 = 1900	600 + 1500 = 2100	

Initial feasible solution is:

	S_1	S_2	D_1	D_2	D_3	Supply	Row Penalty
S_1	(1500) 0	3	12	(400) 4	(400) 12	2300	3 3 3 1 1 1 8 12 –
S_2	5	(1500) 0	(500) 3	6	(200) 10	2200	3 3 3 3 3 6 4 10 10
D_1	8	10	(1500) 0	4	20	1500	4 4 4 4 – – – – –
D_2	20	12	5	(1500) 0	15	1500	5 5 – – – – – – –
D_3	8	10	30	8	(1500) 0	1500	8 – – – – – – – – –
Demand	1500	1500	2000	1900	2100		

Column Penalty	S_1	S_2	D_1	D_2	D_3
	5	3	3	4	10
	5	3	3	4	2
	5	3	3	0	2
	–	3	3	0	2
	–	3	9	2	2
	–	3	–	2	2
	–	–	–	2	2
	–	–	–	–	2
	–	–	–	–	10

The minimum total transportation cost $= 0 \times 1500 + 4 \times 400 + 12 \times 400 + 0 \times 1500 + 3 \times 500 + 10 \times 200 + 0 \times 1500 + 0 \times 1500 + 0 \times 1500 = 9900$

Here, the number of allocated cells = 9 is equal to $m + n - 1 = 5 + 5 - 1 = 9$

$\therefore$ This solution is non-degenerate.

Optimality Test Using MODI method

Allocation table is:

	S_1	S_2	D_1	D_2	D_3	Supply
S_1	(1500) 0	3	12	(400) 4	(400) 12	2300
S_2	5	(1500) 0	(500) 3	6	(200) 10	2200
D_1	8	10	(1500) 0	4	20	1500
D_2	20	12	5	(1500) 0	15	1500
D_3	8	10	30	8	(1500) 0	1500
Demand	1500	1500	2000	1900	2100	

We find u_i and v_j for all occupied cells (i, j), where $c_{ij} = u_i + v_j$

Substituting, $u_1 = 0$, we get

$c_{11} = u_1 + v_1 \Rightarrow v_1 = 0 - 0 \Rightarrow v_1 = 0$

$c_{14} = u_1 + v_4 \Rightarrow v_4 = 4 - 0 \Rightarrow v_4 = 4$

$c_{44} = u_4 + v_4 \Rightarrow u_4 = 0 - 4 \Rightarrow u_4 = -4$

$c_{15} = u_1 + v_5 \Rightarrow v_5 = 12 - 0 \Rightarrow v_5 = 12$

$c_{25} = u_2 + v_5 \Rightarrow u_2 = 10 - 12 \Rightarrow u_2 = -2$

$c_{22} = u_2 + v_2 \Rightarrow v_2 = 0 + 2 \Rightarrow v_2 = 2$

$c_{23} = u_2 + v_3 \Rightarrow v_3 = 3 + 2 \Rightarrow v_3 = 5$

$c_{33} = u_3 + v_3 \Rightarrow u_3 = 0 - 5 \Rightarrow u_3 = -5$

$c_{55} = u_5 + v_5 \Rightarrow u_5 = 0 - 12 \Rightarrow u_5 = -12$

	S_1	S_2	D_1	D_2	D_3	Supply	u_i
S_1	(1500) 0	3	12	(400) 4	(400) 12	2300	0
S_2	5	(1500) 0	(500) 3	6	(200) 10	2200	–2
D_1	8	10	(1500) 0	4	20	1500	–5
D_2	20	12	5	(1500) 0	15	1500	–4
D_3	8	10	30	8	(1500) 0	1500	–12
Demand	1500	1500	2000	1900	2100		
v_j	0	2	5	4	12		

Now we find d_{ij} for all unoccupied cells (i, j), where $d_{ij} = c_{ij} - (u_i + v_j)$

$d_{12} = c_{12} - (u_1 + v_2) = 3 - (0 + 2) = 1$
$d_{13} = c_{13} - (u_1 + v_3) = 12 - (0 + 5) = 7$
$d_{21} = c_{21} - (u_2 + v_1) = 5 - (-2 + 0) = 7$
$d_{24} = c_{24} - (u_2 + v_4) = 6 - (-2 + 4) = 4$
$d_{31} = c_{31} - (u_3 + v_1) = 8 - (-5 + 0) = 13$
$d_{32} = c_{32} - (u_3 + v_2) = 10 - (-5 + 2) = 13$
$d_{34} = c_{34} - (u_3 + v_4) = 4 - (-5 + 4) = 5$
$d_{35} = c_{35} - (u_3 + v_5) = 20 - (-5 + 12) = 13$
$d_{41} = c_{41} - (u_4 + v_1) = 20 - (-4 + 0) = 24$
$d_{42} = c_{42} - (u_4 + v_2) = 12 - (-4 + 2) = 14$
$d_{43} = c_{43} - (u_4 + v_3) = 5 - (-4 + 5) = 4$
$d_{45} = c_{45} - (u_4 + v_5) = 15 - (-4 + 12) = 7$
$d_{51} = c_{51} - (u_5 + v_1) = 8 - (-12 + 0) = 20$
$d_{52} = c_{52} - (u_5 + v_2) = 10 - (-12 + 2) = 20$
$d_{53} = c_{53} - (u_5 + v_3) = 30 - (-12 + 5) = 37$
$d_{54} = c_{54} - (u_5 + v_4) = 8 - (-12 + 4) = 16$

	S_1	S_2	D_1	D_2	D_3	Supply	u_i
S_1	(1500) 0	3 (1)	12 (7)	(400) 4	(400) 12	2300	0
S_2	5 (7)	(1500) 0	(500) 3	6 (4)	(200) 10	2200	–2
D_1	8 (13)	10 (13)	(1500) 0	4 (5)	20 (13)	1500	–5
D_2	20 (24)	12 (14)	5 (4)	(1500) 0	15 (7)	1500	–4
D_3	8 (20)	10 (20)	30 (37)	8 (16)	(1500) 0	1500	–12
Demand	1500	1500	2000	1900	2100		
v_j	0	2	5	4	12		

Since all $d_{ij} \geq 0$. So final optimal solution is arrived.

	S_1	S_2	D_1	D_2	D_3	Supply
S_1	(1500) 0	3	12	(400) 4	(400) 12	2300
S_2	5	(1500) 0	(500) 3	6	(200) 10	2200
D_1	8	10	(1500) 0	4	20	1500
D_2	20	12	5	(1500) 0	15	1500
D_3	8	10	30	8	(1500) 0	1500
Demand	1500	1500	2000	1900	2100	

Minimum Total Transportation Cost = $0 \times 1500 + 4 \times 400 + 12 \times 400 + 0 \times 1500 + 500 + 10 \times 200 + 0 \times 1500 + 0 \times 1500 + 0 \times 1500 = 9900$

2.2. ASSIGNMENT MODELS

2.2.1. Introduction

A problem in which n different facilities are assigned to n different tasks, such a problem is known as an assignment problem. **For example**, if there is availability of three men and there are three jobs to be done where each man has capability of doing any job but because of individual quality variation, it takes different amount of time for each of them to carry out each job. Here the problem is how to bring out the assignment of men to the jobs such that total time spent on jobs can be minimised. Assignment model is basically a special type of linear programming in which the main objective is to bring out assignment of a number of origins to an equal destination number so as to maximise profit or minimise cost. A one-to-one basis needs is to be followed while assigning values.

Assignment Problem is the technique of selecting the best possible assignment of tasks from a number of alternatives.

Assignment problems are associated with matching of objects in two distinct set or bringing out optimal pairing. **For example**, in a children's garment sales depot, there are four sales counter and four salesmen. Now the problem is how assignment of salesmen should be done to the counters in such a manner that the total service time is minimised altogether.

2.2.2. Mathematical Model of Assignment Problem

Given n facilities (resources) and jobs (or activities), and effectiveness (in terms of time, profit, cost, etc.) of each facility (resource) for each job (activity), the problem lies in assignment of each resource to one and only one job (activity) such that there is an optimisation of the given effectiveness measure. For this problem, the data matrix is shown in **table below:**

	Resources		**Supply**
(workers)	**J_1**	**$J_2 \dots J_n$**	
W_1	c_{11}	$c_2 \dots c_{1n}$	1
W_2	c_{21}	$c_{22} \dots c_{2n}$	1
.	.	..	.
.	.	..	.
W_n	c_{n1}	$c_{n2} \dots c_{nn}$	1
Demand	1	$1 \dots 1$	n

It is observed from the table that the data matrix looks exactly like the transportation cost matrix. Only exception being availability (or supply) of each resource and the demand is taken to be one at each destinations. The fact that lies behind this is assignment on a one-to-one basis.

Let x_{ij} denote the assignment of i^{th} facility to j^{th} job such that:

$$x_{ij} = \begin{cases} 1 \text{ if facility is assigned to job j} \\ 0 \text{ otherwise} \end{cases}$$

Then, the mathematical model of the assignment problem can be stated as:

$$\text{Minimise } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = 1, \text{ for all i (resource availability)}$$

$$\sum_{i=1}^{n} x_{ij} = 1, \text{ for all j (activity requirement)}$$

And $x_{ij} = 0$ or 1, for all i and j.

Where c_{ij} represents the cost of assignment of resource i to activity j.

It is clear from the above data that the assignment problem is a special case of transportation problem possessing following two characteristics:

1) The cost matrix is absolutely a square matrix, and
2) The optimal problem solution would be always addressed so that in a given cost matrix's column or row, only one assignment will be there.

2.2.3. Applications of Assignment Model

Assignment model possess several applications. Some of them are as follows:

1) Assignment of operations to job.
2) Machines allocation for optimal space utilization.
3) Salesmen assignment to different sales areas.
4) Employee's assignment to machines.
5) Effectiveness of teachers and subjects.

2.2.4. Solution of Assignment Problem

Following four methods are widely used for solving an assignment problem:

1) **Enumeration Method:** Among the given resources (like men, machines etc.) and activities (like sales area, jobs, etc.), a list of all possible assignments is prepared while using this method. Now, an assignment is selected having minimum cost and maximum profit (time of distance). In case, same minimum cost or maximum profit is possessed by two or more assignments, then there are multiple optimal solutions to the problem. Generally, there are a total of n! Possible assignments for a problem with n job/workers. **For example**, in a problem with n = 5 jobs/workers, one need to evaluate total of 5! or 120 assignments. However, this method is unsuitable in case when n is large because manual calculations are difficult. Hence, Enumeration method is applicable with small n values.

2) **Simplex Method:** Simplex method can be used for solving the assignment problem because each assignment problem can be formulated as a 0 or 1 integer LPP which are solved simply by simplex method. In the general mathematical formulation of assignment problem, it is seen that there are n + n or 2n equalities and n × n decision. **For example,** to solve a problem with 5 workers/jobs, there will be 10 equalities and 25 decision variables which are hard to solve manually.

3) **Transportation Method:** Assignment problem is known to be a special case of transportation problem and thus it can be solved by the use of transportation methods. However, for a general assignment problem, every basic feasible solution having a square payoff matrix of order n must have assignments m + n –1= n + n – 1 = 2n – 1. But any solution cannot possess more than n assignments due to special structure of the problem. Thus, there is inherent degeneration of the assignment problem. So, for removing the degeneracy, dummy allocation of (n – 1) number is required so as to proceed with such transportation method. Thus, it can be evaluated that degeneracy problem at each solution makes the method of transportation inefficient computationally for assignment problem solution.

4) **Hungarian Assignment Method (HAM):** It is observable that none of the above mentioned three working methods to solve an assignment problem is efficient. A method specially designed to handle assignment problems is an efficient way that is based on the opportunity cost concept, known as **Hungarian Assignment Method.**

   This HAM method modifies successively the columns and rows of the effectiveness matrix until there is observance of at least one zero component in each column and row so that a complete assignment can be made corresponding to these zeros. When applied to the original effectiveness matrix, the complete assignment comes out to be an optimal assignment in which the resulting total effectiveness comes out to be minimal. There will always be a convergence of this method to optimal assignments in finite steps which are technically termed as **assignment algorithm.**

2.2.5. Hungarian Method/Flood's Technique

It can be observed that none of the above mentioned three working methods to solve an assignment problem is efficient. A **Hungarian Assignment Method (HAM)** method specially designed to handle assignment problems is an efficient way that is based on the opportunity cost concept.

This HAM method modifies successively the columns and rows of the effectiveness matrix until there is observance of at least one zero component in each column and row so that a complete assignment can be made corresponding to these zeros. When applied to the original effectiveness matrix, the complete assignment comes out to be an optimal assignment in which the resulting total effectiveness comes out to be minimal. There will be always a convergence of this method to optimal assignments in finite steps which are technically termed as **assignment algorithm**.

Following are the various steps which are involved in the assignment algorithm: **(figure 2.5)**

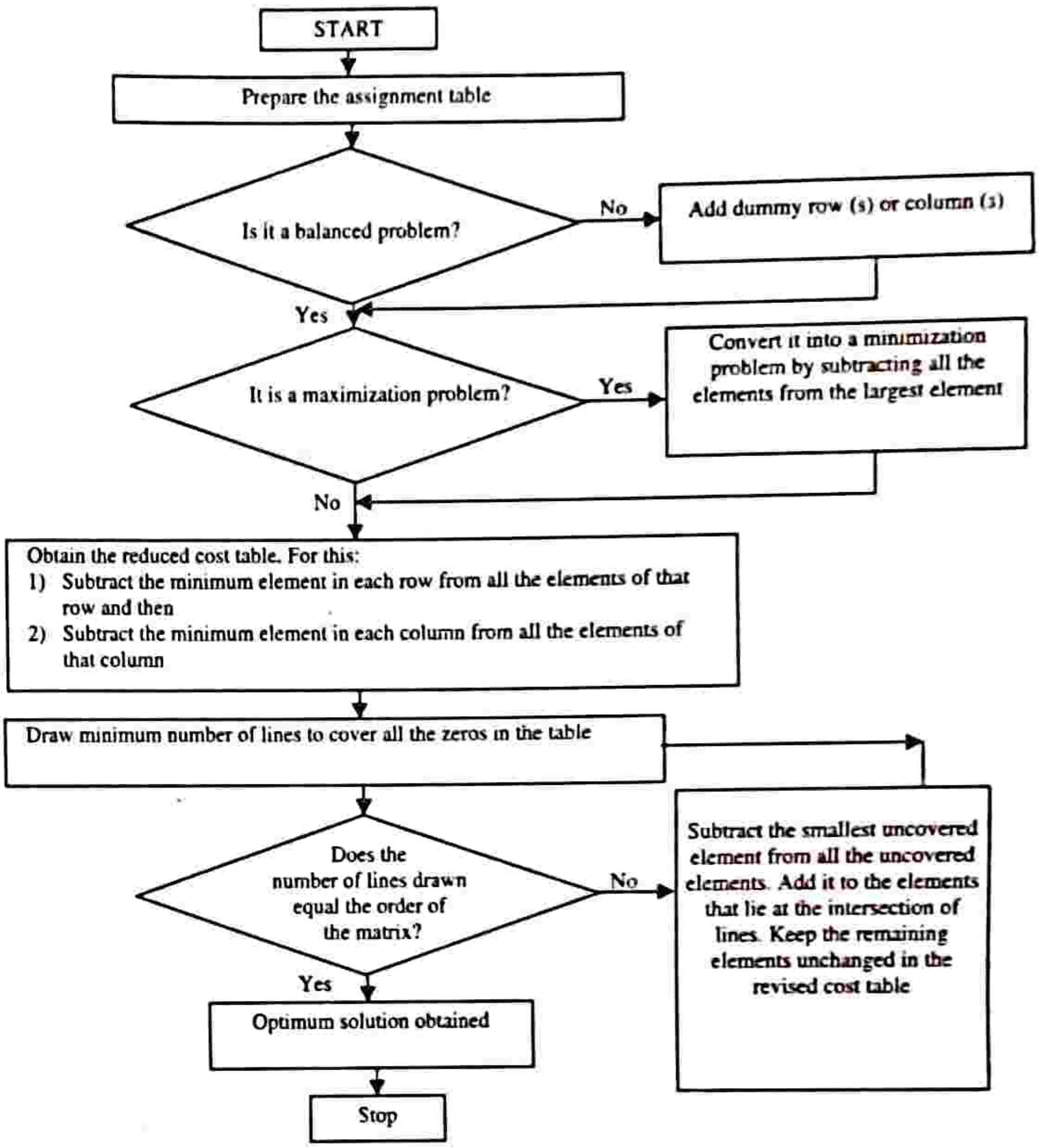


Figure 2.5: Flow Chart for the Hungarian Method

Step 1: Row Reduction: Lowest cost in the row is subtracted from each row.

Step 2: Column Reduction: Lowest cost in the column is subtracted from each column of this new cost matrix.

Step 3: Minimum number of vertical and horizontal lines are drawn so as to necessarily cover all zeros at least once. If in an n × n matrix, minimum number of lines is n, then for the given assignment problem, one has an optimal solution corresponding to a set of independent zeros i.e., a set of zeros where no two zeros in the set occur in the same row or column.

Step 4: In the case when minimum number of lines in n × n matrix is less than n, then smallest number is selected which does not have a line through it.

Step 5: This number is then subtracted from all elements that are covered by lines and added to the elements which are located at the intersection of two lines. Step (3) is applied again to find a solution among the new set of zeros. In case, no result is found, this step is repeated and usage of step (3) and (4) is continued until we find a solution of the assignment problem.

Step 6: Optimality Criterion: The job assignments are made with the help of given optimal solution as indicated by the 'zero' elements. The procedure for this is as follows:

i) A row is located which is containing only one 'zero' element. The job corresponding to this element is assigned to its corresponding person. All zeros in the column corresponding to the element are crossed out, if any. This indicates the fact that there is no more availability of particular person or job.

ii) Step (i) is repeated for each of such rows that contains only one zero in that. The same operation is performed similarly with respect to each column containing only one 'zero' element in the row in which there is element, if any.

iii) In case when there is no column or row with only one 'zero' element left in them, then arbitrarily selection of a row/column is made and one of the jobs or person is chosen to make assignment. Now the remaining zeros are crossed in the row and column with respect to which we have done the assignment.

iv) Repeat steps (i) to (iii) until we have done all the assignments.

v) Total cost is determined with reference to the original cost table.

Example 33: Solve the following assignment problem using Hungarian Method.

Men	Tasks			
	A	B	C	D
1	45	40	50	67
2	57	42	63	55
3	49	51	48	64
4	41	45	60	55

Solution: Applying Hungarian Method, we have the following steps:

Step 1) Row Reduction: Select the lowest element of each row and subtract it from the other elements of that row as follows:

Men	Tasks			
	A	B	C	D
1	5	0	10	27
2	15	0	21	13
3	1	3	0	16
4	0	4	19	14

Step 2) Column Reduction: Select the lowest element of each column and subtract it from the other elements of that column as follows:

Men	Tasks			
	A	B	C	D
1	5	0	10	14
2	15	0	21	0
3	1	3	0	3
4	0	4	19	1

Step 3) Draw Lines: Now we have to draw minimum number of lines in such a manner that it covers all zeros. For this one has to first cover those rows/columns that contain maximum zeros.

Men	Tasks			
	A	B	C	D
1	5	0	10	14
2	15	0	21	0
3	1	3	0	3
4	0	4	19	1

As the total number of lines drawn is 4 which are equal to rows/columns (n = 4), so the solution is optimal. The assignment can be made by scanning all zeros. The assignments are shown by square symbol ( ) as shown in **table 2.24**.

Table 2.24: Assignment of Tasks

Men	Tasks			
	A	B	C	D
1	5	[0]	10	14
2	15	X	21	[0]
3	1	3	[0]	3
4	[0]	4	19	1

Step 4) Optimal Solution: The assignment made occur in the following sequence. Since the first, third and fourth rows have only one zero, hence assignment will be 1–B, 3– C and 4–A. As the task B is assigned for the men 1, we cross-out other zeros occur in the second column. The only zero occur in the second row is for men 2 and task D. Hence the assignment will be 2–D. The optimal solution will be as follows:

Men	Tasks	Cost
1	B	40
2	D	55
3	C	48
4	A	41
	Total	184

Example 34: Find the optimal solution for the following assignment problem by using Hungarian method.

Worker	Job			
	A	B	C	D
1	45	40	51	67
2	57	42	63	55
3	49	52	48	64
4	41	45	60	55

Solution: Applying Hungarian Method,

Step 1) Row Reduction: Subtract from each element of each row the lowest cost in that row.

Worker	Job			
	A	B	C	D
1	5	0	11	27
2	15	0	21	13
3	1	4	0	16
4	0	4	19	14

Step 2) Column Reduction: Subtract form each element of each column the lowest cost in that column.

Worker	Job			
	A	B	C	D
1	5	0	11	14
2	15	0	21	0
3	1	4	0	3
4	0	4	19	1

Step 3) Draw Lines: Cover all the zeros by drawing the minimum number of lines. To do this first cover all those rows/columns that hold maximum number of zeros.

Worker	Job			
	A	B	C	D
1	5	0	11	14
2	15	0	21	0
3	1	4	0	3
4	0	4	19	1

Here the number of drawn lines is 4 which is equal to the n. This implies that we got the optimal solution. When scanning of rows and columns to check the unit zeros is completed then make the assignments. In the **table 2.25**, the assignments are shown by the use of squares.

Table 2.25: Assignment of Jobs

Worker	Job			
	A	B	C	D
1	5	[0]	11	14
2	15	X	21	[0]
3	1	4	[0]	3
4	[0]	4	19	1

Step 4) Optimal Solution: The order of the assignment is made as follows. Because rows number 1, 3 and 4 occupy only one zero each, so the assignment is 1-B, 3-C and 4-A. As job B has been assigned to the worker 1, so the zero present in the second column and second row will be crossed. When these assignments are done, the only left worker is 2 and job is D. Hence the final assignment is given as:

1 ⟶ B
2 ⟶ D
3 ⟶ C
4 ⟶ A

Example 35: Solve the following assignment problem:

Cost Matrix Table

Machines

Jobs	A	B	C	D	E
1	11	17	8	16	20
2	9	7	12	6	15
3	13	16	15	12	16
4	21	24	17	28	26
5	14	10	12	11	13

Solution: Applying the Hungarian method, the appropriate assignments are made as follows:

Step 1) Row Reduction: Find the minimum value from the each row and then subtract it from all elements of that row. This is illustrated in the following table:

Machines

Jobs	A	B	C	D	E
1	3	9	0	8	12
2	3	1	6	0	9
3	1	4	3	0	4
4	4	7	0	11	9
5	4	0	2	1	3

Step 2) Column Reduction: Do the same procedure along columns as done in rows. This is illustrated in the following table:

Machines

Jobs	A	B	C	D	E
1	2	9	0	8	9
2	2	1	6	0	6
3	0	4	3	0	1
4	3	7	0	11	6
5	3	0	2	1	0

Step 3) Drawing Line: Draw the minimum number of horizontal and vertical lines in order to cover all zeros.

Machines

Jobs	A	B	C	D	E
1	2	9	0	8	9
2	2	1	6	0	6
3	0	4	3	0	1
4	3	7	0	11	6
5	3	0	2	1	0

To cover all zeros here only 4 lines are enough. But here the matrix size is 5×5, which is greater than the number of lines. So it cannot give an optimal solution. The minimum number from which the line does not pass is 1. Now subtract it from each uncovered values and add it to all those values which are present at cross-section point. After doing this, again draw lines.

Machines

Jobs	A	B	C	D	E
1	1	8	0	8	8
2	1	0	6	0	5
3	0	4	4	1	1
4	2	6	0	11	5
5	3	0	3	2	0

Again the numbers of drawn lines that cover all zeros are 4 which is less than order of matrix i.e. 5. So it does not give the optimal solution. Here the minimum uncovered value is 1. Subtract it from each of the uncovered values and add it to all those values which are present at cross-section points.

Machines

Jobs	A	B	C	D	E
1	0	7	0	7	7
2	1	0	7	0	5
3	0	4	5	1	1
4	1	5	0	10	4
5	3	0	4	2	0

Again continue with the same procedure, we get the following table:

Machines

Jobs	A	B	C	D	E
1	0	6	0	6	6
2	2	0	8	0	5
3	0	3	5	0	0
4	1	4	0	9	3
5	4	0	5	2	0

Step 4) Optimal Solution: Now the order of the matrix and the number of drawn lines are same i.e. 5.

Machines

		A	B	C	D	E
Jobs	1	[0]	6	~~0~~	6	6
	2	2	[0]	8	~~0~~	5
	3	~~0~~	3	5	[0]	~~0~~
	4	1	4	[0]	9	3
	5	4	~~0~~	5	2	[0]

The order of the assignment made is as follows:

Jobs		Machine	Cost
1	→	A	11
2	→	B	7
3	→	D	12
4	→	C	17
5	→	E	13

Minimum Cost = 11 + 7 + 12 + 17 + 13 = ₹60

Example 36: Solve the following assignment:

	X_1	X_2	X_3	X_4	X_5
A	15	29	35	20	38
B	21	27	33	17	36
C	17	25	37	15	42
D	14	31	39	21	40
E	19	30	40	19	18

Solution: Applying Hungarian Method, the **steps** are as following:

Step 1) Row Reduction: Subtract the lowest element in each row from all the elements of that row.

	X_1	X_2	X_3	X_4	X_5
A	0	14	20	5	23
B	4	10	16	0	19
C	2	10	22	0	27
D	0	17	25	7	26
E	1	12	22	1	0

Step 2) Column Reduction: Subtract the lowest element in each column from all the elements of that column.

	X_1	X_2	X_3	X_4	X_5
A	0	4	4	5	23
B	4	0	0	0	19
C	2	0	6	0	27
D	0	7	9	7	26
E	1	2	6	1	0

Step 3) Draw Lines: Making the minimum number of lines covering all zeros. As a general rule, one should first cover those rows/columns which contain larger number of zeros.

	X_1	X_2	X_3	X_4	X_5
A	0	4	4	5	23
B	4	0	0	0	19
C	2	0	6	0	27
D	0	7	9	7	26
E	1	2	6	1	0

Since there are only four lines which cover all zeros and it is less than the order of the cost matrix (5), the current assignment is not optimal. Now we subtract the least uncovered value from all uncovered values and add intersection point of lines.

	X_1	X_2	X_3	X_4	X_5
A	0	3	3	4	23
B	5	0	0	0	20
C	3	0	6	0	28
D	0	6	8	6	26
E	1	1	5	0	0

Now again we draw the lines which cover the entire zero.

	X_1	X_2	X_3	X_4	X_5
A	0	3	3	4	23
B	5	0	0	0	20
C	3	0	6	0	28
D	0	6	8	6	26
E	1	1	5	0	0

Here number of lines are not equal to order of matrix hence assignments are not possible. So, we repeat the above process. Now again drawing lines we have following matrix:

	X_1	X_2	X_3	X_4	X_5
A	0	0	0	1	20
B	8	0	0	0	20
C	6	0	6	0	28
D	0	3	5	3	23
E	4	1	5	0	0

Step 4) Optimal Solution: Now the numbers of lines are 5. Hence it gives the optimal solution.

	X_1	X_2	X_3	X_4	X_5
A	~~0~~	~~0~~	[0]	1	20
B	8	[0]	~~0~~	~~0~~	20
C	6	~~0~~	6	[0]	28
D	[0]	3	5	3	23
E	4	1	5	~~0~~	[0]

Assignments are made in the following order.

A	→	X_3	35
B	→	X_2	27
C	→	X_4	15
D	→	X_1	14
E	→	X_5	18
		Total =	109

Example 37: Find the least cost allocation for the following data:

Typist	Job				
	P	Q	R	S	T
A	85	75	65	125	75
B	90	78	66	132	78
C	70	66	57	114	69
D	80	72	60	120	72
E	76	64	56	112	68

Solution: Applying Hungarian Method, we have the following steps:

Step 1) Row Reduction: Select the lowest element of each row and subtract it from the other elements of that row as follows:

Typist	Job				
	P	Q	R	S	T
A	20	10	0	60	10
B	24	12	0	66	12
C	13	9	0	57	12
D	20	12	0	60	12
E	20	8	0	56	12

Step 2) Column Reduction: Select the lowest element of each column and subtract it from the other elements of that column as follows:

Typist	Job				
	P	Q	R	S	T
A	7	2	0	4	0
B	11	4	0	10	2
C	0	1	0	1	2
D	7	4	0	4	2
E	7	0	0	0	2

Step 3) Draw Lines: Now we have to draw minimum number of lines in such a manner that it covers all zeros. For this one has to first cover those rows/columns that contain maximum zeros.

Typist	Job				
	P	Q	R	S	T
A	7	2	0	4	0
B	11	4	0	10	2
C	0	1	0	1	2
D	7	4	0	4	2
E	7	0	0	0	2

As the total number of lines drawn is 4 which are not equal to rows/columns (n = 5), so the solution is non-optimal.

For obtaining the optimal solution, we have to subtract the minimum uncovered value of cost matrix from other uncovered elements of matrix, except intersection point of lines where the values are added.

Typist	Job				
	P	Q	R	S	T
A	7	2	2	4	0
B	9	2	0	8	0
C	0	1	2	1	2
D	5	2	0	2	0
E	7	0	2	0	2

Now, again lines are drawn in order to cover all zeros.

Typist	Job				
	P	Q	R	S	T
A	7	2	2	4	0
B	9	2	0	8	0
C	0	1	2	1	2
D	5	2	0	2	0
E	7	0	2	0	2

Since again the total number of lines that covering the all zeros (4) is not equal to order of the matrix (n=5), hence the solution is not optimal.

Again deduct the minimum uncovered value from all other uncovered values, except from the values of point of intersection lines, where the values are added.

Typist	Job				
	P	Q	R	S	T
A	7	1	2	3	0
B	9	1	0	7	0
C	0	0	2	0	2
D	5	1	0	1	0
E	8	0	3	0	3

Now, again lines are drawn in order to cover all zeros.

Typist	Job				
	P	Q	R	S	T
A	7	1	2	3	0
B	9	1	0	7	0
C	0	0	2	0	2
D	5	1	0	1	0
E	8	0	3	0	3

Now again the total number of lines that covering the all zeros (4) is not equal to order of the matrix (n=5), hence the solution is not optimal. Again deduct the minimum uncovered value from all other uncovered values, except from the values of point of intersection lines, where the values are added.

Typist	Job				
	P	Q	R	S	T
A	6	0	2	2	0
B	8	0	0	6	0
C	0	0	3	0	3
D	4	0	0	0	0
E	8	0	4	0	4

Now, again lines are drawn in order to cover all zeros.

Typist	Job				
	P	Q	R	S	T
A	6	0	2	2	0
B	8	0	0	6	0
C	0	0	3	0	3
D	4	0	0	0	0
E	8	0	4	0	3

Now, the numbers of line (5) is equal to order of matrix (n =5). Hence this provides optimal solution. The assignment will be as follows:

Typist	Job				
	P	Q	R	S	T
A	6	X	2	2	[0]
B	8	X	[0]	6	X
C	[0]	X	3	X	3
D	4	X	X	[0]	X
E	8	[0]	4	X	4

Step 4) **Optimal Solution:** The sequence of assignments will be as below:

Typist	Job	Cost
A	T	75
B	R	66
C	P	70
D	S	120
E	Q	64
	Total	395

2.2.6. Branch and Bound Algorithms

Generally smaller and smaller subsets are to be generated by repeatedly dividing the space of all feasible solutions. This is known as **branching**. Within each subset of solution, a lower bound is calculated for cost solutions to minimise the problem. This is known as **bounding**. When we make the subsets, we eliminate those subsets for further partitioning whose bound is more than the cost of known feasible solution. This is done after each partition of subsets.

In this way, a big part of subsets of solutions may be eliminated without any test of each solution from consideration. This process of partitioning continues till that we do not find the feasible solution. The cost of this feasible solution should not be greater than the bound for any subset. In this technique, more the number of solutions examined before getting the optimal solution, more is the success rate.

With the finite number (ordinarily big enough) of feasible solutions, the integer programming problems, scheduling problems, plant location problems, assignment problems, traveling salesman problems, knapsack problems, and some more problems can be solved by using branch-and-bound algorithm with some amount of success.

A certain process to the find the optimal solution is referred by this algorithm. In different kinds of problems, it is used differently. It is basically used in combinatorial problems where the number of solutions is finite. With the use of few rules, these solutions are partitioned into two parts:

1) One keeps the optimal solution most probably and so forwarded for the further examination.
2) The other one does not contain the optimal solution, so it is left-out for further process.

Solution to Assignment Problems

As referred in this technique, initially all the possible solutions to the problem are listed and then from this list, the most optimised one is selected. When the people and jobs are added to the problem, the number of solution also increases. Hence it is very difficult to handle the problems. Even though the computers may be used to handle it, but it becomes extremely costly and time consuming.

For example, take a problem having ten workers and ten jobs needs a consideration of 36, 28, 800 numbers of feasible solutions. An enumeration is assigned by the branch and bound technique but this is done so efficiently that there is only few number of possible solutions needs to be examined individually.

Branch and Bound Algorithm

Step 1) First we consider the relation and assume the distances to which required rows and columns are added for making the matrix to be square.

Step 2) Now to make atleast one zero in each row and each column, subtract the least value among its elements from each row and then from each column.

Step 3) Now we get the sum of values subtracted from rows and columns. This sum constructs the value of the root of a directed tree.

Step 4) For making a first bi-partition, every zero of the relation is assigned an amount which is equal to the sum of lowest value of the row and the lowest value of the column whose member is zero. These numbers are framed in boxes.

Step 5) Now the directed tree is constructed by inputting the two vertices with denomination of the element having the largest value with each frame. A value equal to root plus amount found for the related element is given to which that is represented by a negative sign. The other is represented by positive number.

Step 6) If the 0 is the highest framed value is the member of the row and column then that row and column is removed from the relation of distances. Now one gets the new relation of a lower order.

Step 7) Follow the step 2, in order to obtain a relation having atleast one zero in each row and column.

Step 8) For calculating the value of vertex with a positive symbol, corresponding to the bi-partition described in step 5, one adds the sum of the amounts previously subtracted from rows and columns, to the previous vertex (i.e., root of directed tree).

Step 9) New bi-partition will be formed from the hanging vertex which processes the lowest value.

Step 10) The process is continued by following step 4, until the matrix of order 1×1 is not formed.

Example 38: Following table illustrates the cost to accomplish various jobs by different workers:

Worker	Job			
	1	2	3	4
A	90	18	48	50
B	72	28	85	80
C	53	92	12	78
D	20	70	70	25

Find the optimal assignment of jobs to workers.

Solution: There are 4 workers A, B, C and D. Each of them can do any of the jobs 1, 2, 3 and 4. To find the least cost assignment by solving the problem with the use of branch and bound technique is the main goal. This can be done by using the following steps:

Step 1: The possible solutions of this problem are 4! = 24. On the total cost of the assignment, a lower bound (total cost would not less than this cost) is obtained. The initial lower bound computation tries to give a floor value which cannot be greater than the cost of doing all the jobs. The fastest method to do this is to sum-up the minimum cost value in each of the given four columns as follows:

$$20 + 18 + 12 + 25 = 75$$

This shows D get the job 1, A get the job 2, C get the job 3 and D get the job 4. As it is shown that worker D is scheduled to get the two jobs so it cannot be the feasible solution.

Step 2: At this step, a partition is done of the process of looking for solutions. Go with the assignment of job 1 to each of the 4 worker in stepwise. Now take the least value from each of the left three columns without thinking about the feasibility of this assignment.

Start assignment of job 1 to worker A having the cost of ₹90. Now neglect the first row and first column for worker A and job 1 respectively. The value 28, 12, and 25 are the least values of second, third and fourth column respectively. In the same way, job 1 is assign to worker B and second row and first column are neglected. Now take the least values from each column as do above. The following table illustrates all this process:

Table 2.26: Assignment of Job 1 To Each Worker

Assignment	Lower Bound on the Total	Feasibility
1 – A 2 – B 3 – C 4 – D	90 + 28 + 12 + 25 = 155	Feasible
1 – B 2 – A 3 – C 4 – D	72 + 18 + 12 + 25 = 127	Feasible
1 – C 2 – A 3 – A 4 – D	53 + 18 + 48 + 25 = 144	Infeasible
1 – D 2 – A 3 – C 4 – A	20 + 18 + 12 + 50 = 100	Infeasible

The first two assignments are feasible as seen from the table. **Figure 2.6** shows the tree format of the outcomes of the two steps. In the **figure 2.6**, the 75 is marked because it is the lowest cost bound found in the step 1.

This marked value is followed by four branches that represent four solutions found in the step 2. In all these four solutions, the lowest value 100 is marked as shown in the **figure 2.6**.

When job 1 is assigned to worker B, the least cost is 127 which is less than the least cost 155 when job 1 is assigned to worker A. Hence this is the feasible solution. Since 155 is the lowest value so all other solutions which may be obtained by assigning job 1 to worker A can be logically neglected. In the same logic, when job 1 is assigned to worker C, the lowest cost is 144, so we neglect all other solutions which may derive from the branch following assigning job 1 to C.

Step 3: In this step we take least cost branch of the tree found when job 1 is assigned to worker D in step 2. This branch has total cost 100 and does not involve a feasible solution hence searching for other cost feasible solutions are continued.

To do this, assign the job 2 to the workers A, B and C in stepwise. Take the least value from each of the columns without considering first column and fourth row because job 1 is already assigned to worker D. **Table 2.27** illustrates the result of this process:

Table 2.27: Assignment of Job 1 to Worker D and Job 2 to Each Worker

Assignment	Lower-Bound on the Total	Feasibility
1 – D 2 – A 3 – C 4 – A	20 + 18 + 12 + 50 = 100	Infeasible
1 – D 2 – B 3 – C 4 – A	20 + 28 + 12 + 50 = 110	Feasible
1 – D 2 – C 3 – C 4 – A	20 + 92 + 12 + 50 = 174	Infeasible

Hence one feasible solution is produced by this operation that is, Job 1 to worker D, Job 2 to worker B, Job 3 to worker C and Job 4 to worker A. Its total cost is 100. The updated tree is shown in the **figure 2.7**. The value 100 is shaded which shows the least-cost solution.

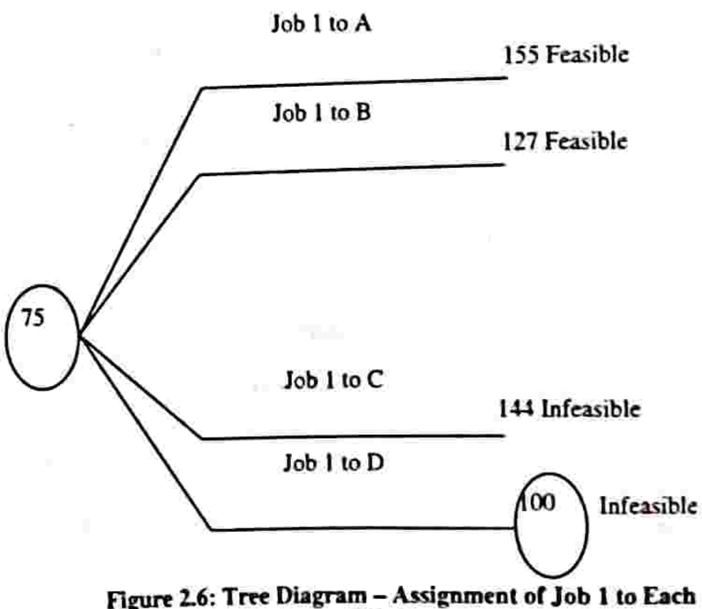


Figure 2.6: Tree Diagram – Assignment of Job 1 to Each Worker

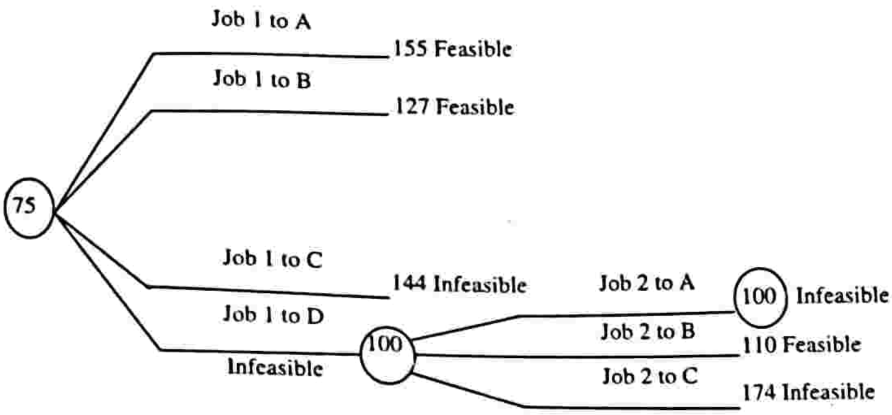


Figure 2.7: Tree Diagram – Assignment of Job 1 to D, Job 2 to Each Worker

The minimum cost of the possible solution when job 1 is assigned to worker D and job 2 is assigned to worker C is 174. This do not have to investigate further because lower cost solution finding from the branch illustrating the assignment of job 1 to D and job 2 to B consists the total cost of 110, that is feasible also.

Step 4: Continue with the branch that shows an infeasible solution with the assignment of job 1 to worker D and job 2 to worker A having the total cost 100 test if it is possible to get some feasible solution that is lower than 100. With the assignments of these two jobs there are two possibilities of assigning the job 3 and 4. Either assign the job 3 to worker B and job 4 to worker C or assign the job 3 to worker C and job 4 to worker B. The costs of each of the possibilities are 20 + 18 + 85 + 78 = 201 and 20 + 18 + 12 + 80 = 130 respectively.

Figure 2.8 shows the complete branch and bound tree.

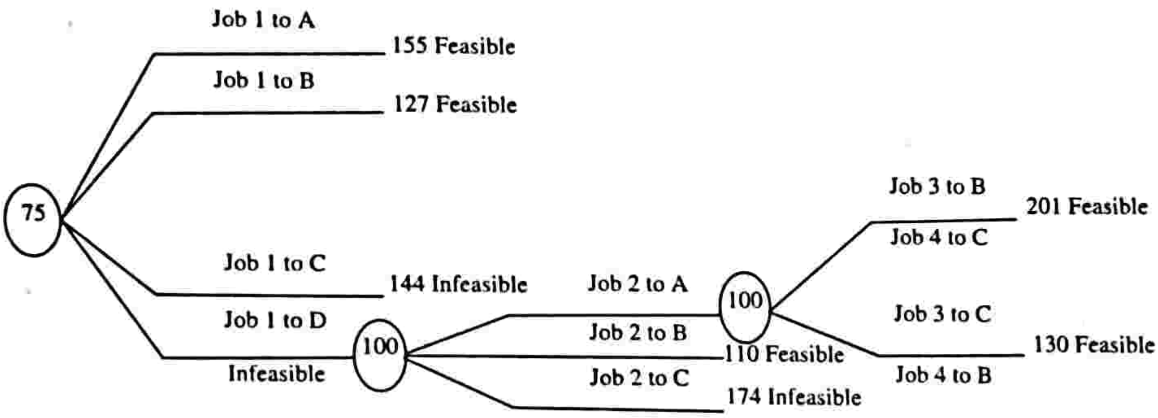


Figure 2.8: Branch and Bound Tree – Optimal Assignment

One can see that each of the solution is feasible but the cost related with the feasible solution which we found in step 3 is 110 which is less than cost of other feasible solutions. So we neglect all these solutions. Hence 110 is the total cost of the optimal solution for this assignment problem. Its solution is as follows:

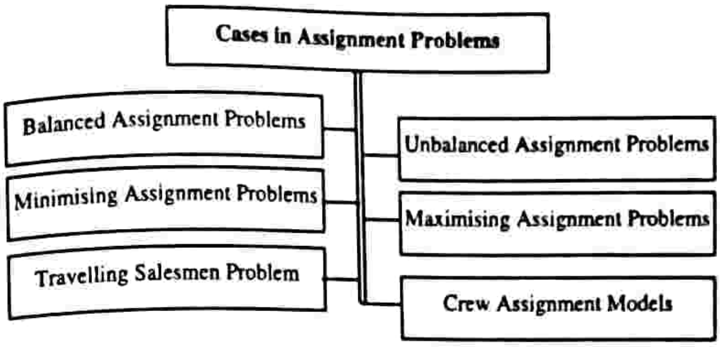
Table 2.28

Worker	Job	Cost
A	4	50
B	2	28
C	3	12
D	1	20
	Total	**110**

The optimal solution may be verified by using Hungarian Assignment method.

2.2.7. Cases in Assignment Problems

There are various basic categories for the assignment problems. These are as follows:



2.2.7.1. Balanced Assignment Problems

A problem with equal number of rows and columns is known as a balanced problem. **For example,** if there are 4 workers and 4 jobs in a problem, such a problem is termed as a balanced assignment problem.

Example 39: Solve the following assignment problem using Hungarian method:

	Job			
Typist	**A**	**B**	**C**	**D**
P	85	55	30	40
Q	90	40	70	45
R	70	60	60	50
S	75	40	35	55

Solution: Applying Hungarian Method, we have the following steps:

Step 1) Row Reduction: Select the lowest element of each row and subtract it from the other elements of that row as follows:

	A	B	C	D
P	55	25	0	10
Q	50	0	30	5
R	20	10	10	0
S	40	5	0	20

Step 2) Column Reduction: Select the lowest element of each column and subtract it from the other elements of that column as follows:

	A	B	C	D
P	35	25	0	10
Q	30	0	30	5
R	0	10	10	0
S	20	5	0	20

Step 3) Draw Lines: Now we have to draw minimum number of lines (horizontal and vertical) in such a manner that it covers all zeros.

	A	B	C	D
P	35	25	0	10
Q	30	0	30	5
R	0	10	10	0
S	20	5	0	20

Here the number of lines drawn (3) is less than the order of cost matrix (n = 4), hence this solution is not optimal. For obtaining the optimal solution, we have to subtract the minimum uncovered value of cost matrix from other uncovered elements of matrix, except intersection point of lines where the values are added.

	A	B	C	D
P	30	25	0	5
Q	25	0	30	0
R	0	15	15	0
S	15	5	0	15

Now, again lines are drawn in order to cover all zeros.

	A	B	C	D
P	30	25	0	5
Q	25	0	30	0
R	0	15	15	0
S	15	5	0	15

Since again the total number of lines that covering the all zeros (3) is not equal to order of the matrix (n=4), hence the solution is not optimal. Again deduct the minimum uncovered value from all other uncovered values, except from the values of point of intersection lines, where the values are added.

	A	B	C	D
P	25	20	0	0
Q	25	0	35	0
R	0	15	20	0
S	10	0	0	10

Now, again lines are drawn in order to cover all zeros.

	A	B	C	D
P	25	20	0	0
Q	25	0	35	0
R	0	15	20	0
S	10	0	0	10

Now, the numbers of line (4) is equal to order of matrix (n =4). Hence this provides optimal solution. The assignment will be as follows:

	A	B	C	D
P	25	20	~~0~~	[0]
Q	25	[0]	35	~~0~~
R	[0]	15	20	~~0~~
S	10	~~0~~	[0]	10

Step 4) Optimal Solution: The sequence of assignments will be as below:

Typist	Job	Cost
P	D	40
Q	B	40
R	A	70
S	C	35
Total		185

Example 40: Given the following data, determine the least cost allocation of the available workers to the five jobs.

	J_1	J_2	J_3	J_4	J_5
W_1	8	4	2	6	1
W_2	0	9	5	5	4
W_3	3	8	9	2	6
W_4	4	3	1	0	3
W_5	9	5	8	9	5

Solution: Applying the Hungarian method, the steps are as follows:

Step 1: Row Reduction: Finding the minimum values along each rows and subtract that to all elements of corresponding rows accordingly. The next table is as below:

Persons \ Jobs	J_1	J_2	J_3	J_4	J_5
W_1	7	3	1	5	0
W_2	0	9	5	5	4
W_3	1	6	7	0	4
W_4	4	3	1	0	3
W_5	4	0	3	4	0

Step 2: Column Reduction: Applying same method along columns. Then we have the table as below:

Persons \ Jobs	J_1	J_2	J_3	J_4	J_5
W_1	7	3	0	5	0
W_2	0	9	4	5	4
W_3	1	6	6	0	4
W_4	4	3	0	0	3
W_5	4	0	2	4	0

Step 3: Drawing Lines: Now drawing lines covering all zeros as below:

Persons \ Jobs	J_1	J_2	J_3	J_4	J_5
W_1	7	3	0	5	0
W_2	0	9	4	5	4
W_3	1	6	6	0	4
W_4	4	3	0	0	3
W_5	4	0	2	4	0

Since the number of lines (5) is equal to order of matrix (5). Hence solution is optimal.

Step 4: Optimal Solution: Assigning the zero □ in the row that has only one zero and × cross out other zero in the column. Same method is applying along column.

Persons \ Jobs	J_1	J_2	J_3	J_4	J_5
W_1	7	3	×	5	[0]
W_2	[0]	9	4	5	4
W_3	1	6	6	[0]	4
W_4	4	3	[0]	×	3
W_5	4	[0]	2	4	×

Hence, assignment has been made in the following sequence:

$W_1 \rightarrow J_5$

$W_2 \rightarrow J_1$

$W_3 \rightarrow J_4$

$W_4 \rightarrow J_3$

$W_5 \rightarrow J_2$

Hence minimum cost = 1 + 0 + 2 + 1 + 5 = 9

2.2.7.2. Unbalanced (Non Square Matrix) Assignment Problems

This is the case when number of jobs and number of facilities are not equal and it is termed as unbalanced assignment problem. This type of matrix is not square and since HAM needs a square matrix, so fictitious jobs or facilities are added to matrix and zero costs are assigned to matrix's corresponding cells. Then, these cells are treated like the real cost cells while solving the problem and are termed as **dummy rows columns.**

For example, assume a cost matrix of order 4×3, in such a case, a dummy column needs to be added with zero cost element to make it a square matrix, So that, Hungarian method can be applicable to solve such problems.

Example 41: A computer science faculty of a college decides to arrange special seminars on four modern topics in order to motivate students for academic discussion. These topics are Computer Network, Information System, Operating System and E-Commerce. Each seminar should be held once per week at 12:00 O' Clock. The scheduling of these seminars should be done in such a manner that number of students (not attending) is kept to a minimum. The number of students which are unable to attend a specific seminar of particular day is shown in **table below:**

	Computer Network (CN)	Information System (IS)	Operating System (OS)	E-Commerce (E-Comm)
Monday	50	40	50	20
Tuesday	40	30	40	30
Wednesday	60	20	30	20
Thursday	30	30	20	30
Friday	10	20	10	30

Find the optimal schedule of these seminars. Also, find the number of students who are unable to attend at least one seminar.

Solution: It is an unbalanced problem with five numbers of days and four seminar topics. For balancing this problem, we introduce a dummy column with zero values as follows:

	Computer Network (CN)	Information System (IS)	Operating System (OS)	E-Commerce (E-Comm)	Dummy (E)
Monday	50	40	50	20	0
Tuesday	40	30	40	30	0
Wednesday	60	20	30	20	0
Thursday	30	30	20	30	0
Friday	10	20	10	30	0

Now applying Hungarian methods, we have the following steps:

Step 1) Row Reduction: Select the lowest element of each row and subtract it from the other elements of that row. Since the lowest element in each row is '0' hence there is no change in each row.

Step 2) Column Reduction: Select the lowest element of each column and subtract it from the other elements of that column as follows:

	CN	IS	OS	E-Comm	E
Monday	40	20	40	0	0
Tuesday	30	10	30	10	0
Wednesday	50	0	20	0	0
Thursday	20	10	10	10	0
Friday	0	0	0	10	0

Step 3) Draw Lines: Now we have to draw minimum number of lines (horizontal and vertical) in such a manner that it covers all zeros.

	CN	IS	OS	E-Comm	E
Monday	40	20	40	0	0
Tuesday	30	10	30	10	0
Wednesday	50	0	20	0	0
Thursday	20	10	10	10	0
Friday	0	0	0	10	0

Since the number of lines (4) is less than the order of matrix (n=5), hence this solution is not optimal. For obtaining the optimal solution, we have to subtract the minimum uncovered value of matrix from other uncovered elements of matrix, except intersection point of lines where the values are added.

	CN	IS	OS	E-Comm	E
Monday	30	10	30	0	0
Tuesday	20	0	20	10	0
Wednesday	50	0	20	10	10
Thursday	10	0	0	10	0
Friday	0	0	0	20	10

Now, again lines are drawn in order to cover all zeros.

	CN	IS	OS	E-Comm	E
Monday	30	10	30	0	0
Tuesday	20	0	20	10	0
Wednesday	50	0	20	10	10
Thursday	10	0	0	10	0
Friday	0	0	0	20	10

Now, the numbers of line (5) is equal to order of matrix (n =5). Hence it is optimal solution. The assignment will be as follows:

	CN	IS	OS	E-Comm	E
Monday	30	10	30	[0]	X
Tuesday	20	X	20	10	[0]
Wednesday	50	[0]	20	10	10
Thursday	10	X	[0]	10	X
Friday	[0]	X	X	20	X

Step 4) Optimal Solution: The sequence of assignments will be as follows:

Day	Subject	No. of Students
Monday	E-Commerce(E-Comm)	20
Tuesday	No Seminar	0
Wednesday	Information System(IS)	20
Thursday	Operating System(OS)	20
Friday	Computer Network(CN)	10

Number of Student Missing at Least One Seminar $= 20 + 0 + 20 + 20 + 10 = 70$

Example 42: A company has one surplus truck in each of the cities, A, B, C, D, and E and one deficit truck in each of the cities 1, 2, 3, 4, 5, and 6. The distance between the cities in kilometre is shown in the matrix. Find the assignment of trucks from cities in surplus to cities in deficit so that the total distance covered by vehicles is minimum.

	1	2	3	4	5	6
A	12	10	15	22	18	8
B	10	18	25	15	16	12
C	11	10	3	8	5	9
D	6	14	10	13	13	12
E	8	12	11	7	13	10

Solution: As the situation involves a non-square matrix, it has to be modified to a square matrix by adding dummies. Add a dummy city with surplus vehicle. Since there is no distance associated with it, the corresponding cell values are made all zeros.

Table 2.29

	1	2	3	4	5	6
A	12	10	15	22	18	8
B	10	18	25	15	16	12
C	11	10	3	8	5	9
D	6	14	10	13	13	12
E	8	12	11	7	13	10
Dummy	0	0	0	0	0	0

Now applying the Hungarian method, the steps are as below:

Step 1) Row and Column Reduction: Subtract from each element of each row the lowest cost in that row. Similarly, subtract form each element of each column the lowest cost in that column. We have the following **table 2.30**:

Table 2.30

	1	2	3	4	5	6
A	4	2	7	14	10	0
B	0	8	15	5	6	2
C	8	7	0	5	2	6
D	0	8	4	7	7	6
E	1	5	4	0	6	3
Dummy	0	0	0	0	0	0

Step 2) Drawing Lines: Now one has to draw lines covering all zeros. The minimum number of lines crossing all zeros are given by **table 2.31**:

Table 2.31

	1	2	3	4	5	6
A	4	2	7	14	10	0
B	0	8	15	5	6	2
C	8	7	0	5	2	6
D	0	8	4	7	7	6
E	1	5	4	0	6	3
Dummy	0	0	0	0	0	0

As the minimum number of lines crossing all zeros is 5 not equal to order of matrix 6, hence it is not optimal solution.

Now subtract the least uncovered cell value 2 from each of the uncovered values and adding it to cross points of lines. We have following **table 2.32**:

Table 2.32

	1	2	3	4	5	6
A	6	2	7	14	10	0
B	0	6	13	3	4	0
C	10	7	0	5	2	6
D	0	6	2	5	5	4
E	3	5	4	0	6	3
Dummy	2	0	0	0	0	0

Again draw lines, the table is given **table 2.33**:

Table 2.33

	1	2	3	4	5	6
A	6	2	7	14	10	0
B	0	6	13	3	4	0
C	10	7	0	5	2	6
D	0	6	2	5	5	4
E	3	5	4	0	6	3
Dummy	2	0	0	0	0	0

As the minimum number of lines crossing all zeros is 5(<6), optimal assignment cannot be obtained.

Doing similar steps as shown above, the matrix is as following **table 2.34**:

Table 2.34

	1	2	3	4	5	6
A	6	0	5	12	8	0
B	0	4	11	1	2	0
C	12	7	0	5	2	8
D	0	4	0	3	3	4
E	5	5	4	0	6	5
Dummy	4	0	0	0	0	2

Step 3) Optimal Solution: Again drawing lines and allocating the zeros. The optimal solution is as follows **table 2.35**:

Table 2.35

	1	2	3	4	5	6
A	6	[0]	5	12	8	X
B	X	4	11	1	2	[0]
C	12	7	[0]	5	2	8
D	[0]	4	X	3	3	4
E	5	5	4	[0]	6	5
Dummy	4	X	X	X	[0]	2

The optimal assignment pattern is:
City A should supply the vehicle to city 2,
City B should supply the vehicle to city 6,
City C should supply the vehicle to city 3,
City D should supply the vehicle to city 1,
City E should supply the vehicle to city 4, and
Minimum distance travelled

$$= (10 + 12 + 3 + 6 + 7) \text{ km} = 38\text{km}.$$

No truck is supplied to city 5.

2.2.7.3. Minimising Assignment Problems

Cost, time and **distance** data is involved in a minimisation assignment problem. The main objective of this problem is to minimize the final objective function.

In minimization assignment problems, while a constant quantity is subtracted or added to every column or row in the given cost matrix, an assignment causing minimization of total cost in one of the matrix also leads to minimization of the total cost in other matrix. In such a case, an optimal solution is the one having zero total cost.

Example 43: Find the optimal assignment for the following cost matrix using Hungarian method:

Salesmen	Regions			
	R_1	R_2	R_3	R_4
S_1	35	27	28	37
S_2	28	34	30	40
S_3	35	24	32	33
S_4	24	32	25	82

Solution: After applying the Hungarian method, we have the following steps:

Step 1) Row Reduction: Select the lowest element of each row and subtract it from the other elements of that row as follows:

Salesman	Regions			
	R_1	R_2	R_3	R_4
S_1	8	0	1	10
S_2	0	6	2	12
S_3	11	0	8	9
S_4	0	8	1	58

Step 2) Column Reduction: Select the lowest element of each column and subtract it from the other elements of that column as follows:

Salesman	Regions			
	R_1	R_2	R_3	R_4
S_1	8	0	0	1
S_2	0	6	1	3
S_3	11	0	7	0
S_4	0	8	0	49

Step 3) Draw Lines: Now we have to draw minimum number of lines (horizontal and vertical) in such a manner that it covers all zeros.

Salesman	Regions			
	R_1	R_2	R_3	R_4
S_1	8	0	0	1
S_2	0	6	1	3
S_3	11	0	7	0
S_4	0	8	0	49

Since the number of lines (4) is equal to the order of matrix (n=4), hence solution is optimal. The assignment can be made by scanning all zeros. Assign zero with the symbol □ in the row that contains only one zero and cross out all other zeros of corresponding column. Same process will be done along column. The assignments are shown in **table below**:

Salesman	Regions			
	R_1	R_2	R_3	R_4
S_1	8	[0]	~~0~~	1
S_2	[0]	6	1	3
S_3	11	~~0~~	7	[0]
S_4	~~0~~	8	[0]	49

Step 4) Optimal Solution: The order of assignment will be as follows:

$S_1 \rightarrow R_2$

$S_2 \rightarrow R_1$

$S_3 \rightarrow R_4$

$S_4 \rightarrow R_3$

Hence Minimum Cost = 27 + 28 + 33 + 25
= ₹113.

Example 44: Time taken (in hours) by five employees in performing five jobs is given in the following matrix:

Jobs	Employees A	B	C	D	E
P	10	5	13	15	16
Q	3	9	18	13	6
R	10	7	2	2	2
S	7	11	9	7	12
T	7	9	10	4	12

Find the optimal allocation of job. Will the optimal allocation change if job R cannot be assigned to employee E? Show.

Solution: Applying **Hungarian** Method, we have the following steps:

Step 1) Row Reduction: Select the lowest element of each row and subtract it from the other elements of that row as follows:

Jobs	Employees A	B	C	D	E
P	5	0	8	10	11
Q	0	6	15	10	3
R	8	5	0	0	0
S	0	4	2	0	5
T	3	5	6	0	8

Step 2) Column Reduction: Select the lowest element of each column and subtract it from the other elements of that column as follows:

Jobs	Employees A	B	C	D	E
P	5	0	8	10	11
Q	0	6	15	10	3
R	8	5	0	0	0
S	0	4	2	0	5
T	3	5	6	0	8

Step 3) Draw Lines: Now we have to draw minimum number of lines (horizontal and vertical) in such a manner that it covers all zeros.

Jobs	Employees A	B	C	D	E
P	5	0	8	10	11
Q	0	6	15	10	3
R	8	5	0	0	0
S	0	4	2	0	5
T	3	5	6	0	8

Here the number of lines drawn (4) is less than the order of cost matrix (n = 5), hence this solution is not optimal. For obtaining the optimal solution, we have to subtract the minimum uncovered value of cost matrix from other uncovered elements of matrix, except intersection point of lines where the values are added.

Jobs	Employees A	B	C	D	E
P	7	0	8	12	11
Q	0	4	13	10	1
R	10	5	0	2	0
S	0	2	0	0	3
T	3	3	4	0	6

Now, again lines are drawn in order to cover all zeros.

Jobs	Employees A	B	C	D	E
P	7	0	8	12	11
Q	0	4	13	10	1
R	10	5	0	2	0
S	0	2	0	0	3
T	3	3	4	0	6

Now, the numbers of line (5) is equal to order of matrix (n =5). Hence this provides optimal solution. Starting with row 1, box a single zero, if any, and cross all other zeros in its column. Thus we get:

Employees

Jobs	A	B	C	D	E
P	7	[0]	8	12	11
Q	[0]	4	13	10	1
R	10	5	X	2	[0]
S	X	2	[0]	X	3
T	3	3	4	[0]	6

Step 4) Optimal Solution: The sequence of assignments will be as below:

Job	Employee	Time
P	B	5
Q	A	3
R	E	2
S	C	9
T	D	4
	Total	23 hours

If job R cannot be assigned to employee E then the optimal allocation will change as shown below:

Employees

Jobs	A	B	C	D	E
P	7	[0]	8	12	11
Q	[0]	4	13	10	1
R	10	5	[0]	2	X
S	X	2	X	X	3
T	3	3	4	[0]	6

The solution is not optimal since only four assignments are made.

2.2.7.4. Maximising Assignment Problems

There is the involvement of **sales, revenue** and **profit** data in a maximization assignment problem. The highest profit value is found from the original profit values.

All profit values are then subtracted from the highest profit value and the resulting matrix is **Regret matrix.** Before using the Hungarian method, maximisation problem needs to be changed to minimisation.

The transformation of maximization to minimization matrix is done through any of following two ways:

1) Highest element is selected from the entire assignment table and all other elements are subtracted from the highest element.
2) The signs of all values are changed in the table i.e. the matrix elements are multiplied by −1.

For example, the following matrices show the transformation of profit matrix (maximization) to cost matrix (minimization) using first method:

Profit Matrix

12	17	15
14	16	12
15	18	13

Subtract from 18 ----------→

Cost Matrix

6	1	3
4	2	6
3	0	5

Multiple Optimal Solutions

In the reduced assignment matrix, while an assignment is made, there are two or more ways to strike off a number of certain zeros. This is an indication of multiple optimal solutions for the problem with the same objective function value. Decision maker may be used to obtain more suitable solution in such cases.

Example 45: A company has 4 territories and four salesmen for assignment. The territories are not equally rich in their sales potential. It is estimated that a typical salesman operating in each territory would bring the following annual sales:

Territory	I	II	III	IV
Annual Sales in (₹)	60,000	50,000	40,000	30,000

The four salesmen are also considered to differ in their ability; it is estimated that working under same condition, their yearly sales would be proportionately as follows:

Salesman	A	B	C	D
Proportion	0.1	0.2	0.3	0.4

If the criteria is to maximise expected sales, what is your intuitive answer and verify your answer with Hungarian method.

Solution: Following table illustrates the maximum sales matrix:

Sales in ₹1,000 →		60	50	40	30
Sales Proportion ↓		I	II	III	IV
0.1	A	6	5	4	3
0.2	B	12	10	8	6
0.3	C	18	15	12	9
0.4	D	24	20	16	12

Applying Hungarian method, we have the following steps:

Step 1) Maximisation to Minimisation: It is a maximisation problem so first converts it to minimisation problem. For this take the largest value (here it is 24) and subtract each value of the table from this largest value.

The minimisation form of this problem is illustrated in the table below:

	I	II	III	IV
A	18	19	20	21
B	12	14	16	18
C	6	9	12	15
D	0	4	8	12

Step 2) Row Reduction: Now find the least values from each row and subtract it from all other values of the corresponding row.

	I	II	III	IV
A	0	1	2	3
B	0	2	4	6
C	0	3	6	9
D	0	4	8	12

Step 3) Column Reduction: Like the row reduction, do the same thing with column reduction. Find the least value from all the columns and subtract these values from all other values of the corresponding column.

	I	II	III	IV
A	0	0	0	0
B	0	1	2	3
C	0	2	4	6
D	0	3	6	9

Step 4) Draw Lines: Draw the minimum number of lines that covers all zeros. For this, apply the basic rule, i.e., select first those rows/columns that includes maximum number of zeros.

	I	II	III	IV
A	0	0	0	0
B	0	1	2	3
C	0	2	4	6
D	0	3	6	9

Here the order of the cost matrix is 4 which is greater than the lines covering all zeros (i.e. 2), so this assignment is not optimal. Now take the least value among the uncovered values. Subtract this value from all the remaining uncovered values and add it to those values which are present at the cross section points.

	I	II	III	IV
A	1	0	0	0
B	0	0	1	2
C	0	1	3	5
D	0	2	5	8

Again draw the lines covering all zeros.

	I	II	III	IV
A	1	0	0	0
B	0	0	1	2
C	0	1	3	5
D	0	2	5	8

Here the order of cost matrix (4) is greater than the number of lines covered all zeros (3), so this assignment is not optimal again. Again take the least value among the uncovered values. Subtract this least value from all the remaining uncovered values and add it to those values which are present at the cross section points.

	I	II	III	IV
A	2	0	0	0
B	1	0	1	2
C	0	0	2	4
D	0	1	4	7

Again draw the lines covering all zeros.

	I	II	III	IV
A	2	0	0	0
B	1	0	1	2
C	0	0	2	4
D	0	1	4	7

Here the order of cost matrix (4) is greater than the number of lines covered all zeros (3), so this assignment is not optimal still. Again take the least value among the uncovered values. Subtract this least value from all the remaining uncovered values and add to those values which are present at the cross section points.

	I	II	III	IV
A	3	1	0	0
B	1	0	0	1
C	0	0	1	3
D	0	1	3	6

Again we draw lines for covering all zeros.

	I	II	III	IV
A	3	1	0	0
B	1	0	0	1
C	0	0	1	3
D	0	1	3	6

At this time the order of cost matrix (4) is equal to the number of lines covering all zeros (4), so this provides optimal solution.

	I	II	III	IV
A	3	1	~~0~~	[0]
B	1	~~0~~	[0]	1
C	~~0~~	[0]	1	3
D	[0]	1	3	6

Step 4) Optimal Solution: Following is the optimal assignment:

Salesman	Territory	Sales (in '00 ₹)
A	IV	3000
B	III	8000
C	II	15000
D	I	24000
Total		50,000

Example 46: A company has a team of four salesman and there are four districts where the company wants to start its business. After taking into account the capabilities of salesman and the nature of districts, the company estimates that the profit per day in rupees for each salesman in each district is as below:

		District			
		1	2	3	4
Salesman	A	16	10	14	11
	B	14	11	15	15
	C	15	15	13	12
	D	13	12	14	15

Find the assignment of salesman to various districts which will yield maximum profit.

Ans: First converting the minimisation problem into a maximisation problem by subtracting all the elements from the highest element i.e., 16. The reduced matrix is:

Salesman	District			
	1	2	3	4
A	0	6	2	5
B	2	5	1	1
C	1	1	3	4
D	3	4	2	1

Applying Hungarian Method, the steps are as below:

Step 1) Row Reduction: Subtract from each element of each row to the lowest cost in that row.

Salesman	District			
	1	2	3	4
A	0	6	2	5
B	1	4	0	0
C	0	0	2	3
D	2	3	1	0

Step 2) Column Reduction: Subtract from each element of each column to the lowest cost in that column.

Salesman	District			
	1	2	3	4
A	0	6	2	5
B	1	4	0	0
C	0	0	2	3
D	2	3	1	0

Step 3) Draw Lines: Making the minimum number of lines covering all zeros. As a general rule, one should first cover those rows/columns which contain larger number of zeros.

Salesman	District			
	1	2	3	4
A	0	6	2	5
B	1	4	0	0
C	0	0	2	3
D	2	3	1	0

Step 4) Optimal Solution: Since, the numbers of lines are 4 which is equal to order of matrix. Hence it gives the optimal solution. So allocations are as below:

Salesman	District			
	1	2	3	4
A	[0]	6	2	5
B	1	4	[0]	X
C	X	[0]	2	3
D	2	3	1	[0]

Hence assignments are made in the following order.

A → 1

B → 3

C → 2

D → 4

Profit per day = [16 + 15 + 15 + 15] = ₹61

2.2.7.5. Multiple Assignment Problems

There can be more than one possible assignment combination for an assignment problem and this is termed as multiple optimal solutions.

But in such case, there is same optimal answer for all the possible combinations. Whenever in the final assignment problem table there are multiple zeroes in any rows or columns, it is visible that there are multiple solutions to the specific problem.

Example 47: There are five workers available for five different jobs. The working time (in hours) of each job performed by different persons is well-known from their previous records. It is shown in table below:

		Job				
		J_1	J_2	J_3	J_4	J_5
Worker	A	2	9	2	7	1
	B	6	8	7	6	1
	C	4	6	5	3	1
	D	4	2	7	3	1
	E	5	3	9	5	1

Find the assignment of workers to different jobs in order to minimise the working time. Also calculate the minimum time needs for the completion of jobs.

Solution: Applying Hungarian method, we have the following steps:

Step 1) Row Reduction: First, select the smallest element of each row and subtract it from the other elements of that row. We get following table:

		Job				
		J_1	J_2	J_3	J_4	J_5
Worker	A	1	8	1	6	0
	B	5	7	6	5	0
	C	3	5	4	2	0
	D	3	1	6	2	0
	E	4	2	8	4	0

Step 2) Column Reduction: Similar process is applied on column, that is, select the smallest element of each column and subtract it from the other elements of that column. We get the following table:

		Job				
		J_1	J_2	J_3	J_4	J_5
Worker	A	0	7	0	4	0
	B	4	6	5	3	0
	C	2	4	3	0	0
	D	2	0	5	0	0
	E	3	1	7	2	0

Step 3) Draw Lines: Now we have to draw minimum number of lines (horizontal and vertical) in such a manner that it covers all zeros.

Job

Worker	J_1	J_2	J_3	J_4	J_5
A	0	7	0	4	0
B	4	6	5	3	0
C	2	4	3	0	0
D	2	0	5	0	0
E	3	1	7	2	0

Since the number of lines drawn (4) is less than the order of the matrix (n = 5), hence the solution is not optimal. For obtaining the optimal solution, we have to subtract the minimum uncovered value (1) of matrix from other uncovered elements of matrix, except intersection point of lines where the value is added.

Job

Worker	J_1	J_2	J_3	J_4	J_5
A	0	7	0	4	1
B	3	5	4	2	0
C	2	4	3	0	1
D	2	0	5	0	1
E	2	0	6	1	0

Now, again lines are drawn in order to cover all zeros at least once. We get the following table:

Job

Worker	J_1	J_2	J_3	J_4	J_5
A	0	7	0	4	1
B	3	5	4	2	0
C	2	4	3	0	1
D	2	0	5	0	1
E	2	0	6	1	0

Now, the numbers of line (4) is again less than the order of matrix (n =5). Hence it is not optimal solution. Again we have to apply similar method as shown above. We get the following table:

Job

Worker	J_1	J_2	J_3	J_4	J_5
A	0	9	0	6	3
B	1	5	2	2	0
C	0	4	1	0	1
D	0	0	3	0	1
E	0	0	4	1	0

Again we have to draw minimum number of lines for determining optimal solution.

Job

Worker	J_1	J_2	J_3	J_4	J_5
A	0	9	0	6	3
B	1	5	2	2	0
C	0	4	1	0	1
D	0	0	3	0	1
E	0	0	4	1	0

Now, the numbers of line (5) is equal to order of matrix (n =5). Hence it is optimal solution. The assignment will be as follows:

Job

Worker	J_1	J_2	J_3	J_4	J_5
A	X	9	[0]	6	3
B	1	5	2	2	[0]
C	X	4	1	[0]	1
D	X	[0]	3	X	1
E	[0]	X	4	1	X

Optimal Assignment I

Job

Worker	J_1	J_2	J_3	J_4	J_5
A	X	9	[0]	6	3
B	1	5	2	2	[0]
C	X	4	1	[0]	1
D	[0]	X	3	X	1
E	X	[0]	4	1	X

Optimal Assignment II

Job

Worker	J_1	J_2	J_3	J_4	J_5
A	X	9	[0]	6	3
B	1	5	2	2	[0]
C	[0]	4	1	X	1
D	X	X	3	[0]	1
E	X	[0]	4	1	X

Optimal Assignment III

Step 4) Optimal Solution: There exists three optimal assignments as shown **table 2.36**:

Table 2.36: Optimal Assignments

	Optimal Assignment (I)		Optimal Assignment (II)		Optimal Assignment (III)	
Worker	**Job**	**Working Time**	**Job**	**Working Time**	**Job**	**Working Time**
A	J_3	2	J_3	2	J_3	2
B	J_5	1	J_5	1	J_5	1
C	J_4	3	J_4	3	J_1	4
D	J_2	2	J_1	4	J_4	3
E	J_1	5	J_2	3	J_2	3
Total		13		13		13

Hence from the table above, it is clear that the minimum time required for the completion of jobs is 13 hours.

Example 48: A marketing manager has 5 salesmen and 5 sales districts considering the capabilities of the salesman and the nature of districts the marketing manager estimates the sales per month (in ₹100) for each salesman in each district would be as follows:

Districts

Salesman	A	B	C	D	E
1	30	38	40	28	40
2	40	24	28	21	36
3	41	27	33	30	37
4	22	28	41	36	36
5	29	33	40	35	39

Find the assignment of salesman of districts that will result in maximum sales.

Solution: The given maximisation problem can be converted into a minimisation problem by subtracting from the largest element (i.e., 41) all the elements of the given table. The new cost data so obtained is given in **table**:

Salesman	Districts				
	A	**B**	**C**	**D**	**E**
1	11	3	1	13	1
2	1	17	13	20	5
3	0	14	8	11	4
4	19	13	0	5	5
5	12	8	1	6	2

Applying Hungarian Method, the steps are as below:

Step 1) Row Reduction: Subtract lowest element of each row from all the elements in that row.

Salesman	Districts				
	A	**B**	**C**	**D**	**E**
1	10	2	0	12	0
2	0	16	12	19	4
3	0	14	8	11	4
4	19	13	0	5	5
5	11	7	0	5	1

Step 2) Column Reduction: Subtract lowest element of each column from all the elements in that column.

Salesman	Districts				
	A	**B**	**C**	**D**	**E**
1	10	0	0	7	0
2	0	14	12	14	4
3	0	12	8	6	4
4	19	11	0	0	5
5	11	5	0	0	1

Step 3) Draw Lines: Drawing the minimum number of lines covering all zeros. As a general rule, one should first cover those rows/columns which contain larger number of zeros.

Salesman	Districts				
	A	**B**	**C**	**D**	**E**
1	10	0	0	7	0
2	0	14	12	14	4
3	0	12	8	6	4
4	19	11	0	0	5
5	11	5	0	0	1

Since, there are only four lines which cover all zeros and it is less than the order of the cost matrix (5), hence the current assignment is not optimal. Now we subtract the least uncovered value from all uncovered values and add it to intersection point of lines.

Salesman	Districts				
	A	**B**	**C**	**D**	**E**
1	11	0	1	8	0
2	0	13	12	14	3
3	0	11	8	6	3
4	19	10	0	0	4
5	11	4	0	0	0

Now, we draw the lines which cover all the zeros.

Salesman	District				
	A	**B**	**C**	**D**	**E**
1	11	0	1	8	0
2	0	13	12	14	3
3	0	11	8	6	3
4	19	10	0	0	4
5	11	4	0	0	0

Since, there are only four lines which cover all zeros and it is again less than the order of the cost matrix (5), hence current assignment is not still optimal. Now we subtract the least uncovered value from all uncovered values and add it to intersection point of lines.

Salesman	District				
	A	**B**	**C**	**D**	**E**
1	14	0	1	8	0
2	0	10	9	11	0
3	0	8	5	3	0
4	22	10	0	0	4
5	14	4	0	0	0

Now, we draw the line which covers all the zeros.

Salesman	District				
	A	**B**	**C**	**D**	**E**
1	14	0	1	8	0
2	0	10	9	11	0
3	0	8	5	3	0
4	22	10	0	0	4
5	14	4	0	0	0

Now, the numbers of lines are 5. Hence it gives the optimal solution. Two more alternative solutions exist due to presence of zero element in cells (4, C), (4, D) and cells (5, C), (5, D).

Salesman	District				
	A	**B**	**C**	**D**	**E**
1	14	[0]	1	8	X
2	[0]	10	9	11	X
3	X	8	5	3	[0]
4	22	10	[0]	X	4
5	14	4	X	[0]	X

Salesman	District				
	A	**B**	**C**	**D**	**E**
1	14	[0]	1	8	X
2	X	10	9	11	[0]
3	[0]	8	5	3	X
4	22	10	X	[0]	4
5	14	4	[0]	X	X

Step 4) Optimal Solution: Two optimal assignments are as follows:

	Assignment I		Assignment II	
Salesman	**District**	**Sales (in '00 ₹)**	**District**	**Sales (in '00 ₹)**
1	B	38	B	38
2	A	40	E	36
3	E	37	A	41
4	C	41	D	36
5	D	35	C	40
Total		**191**		**191**

2.2.7.6. Traveling Salesman Problem

In such a problem, a salesman travel in cities and returns back to the starting city by incurring the minimum cost and travelling across one city is allowed only once. The assumption made in traveling salesman problem is that all the cities are connected with each other. The distance between two cities is indicated by the cost incurred.

For example, assume how posts are delivered by the postman to the addressee. All the letters are arranged by him in an order and then only he starts from post office to deliver all the posts and comes back finally. If arrangement of posts in an order is not done, he may need to travel longer distance so as to clear all the posts to the addressee. Same way, a traveling salesman needs to plan his visits in a particular sequence. Let us consider his journey starts from head office to branch offices and then finally back to head office. During his travel, he would not visit the branch once visited and he will be back only after visiting all the branches.

Different types of traveling salesman's problems do exist. Some of these are as follows:

1) **Cyclic Problem:** This type of problem is solved by Assignment method or the Hungarian method. In this, a traveling salesman starts journey from the headquarters and visits all branches and finally comes back to the headquarters.

2) **Acyclic problem:** This is the second type of traveling salesman problem solved by the Dynamic programming method. In this type of traveling salesman problem, journey of salesman starts from the headquarters and he visits all intermediate branches and reaches the last branch finally and stays there. This type of problem is further classified into two types:
   i) **Symmetrical:** When the distance (or time or cost) between every cities pair is not dependent on the journey direction, it is said to be symmetrical acyclic problem.
   ii) **Asymmetrical:** When for one or more cities pair, the distance (or time or cost) is dependent on the journey direction i.e., it changes with the direction, it is said to be asymmetrical acyclic problem. **For example,** while salesman has to visit two cities only,

   A and B, there is no choice as such. But while he has to visit 3 cities, there are 2! Possible routes for the journey.

   While having 4 cities, he has 3! Possible routes for the travel. Generally, to visit n cities there are (n – 1)! Possible routes for travelling along.

There is much similarity in the travelling salesman problem and the assignment problem. The only point where these two varies is an additional restriction in case of travelling salesman problem that is x_{ij} is chosen such that there is no double visit to any city before completion of travel to all the cities.

Mathematical Formulation of Travelling Salesman Problem

Let us consider that C_{ij} be the time or cost or distance of going from city i to city j. The decision variable x_{ij} be 1 if the salesman travels from city i to city j and otherwise 0.

The objective is to minimise the traveling time.

$$Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = 1, i = 2..n.$$

$$\sum_{j=1}^{n} x_{ij} = 1, j = 2..n.$$

and subject to the additional constraint that x_{ij} is so chosen that, no city is visited twice before all the cities are completely visited.

In particular, going from i directly to j is not permitted. Which means $c_{ij} = \infty$, when $i = j$.

One cannot choose diagonal elements and thus can be avoided by filling the diagonal with infinitely large elements. This type of problem is similar to the assignment problem except that in the former case, there is an additional restriction; the x_{ij} is so chosen that no city is visited twice before the tour of all the cities is completed.

Example 49: Solve the travelling-salesman problem given in the matrix form as shown below:

		To			
		A	B	C	D
From	P	∞	46	16	40
	Q	41	∞	50	40
	R	82	32	∞	60
	S	40	40	36	∞

Solution: Using the Hungarian method, we get the following steps:

Step 1) Row Reduction: Choose the smallest element of each row and subtract it from the other elements of that row. We gt following table:

	A	B	C	D
P	∞	30	0	24
Q	1	∞	10	0
R	50	0	∞	28
S	4	4	0	∞

Step 2) Column Reduction: Similar method is applied on column, that is, select the smallest element of each column and subtract it from the other elements of that column. We get the following table:

	A	B	C	D
P	∞	30	0	24
Q	0	∞	10	0
R	49	0	∞	28
S	3	4	0	∞

Step 3) Draw Lines: Now we have to draw minimum number of lines (horizontal and vertical) in such a manner that it covers all zeros. The general rule is first cover those rows or columns that contain maximum number of zeros.

	A	B	C	D
P	∞	30	0	24
Q	0	∞	10	0
R	49	0	∞	28
S	3	4	0	∞

Since the number of lines drawn (3) is less than the order of the matrix (n = 4), hence the solution is not optimal. For obtaining the optimal solution, we have to subtract the minimum uncovered value (3) of matrix from other uncovered elements of matrix, except intersection point of lines where the values are added.

	A	B	C	D
P	∞	27	0	21
Q	0	∞	13	0
R	49	0	∞	28
S	0	1	0	∞

Again we have to draw minimum number of lines for determining optimal solution.

	A	B	C	D
P	∞	27	0	21
Q	0	∞	13	0
R	49	0	∞	28
S	0	1	0	∞

Now, the numbers of line (4) is equal to order of matrix (n =4). Hence it is optimal solution. Now the assignment will be as follows:

	A	B	C	D
P	∞	27	[0]	21
Q	X	∞	13	[0]
R	49	[0]	∞	28
S	[0]	1	X	∞

Step 4) Optimal Solution: Now the sequence of assignments will be as follows:

P → C, Q → D,
R → B, and S → A

Hence, minimum cost = 16 + 40 + 32 + 40 = ₹128.

Example 50: A company has four sales territories. These territories are assigned to four sales representatives of a company. The following table illustrates the monthly sales increase (in lac rupees) estimated for every representative for different territories:

	Sales Territory			
Sales Representatives	**1**	**2**	**3**	**4**
A	200	150	170	220
B	160	120	150	140
C	190	195	190	200
D	180	175	160	190

Find the optimal solution and the total maximum sale increase per month.

Solution: It is the maximisation problem. First convert it into minimisation problem. For this, take the largest value (220) and subtract each value of the table from this largest value. The minimisation form of this problem is illustrated in the table below:

Table 2.37: Opportunity Loss Matrix

	I	II	III	IV
A	20	70	50	0
B	60	100	70	80
C	30	25	30	20
D	40	45	60	30

Now applying the Hungarian method, we have the following steps:

Step 1) Row Reduction: Now find the least values from each row and subtract these least values from all other values of the corresponding row. **Table 2.38** shows the Reduced Cost Table (RCT 1).

Table 2.38: Reduced Cost Table(RCT) 1

	I	II	III	IV
A	20	70	50	0
B	0	40	10	20
C	10	5	10	0
D	10	15	30	0

Step 2) Column Reduction: Like the row reduction do the same thing with column reduction. Find the least values from all the columns and subtract these values from all other values of the corresponding column. **Table 2.39** shows the Reduced Cost Table (RCT 2).

Table 2.39: Reduced Cost Table 2

	I	II	III	IV
A	20	65	40	0
B	0	35	0	20
C	10	0	0	0
D	10	10	20	0

Step 3) Draw Lines: Draw the minimum number of lines that covers all zeros. For this apply the basic rule, i.e., select first those rows/columns that includes maximum number of zeros.

Table 2.40

	I	II	III	IV
A	20	65	40	0
B	0	35	0	20
C	10	0	0	0
D	10	10	20	0

Here the order of the cost matrix is 4 which is greater than the lines covering all zeros (i.e. 3), so this assignment is not optimal. Now take the least value among the uncovered values. Subtract this least value from all the remaining uncovered values and add it to those values which are present at the cross section points. This is illustrated in the **table 2.41**.

Table 2.41

	I	II	III	IV
A	10	55	30	0
B	0	35	0	30
C	10	0	0	10
D	0	0	10	0

Again lines are drawn for covering all zeros as shown in **table 2.42**.

Table 2.42

	I	II	III	IV
A	10	55	30	0
B	0	35	0	30
C	10	0	0	10
D	0	0	10	0

Now the order of cost matrix (4) is equal to the number of lines (4) covering all zeros. Hence the solution is optimal. Now the assignments are as follows:

Table 2.43

	I	II	III	IV
A	10	55	30	[0]
B	[0]	35	~~0~~	30
C	10	~~0~~	[0]	10
D	~~0~~	[0]	10	~~0~~

Optimal Solution: This problem contain two optimal solutions, both are shown below:

Alternative 1		
Salesman	**Territory**	**Sales**
A	IV	220
B	I	160
C	III	190
D	II	175
	Total	**745**

Alternative 2		
Salesman	**Territory**	**Sales**
A	IV	220
B	III	150
C	II	195
D	I	180
	Total	**745**

2.2.7.7. Crew Assignment Models

An optimal assignment of desired crew members to the each flight segment of a provided time interval as long as obeying the various work regulations and a group of agreements is the typical problem arises in airline crew management.

This problem is known is **Crew Assignment Problem (CAP)**. It is partitioned into two independent sub-problems:

1) Well-known Crew Pairing Problem, which is followed by,
2) Working Schedules Construction Problem

These sub-problems are modelled and solved sequentially. A group of legal minimum-cost pairings is created in the first sub-problem that covers all the planned flight segments. A working schedule that consists of pairings, rest interval, training interval, annual leaves, etc., is created in the second sub-problem. After that it is assigned to crew members.

Example 51: A time table of a Prachi Airlines is shown in **table below**. It works 7 days in a week. In this airline, a minimum 6 hour lay-over is provided to the crews between flights. Find the flights pairing that minimizes the lay-over time away from home.

Flight No.	**Chennai Depart**	**Mumbai Arrival**	**Flight No.**	**Mumbai Depart**	**Chennai Arrival**
1	7.00 A.M.	9.00 A.M.	101	9.00 A.M.	11.00 A.M.
2	9.00 A.M.	11.00 A.M.	102	10.00 A.M.	12.00 Noon
3	1.30 P.M.	3.30 P.M.	103	3.30 P.M.	5.30 P.M.
4	7.30 P.M.	9.30 P.M.	104	8.00 P.M.	10.00 P.M.

For every pair also mention the town where the crew should be based.

Solution: If Chennai is the location for the crew, then we have the following table:

	101	102	103	104
1	24	25	6.5	11
2	22	23	28.5	9
3	17.5	18.5	24	28.5
4	11.5	12.5	18	22.5

If Mumbai is the location for the crew, then we have the following table:

Minimum Lay-Over Table

	101	102	103	104
1	20	19	13.5	9
2	22	21	15.5	11
3	26.5	25.5	20	15.5
4	8.5	7.5	26	21.5

	101	102	103	104
1	20^M	19^M	6.5^C	9^M
2	22^{CM}	21^M	15.5^C	9^C
3	17.5^C	18.5^C	20^M	15.5^M
4	8.5^M	7.5^M	18^C	21.5^M

Now applying the Hungarian method to this problem, the steps are as follows:

Step 1) Row Reduction: Choose the smallest element of each row and subtract it from the other elements of that row.

	101	102	103	104
1	13.5	12.5	0	2.5
2	13	12	6.5	0
3	2	3	4.5	0
4	1	0	10.5	14

Step 2) Column Reduction: Choose the smallest element of each column and subtract it from the other elements of that column.

	101	102	103	104
1	12.5	12.5	0	2.5
2	12	12	6.5	0
3	1	3	4.5	0
4	0	0	10.5	14

Step 3) Draw Lines: Draw the minimum number of lines that covers all zeros. For this apply the basic rule, i.e., select first those rows/columns that includes maximum number of zeros.

	101	102	103	104
1	12.5	12.5	0	2.5
2	12	11	5.5	0
3	1	2	4.5	0
4	0	0	10.5	14

Here the order of the cost matrix is 4 which is not equal to the lines covering all zeros (i.e. 3), so this solution is not optimal. Find the smallest uncovered value and subtract it from all uncovered values as well as add it to all those values that are present at the cross-section points. We get the following table:

	101	102	103	104
1	12.5	12.5	0	3.5
2	11	11	5.5	0
3	0	2	3.5	0
4	0	0	10.5	15

Again minimum number of lines is drawn for covering all zeros.

	101	102	103	104
1	12.5	12.5	0	3.5
2	11	11	5.5	0
3	0	2	3.5	0
4	0	0	10.5	15

Here the order of the cost matrix is 4 which is equal to the lines covering all zeros (i.e. 4), so this solution is optimal. The optimal assignments are as follows:

	101	102	103	104
1	12.5	12.5	[0]	3.5
2	11	11	5.5	[0]
3	[0]	2	3.5	~~0~~
4	~~0~~	[0]	10.5	15

Step 4) Optimal Solution: The optimum assignment schedule is as follows:

Flight	Crew Base	Lay-Over Time
1 → 103	Chennai	6.5
2 → 104	Chennai	9
3 → 101	Chennai	17.5
4 → 102	Mumbai	7.5
	Total	40.5 hours

Example 52: An airline that operates 7 days a week has the time table shown below. Crew must have minimum layover of 5 hours between flights. Obtain the pairing of flights that minimises layover time away from home assuming that the crew can be based at either of the two cities. The crew will be based at the city that results in smaller layover.

Table 2.44

Delhi-Jaipur			Jaipur-Delhi		
Flight no.	**Depart**	**Arrive**	**Flight no.**	**Depart**	**Arrive**
1	7 AM	8 AM	101	8 AM	9.15 AM
2	8 AM	9 AM	102	8.30 AM	9.45 AM
3	1.30 AM	2.30 AM	103	12 Noon	1.15 PM
4	6.30 PM	7.30 PM	104	5.30 PM	6.45 PM

Solution: If all the crew start from Delhi, then layover time at Jaipur for various pairing of flights is given by **table 2.45.**

Table 2.45
Layover time at Jaipur

	101	102	103	104
1	24.0	24.5	28.0	9.5
2	23.0	23.5	27.0	8.5
3	17.5	18.0	21.5	27.0
4	12.5	13.0	16.5	22.0

Similarly, if all the crew is based at Jaipur, then layover time at Delhi for various pairing of flights is given by **table 2.46:**

Table 2.46
Layover time at Delhi

	101	102	103	104
1	21.75	21.25	17.75	12.25
2	22.75	22.25	18.75	13.25
3	28.25	27.75	24.25	18.75
4	9.25	8.75	5.25	23.75

As the crew can be either based at Delhi or Jaipur, they will obviously be based at the city that results in smaller layover time. We, therefore, pick-up the smaller layover time for all pairing of flights from **tables 2.45** and **2.46** and represent them in **table 2.47:**

Table 2.47

	101	102	103	104
1	21.75	21.25	17.75	9.5
2	22.75	22.25	18.75	8.5
3	17.50	18.0	21.50	18.75
4	9.25	8.75	5.25	22.0

The Hungarian method is now applied for finding the optimal pairings and successive tables obtained are given below:

Step 1) Row Reduction: Select the lowest element of each row and subtract it from the other elements of that row as follows:

Table 2.48

	101	102	103	104
1	12.25	11.75	8.25	0
2	14.25	13.75	10.25	0
3	0	0.50	4.00	1.25
4	4.00	3.50	0	16.75

Step 2) Column Reduction: Select the lowest element of each column and subtract it from the other elements of that column as follows:

Table 2.49

	101	102	103	104
1	12.25	11.25	8.25	0
2	14.25	13.25	10.25	0
3	0	0	4.00	1.25
4	4.00	3	0	16.75

Step 3) Draw Lines: Now we have to draw minimum number of lines in such a manner that it covers all zeros. For this one has to first cover those rows/columns that contain maximum zeros.

Table 2.50

	101	102	103	104
1	12.25	11.25	8.25	0
2	14.25	13.25	10.25	0
3	0	0	4	1.25
4	4	3	0	16.75

As the total number of lines drawn is 3 which are not equal to rows/columns (n = 4), so the solution is non-optimal. For obtaining the optimal solution, we have to subtract the minimum uncovered value of cost matrix from other uncovered elements of matrix, except intersection point of lines where the values are added.

Table 2.51

	101	102	103	104
1	4	3	0	0
2	6	5	2	0
3	0	0	4	9.5
4	4	3	0	25

Now, again lines are drawn in order to cover all zeros.

Table 2.52

	101	102	103	104
1	4	3	0	0
2	6	5	2	0
3	0	0	4	9.5
4	4	3	0	25

As the total number of lines drawn is 3 which are not equal to rows/columns (n = 4), so the solution is non-optimal. For obtaining the optimal solution, we have to subtract the minimum uncovered value of cost matrix from other uncovered elements of matrix, except intersection point of lines where the values are added.

Table 2.53

	101	102	103	104
1	1	0	0	0
2	3	2	2	0
3	0	0	7	12.5
4	1	0	0	25

Now, again lines are drawn in order to cover all zeros.

Table 2.54

	101	102	103	104
1	1	0	0	0
2	3	2	2	0
3	0	0	7	12.5
4	1	0	0	25

Now, the numbers of line (4) is equal to order of matrix (n =4). Hence this provides optimal solution.

Step 4) Optimal Solution: The assignment will be as follows:

Table 2.55

	101	102	103	104
1	1	0	[0]	X
2	3	2	2	[0]
3	[0]	X	7	12.5
4	1	[0]	X	25

The best pairing of flights that result in minimum total layover time is given in **table 2.56:**

Table 2.56

Flight Pair	Crew Based at City	Layover Time (Hours)
103-1	Jaipur	17.75
2-104	Delhi	8.50
3-101	Delhi	17.50
102-4	Jaipur	8.75
	Total	52.50

The problem has alternate optimal pairing of flights also.

Example 53: The Owner of a bus company is planning to provide accommodation for his crew. He has five buses which ply between Chennai and Coimbatore with three crew members in each return trip. The seating capacity in each bus is 50.

The crew can either stay in Chennai or in Coimbatore. Suggest an appropriate decision model for this case where the crew can have a home to reside or a temporary place to stay during a trip. Show an illustration with hypothetical data. Make and state the assumptions regarding the time schedules of the trips.

Solution: Let us assume that a trip from Chennai to Coimbatore takes six hours by bus. A typical time table of the bus service in both the direction is given in the **Table 2.57**. The constraint is that every crew should be provided with more than 4 hours of rest before the return trip again and should not wait for more than 24 hours for the return trip.

Table 2.57

Departure from Chennai	Route Number	Arrival at Coimbatore	Arrival at Chennai	Route Number	Departure from Coimbatore
06.00	1	12.00	11.30	a	05.30
07.30	2	13.30	15.00	b	09.00
11.30	3	17.30	21.00	c	15.00
19.00	4	01.00	00.30	d	18.30
00.30	5	06.30	06.00	e	00.00

The service time is constant for every line so that it does not involve directly in caculation. Let us consider that if every crew member boards at Channia and then the waiting times at Coimbatore for different routes are shown in **table 2.58**.

In case rounte 1 is joined with route a, the crew after arriving at at Coimbatore at 12 Noon start.at 5.30 next morning. Hence the wainting time is 17.5 hours. Route c leaves Coimbatore at 15.00 hours.

Thus the crew of route 1 arriving Coimbatore at 12 Noon are not able to take the minimum required rest of 4 hours if they are ready to leave by route c. Thus, the route 1-c is considered as infeasible assignment. Thus it M (a large positive number) cost is assigned.

Table 2.58

Route	a	b	c	d	e
1	17.5	21	M	6.5	12
2	16	19.5	M	5	10.5
3	12	15.5	21.5	M	6.5
4	4.5	8	4	17.5	23
5	23	M	8.5	12	17.5

Similarly, if the crews are boarding at Coimbatore then the waiting times of the crew in hours at Chennai for different route combinations are given below in **Table 2.59.**

Table 2.59

Route	a	b	c	d	e
1	18.5	15	9	5.5	M
2	20	16.5	10.5	7	M
3	M	20.5	14.5	11	5.5
4	7.5	M	22	18.5	13
5	13	9.5	M	M	18.5

Suppose, if the crew are asked to reside either at Chennai or at Coimbatore, minimum waiting time from the above can be calculated for different route combination by choosing the minimum of the two waiting times (shown in the **table 2.58** and **table 2.59**). This is given in the following **table 2.60.**

Table 2.60: Cost Matrix

Route	a	b	c	d	e
1	17.5*	15	9	5.5	12*
2	16*	16.5	10.5	5*	10.5*
3	12*	15.5*	14.5	11	5.5
4	4.5*	8*	14*	17.5*	13
5	13	9.5	8.5*	12*	17.5*

Note: The * marked denotes that the crew are based at Chennai.

Now we can solve the assignment problem (presented in **Table 4)** using Hungarian Method.

Step 1) Row Reduction: Select the lowest element of each row and subtract it from the other elements of that row as follows(**Table 2.61**):

Table 2.61

Route	a	b	c	d	e
1	12	9.5	3.5	0	6.5
2	11	11.5	5.5	0	5.5
3	6.5	10	9	5.5	0
4	0	3.5	9.5	13	8.5
5	4.5	1	0	3.5	9

Step 2) Column Reduction: Select the lowest element of each column and subtract it from the other elements of that column as follows(**Table 2.62**):

Table 2.62

Route	a	b	c	d	e
1	12	8.5	3.5	0	6.5
2	11	10.5	5.5	0	5.5
3	6.5	9	9	5.5	0
4	0	2.5	9.5	13	8.5
5	4.5	0	0	3.5	9

Step 3) Draw Lines: Now we have to draw minimum number of lines (horizontal and vertical) in such a manner that it covers all zeros **(Table 2.63):**

Table 2.63

Route	a	b	c	d	e
1	12	8.5	3.5	0	6.5
2	11	10.5	5.5	0	5.5
3	6.5	9	9	5.5	0
4	0	2.5	9.5	13	8.5
5	4.5	0	0	3.5	9

Step 4) Since the numbers of lines (4) are less than the order of matrix (5). Hence this is not optimal solution. For obtaining the optimal solution, we have to subtract the minimum uncovered value of cost matrix from other uncovered elements of matrix, except intersection point of lines where the values are added.

Table 2.64

Route	a	b	c	d	e
1	8.5	5	0	0	6.5
2	7.5	7	2	0	5.5
3	3	5.5	5.5	5.5	0
4	0	2.5	9.5	16.5	12
5	4.5	0	0	7	12.5

Now, again lines are drawn in order to cover all zeros.

Table 2.65

Route	a	b	c	d	e
1	8.5	5	0	0	6.5
2	7.5	7	2	0	5.5
3	3	5.5	5.5	5.5	0
4	0	2.5	9.5	16.5	12
5	4.5	0	0	7	12.5

Since the numbers of lines (5) are equal to the order of matrix (5). Hence this is optimal solution. The assignment can be made by scanning all zeros.

Assign zero with the symbol □ in the row that contains only one zero and cross out all other zeros of corresponding column. Same process will be done along column. The assignments are shown in **table 2.66**:

Table 2.66

Route	a	b	c	d	e
1	8.5	5	[0]	X	6.5
2	7.5	7	2	[0]	5.5
3	3	5.5	5.5	5.5	[0]
4	[0]	2.5	9.5	16.5	12
5	4.5	[0]	X	7	12.5

Step 5) Optimal Solution: The assignment will be as follows:

Table 2.67

Routes	Residence of the Crew	Waiting Time
1 – c	Coimbatore	9
2 – d	Chennai	5
3 – e	Coimbatore	5.5
4 – a	Chennai	4.5
5 – b	Coimbatore	9.5
	Total	33.5

2.2.8. Comparison between Transportation and Assignment Problem

Table 2.68 shows the comparison between Transportation and Assignment Problem:

Table 2.68: Comparison between Transportation and Assignment Problem

Transportation Problem	Assignment Problem
The matrix of such a problem may be square or rectangular one.	There must be a square matrix for such a problem, or if not square it must be converted to a square matrix.
Depending on the rim conditions, such problems may have any number of allocations of row and columns.	There need to be one to one allocation of rows and columns. Due to this reason, matrix must be a square.
The methods used to find the solution may be VAM or North-west corner method or the matrix minimum method.	The methods used to find the optimal solution may be Assignment algorithm or Hungarian method or Flood's technique.
MODI test or stepping stone test may be used for checking the optimality.	Drawing minimum number of vertical and horizontal lines to cover all zeros in the objects is used for checking the optimality.
The allocation for basic feasible solution must be (m + n – 1).	There must be at least one zero in every row and column. There must be assignment of one machine to one job and *vice versa*.
There may be any positive numbers in the rim as requirement.	For each row and each column, there is rim requirement to be always 1.
There is dealing with just one commodity that is moved from varying origins to varying destinations in a transportation problem.	In an assignment problem, representation of jobs or machines is through rows whereas a column represents machines or jobs.

2.3. EXERCISE

2.3.1. Short Answer Type Questions

1) What is transportation model?
2) Explain North-West Corner Method (NWCM) method of transportation problem?
3) What are advantages of North West Corner Method?
4) What are advantages of Least Cost Method?
5) What are advantages of Vogel's Approximation Method?
6) What is an unbalanced case in a transportation model?
7) Explain the steps in Vogel's approximation method.
8) Explain the branch and bound algorithm for assignment problems.
9) What are the advantages of assignment problem?
10) What are the limitations of assignment problem?
11) What is an unbalanced case in an assignment model?
12) Explain Hungarian method of assignment problems?

2.3.2. Long Answer Type Questions

1) Padma Iron and Steel Company (PISCO) has three open hearth furnaces and five rolling mill Transportation cost (₹Per quintal) for shipping steel from furnaces to rolling mills is shown in the following table:

		Mills					
		M_1	M_2	M_3	M_4	M_5	Capacities
	F_1	4	2	3	2	6	8
Furnaces	F_2	5	4	5	2	1	12
	F_3	6	5	4	7	7	14
Requirements		4	4	6	8	8	

Compute transportation cost (optimal).

[Ans: TC = 80]

2) Find a feasible solution to the following transportation problem using NWC rule:

		To			
		x	y	z	Demand
	P	18	12	3	150
From	Q	9	24	21	120
	R	12	12	6	180
Supply		60	285	105	

[Ans: TC = 6570]

3) Solve the following Transportation Problem by Vogel's method:

	D_1	D_2	D_3	D_4	Supply
W_1	10	12	15	8	130
W_2	14	11	9	10	150
W_3	20	5	7	18	170
Demand	90	10	140	120	

[Ans: TC = 3,640]

4) The transportation cost matrix for a given situation for supply of the commodity from sources A, B, C to points of usage P, Q, R is given below:

	P	Q	R	Supply
A	4	8	8	76
B	16	24	16	82
C	8	16	24	77
Demand	72	102	41	

i) Work-out the initial basic feasible solution by VAM.
ii) Does this problem have more than one optimal solution? If so, show all of them.

[Ans: There is only one optimal solution, Total Cost = ₹2,424**]**

5) Determine on IFBS using:
i) North-West Corner Rule
ii) VAM

	D_1	D_2	D_3	D_4	D_5	Supply
A	2	11	10	3	7	4
B	1	4	7	2	1	8
C	3	9	4	8	12	9
Demand	3	3	4	5	6	21

[Ans: i) North-West Corner Rule – $x_{11} = 3$, $x_{12} = 1$, $x_{22} = 2$, $x_{23} = 4$, $x_{24} = 2$, $x_{34} = 3$, $x_{35} = 6$; Cost = ₹153
ii) Vogel's Method – $x_{14} = 4$, $x_{22} = 2$, $x_{25} = 6$, $x_{24} = 3$, $x_{32} = 1$, $x_{33} = 4$, $x_{34} = 1$; Cost = ₹68**]**

6) Five Salesmen are to be assigned to five Territories based on the past performance; the following table shows the annual sales (in rupees of lac) that can be generated by each salesman in each territory. Find the optional assignment.

Salesmen / Territory	T_1	T_2	T_3	T_4	T_5
S_1	26	14	10	12	9
S_2	31	27	30	14	16
S_3	15	18	16	25	30
S_4	17	12	21	30	25
S_5	20	19	25	16	10

[Ans: $S_1 \rightarrow T_3$, $S_2 \rightarrow T_4$, $S_3 \rightarrow T_1$, $S_4 \rightarrow T_2$, $S_5 \rightarrow T_5$**]**

7) Solve the assignment problem for optimal solution using HAM.

Worker / Job	A	B	C	D
1	45	40	51	67
2	57	42	63	55
3	49	52	48	64
4	41	45	60	55

[Ans: 1→B, 2→D, 3→C, 4→A**]**

8) Solve the problem of assignment for the given table to maximise the sales:

Jobs / Machines	A	B	C	D	E
1	32	38	40	28	40
2	40	24	28	21	36
3	41	27	33	30	37
4	22	38	41	36	36
5	29	33	40	35	39

The five jobs are to be processed and five machines are available. Any machine can process any job with reducing profit (in rupees) is given above.

[Ans: 1 → B; 2 → A; 3 → E; 4 → C; 5 → D**]**

9) Solve the following assignment problem:

Table

	I	II	III	IV	V
1	11	17	8	16	20
2	9	7	12	6	15
3	13	16	15	12	16
4	21	24	17	28	26
5	14	10	12	11	15

[Ans: 1 – I, 2 – IV, 3 – V, 4 – III, 5 – II; $Z_{min} = 60$**]**

10) A team of 5 horses and 5 riders has entered a jumping show contest. The number of penalty points to be expected when each rider rides any horse is shown below:

Table: Rider

Horse	R_1	R_2	R_3	R_4	R_5
H_1	5	3	4	7	1
H_2	2	3	7	6	5
H_3	4	1	5	2	4
H_4	6	8	1	2	3
H_5	4	2	5	7	1

How should the horses be allotted to the riders so as to minimise the expected loss of the team?

[Ans: $H_1 - R_5$, $H_2 - R_1$, $H_3 - R_4$, $H_4 - R_3$, $H_5 - R_2$; $Z_{min} = 8$**]**

11) Five employees of a company are to be assigned to five jobs, which can be done by any of them. The workers get different wages per hour. These are ₹5 per hour for A, B and C each and ₹3 per hour for D and E each. The amount of time in hours taken by each employee to do a given job is given in the **table below.** Determine the assignment pattern that:
i) Minimises the total time taken,
ii) Minimises the total cost of getting the five jobs done.

Table: Employee

Job	A	B	C	D	E
1	7	9	3	3	2
2	6	1	6	6	5
3	3	4	9	10	7
4	1	5	2	2	4
5	6	6	9	4	2

[Ans:
i) 1 – C, 2 – B, 3 – A, 4 – D, 5 – E; 11 hours.
ii) 1 – D, 2 – B, 3 – A, 4 – C, 5 – E; E; ₹45**]**

12) Solve the following assignment problem for minimum optimal cost:

Table: To City

From City	1	2	3	4	5	6
A	12	10	15	22	18	8
B	10	18	25	15	16	12
C	11	10	3	8	5	9
D	6	14	10	13	13	12
E	8	12	11	7	13	10

[Ans: A – 2, B – 6, C – 3, D – 1, E – 4; 38**]**

13) A company has four machines on which three jobs have to be done. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table:

Machine

Job	P	Q	R	S
A	18	24	28	32
B	8	13	17	19
C	10	15	19	22

What are the job assignments which will minimise the cost?

[Ans: A → P, B → Q, C → R, D → S; Minimum Cost = 50**]**

Unit 3

Decision and Game Theories

3.1. DECISION THEORY

3.1.1. Introduction

A process is used to determine the best possible decision among the various available options is known as decision theory. Depending upon its degree of certainty, a decision maker is facilitated to evaluate and examine the best decision. The range of this degree is from completely certain (positive) to completely uncertain (negative) involving the mid-range risk factors.

The prime reason in the making of a new decision is either due to the dissatisfaction of an individual or an organisation regarding the existing conditions and also having possible options in improving the existing conditions. For a decision maker, the quantitative approach depends upon data, facts, information and logic.

This approach involves in solving problems objectively and scientifically by organising, defining, and analysing them in a systematic framework.

Decision theory can be considered as the procedure of logical and quantitative analysis of all those factors that affect the decision problems. It helps the decision maker to analyse the decision problems with several procedures of action and results.

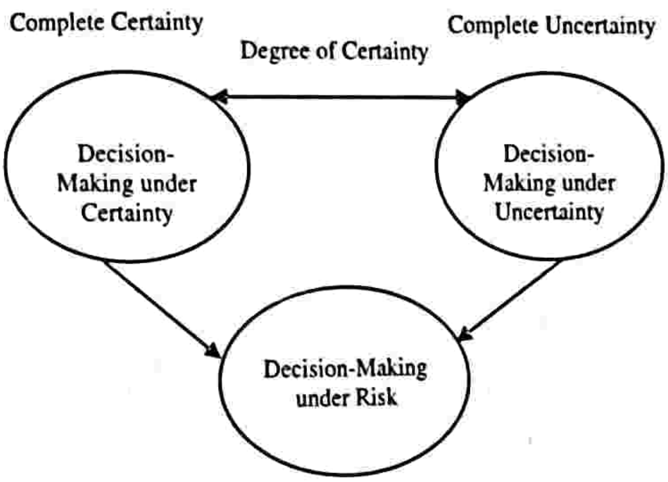


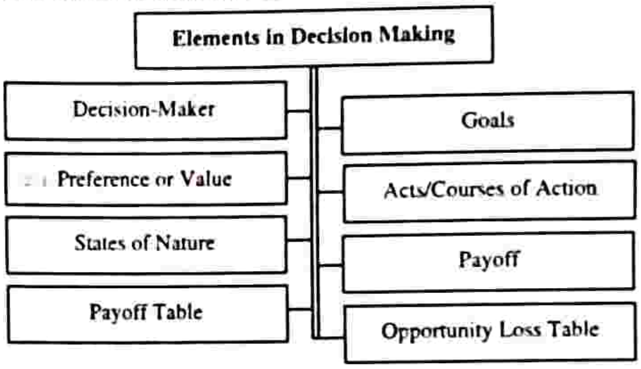
Figure 3.1: Decision-Making Situations

The classification of decision models is in accordance with the degree of certainty. The specific region falling between the two extremities corresponds to the decision making under the possibility of risk is shown in the **figure 3.1.**

According to Fishburn, "solving the decision model consists of finding a strategy for action, the expected related value of which is atleast as great as the expected relative value of any other strategy in a specified set. The perspective criteria of a strategy will be maximisation of decision maker's total expected relative value."

3.1.2. Elements in Decision Making

The different elements of decision making are as follows:



1) **Decision-Maker:** The decision maker is either an individual or a group of individuals, who are responsible to choose a suitable course of action from the available courses of action.

2) **Goals:** Certain objectives that a decision maker is willing to attain through his actions are known as goals.

3) **Preference on Value System:** This refers to the norms in deciding the best course of action by the decision maker that includes maximisation of income, utility, profit, etc.

4) **Acts/Courses of Action:** The courses of action or decision choices is also known as acts. Hence, for any difficult situation every possible plans of action must be included. Whether these possible courses of action are large or small, it should be emphasized that these must be under the control of the decision maker.

   For example, a decision maker decides what course of action is possible in anticipation of a strike. A course of action can have a decision to raise a stockpile that may consist of numerical description of stocking, 100 units of a particular item or a non-numerical description such as a decision to build up a stockpile.

5) **States of Nature:** The decision-maker must develop a complete list of possible future events before the application of decision theory. Though, there is no direct control of the decision-maker over the happening of a particular event. These future events

are known as states of nature and it is assumed to be mutually exclusive and collectively exhaustive. These states of nature can be numerically descriptive, as demand for some units of a particular item or a non-numerical description of an employees' strike.

6) **Payoff:** Payoffs are also known as **conditional values or profits**, and its effectiveness is linked with a particular combination of a plan of action and state of nature.

For example, for an action, a conditional profit of ₹10 is associated in stocking of 15 units of an item whereupon the demand of 12 units of that item is its outcome then the costs is handled as negative profit.

7) **Payoff Table:** A payoff table is a record of the states of nature (outcomes), that is mutually exclusive and collectively exhaustive with a set of likely courses of action. The payoff is calculated for each combination of nature and course of action.

The association of the weighted profit with the specified combination of state of nature and course of action is acquired by the multiplication of the payoff, for that specific state of nature and course of action with the probability of happening of the given outcome. The **table** shown below is an example of payoff table.

According to this table, $O_1, O_2......O_m$ represents 'm' states of nature, as regards to 'n' courses of action S_1, S_2 .... S_n. Thus, for a specified combination of state of nature and course of action, a_{ij} denotes the matching payoff.

States of Nature	Courses of Action			
	S_1	S_2	S_j	S_n
O_1	a_{11}	$a_{12}\cdots$	$a_{1j}\cdots$	$\cdots a_{1n}$
O_2	a_{21}	$a_{22}\cdots$	$\cdots a_{2j}\cdots$	$\cdots a_{2n}$
	⋮	⋮	⋮	⋮
O_i	A_{i1}	$A_{i2}\cdots$	$\cdots a_{ij}\cdots$	$\cdots a_{in}$
	⋮	⋮	⋮	⋮
O_m	a_{m1}	$A_{m2}\cdots$	$a_{mj}\cdots$	$\cdots a_{mn}$

Example 1: A shopkeeper sells a specific face cream for ₹120. He/she buy it for ₹100 per piece. As the new tax laws will be applied at every year, the pieces now become obsolete at the completion of a year and hence it may be disposed-off for ₹50 per piece. The yearly demand for this cream is in between 18 and 23, according to shopkeeper previous experience. The probabilities will be as follows:

Demand	18	19	20	21	22	23
Probability	0.05	0.10	0.30	0.40	0.10	0.05

Solution:

i) **Events:** Yearly demand ranges between 18 and 23 pieces.

D_1 = 18 pieces are demanded	D_4 = 21 pieces are demanded
D_2 = 19 pieces are demanded	D_5 = 22 pieces are demanded
D_3 = 20 pieces are demanded	D_6 = 23 pieces are demanded

ii) **Course of Action:** The available course of action will be in between 18 to 23 numbers of pieces because the demand cannot be less than 18 and greater than 23 pieces. Hence shopkeeper does not stock item (cream) less than 18 or greater than 23 pieces.

So there are six course of action as shown in table below:

A_1 = purchase 18 pieces	A_4 = purchase 21 pieces
A_2 = purchase 19 pieces	A_5 = purchase 22 pieces
A_3 = purchase 20 pieces	A_6 = purchase 23 pieces

iii) **Payoff Table:** The payoff table for this will be as follows:

Payoff Table

	A_1 : 18	A_2 : 19	A_3 : 20	A_4 : 21	A_5 : 22	A_6 : 23
D_1: 18	360	310	260	210	160	110
D_2: 19	360	380	330	280	230	180
D_3: 20	360	380	400	350	300	250
D_4: 21	360	380	400	420	370	320
D_5: 22	360	380	400	420	440	390
D_6: 23	360	380	400	420	440	460

For example, if the 22 pieces of cream are stocked by shopkeeper and 23 pieces of cream are demanded then the Profit =22 (pieces of cream stocked) × 20(profit per piece) = ₹440.

While if the 23 pieces of cream are stocked and 21 pieces of cream are demanded, then Profit = 21(pieces of cream sold-out) × 20(profit per piece) – 2 (piece disposed out) × 50 (loss per piece) = 420 – 100 = ₹320. **Table above** shows all combinations of pay-offs.

8) **Opportunity Loss Table:** The failure of not choosing the most positive course of action or approach is responsible for incurring an opportunity loss.

For each state of nature or outcome, opportunity loss values are calculated individually that is, by initially finding the most positive course of action for that particular outcome.

Then by stating the difference between that payoff value for that course of action and the payoff value for the best feasible course of action should be selected.

The regret table for above example is **shown below:**

Regret Table

	A_1 : 18	A_2 : 19	A_3 : 20	A_4 : 21	A_5 : 22	A_6 : 23
D_1: 18	0	50	100	150	200	250
D_2: 19	20	0	50	100	150	200
D_3: 20	40	20	0	50	100	150
D_4: 21	60	40	20	0	50	100
D_5: 22	80	60	40	20	0	50
D_6: 23	100	80	60	40	20	0

3.1.3. Steps in Decision Making

Figure 3.2 depicts the following steps that are involved in the decision theory approach:

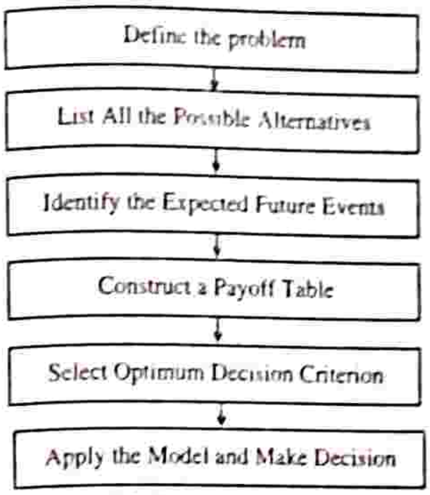


Figure 3.2: Steps of Decision Making

Step 1) Define the Problem: The first action has to be taken by the decision maker is to define the problem clearly. In decision making, it is a very crucial step as ambiguously defined problems will only produce poor results. It is well said "a problem well defined is half solved". Thus, problems which are not clearly defined, will hinder the decision maker to decide correctly.

Step 2) List All Possible Alternatives: For a problem that has been clearly defined its second step will be to make a list of those possible options that have been considered for its decision. **For example,** Following are the three options for the company:

i) Expansion of the existing plant.
ii) Construction of a new plant.
iii) Hiring production for further demand.

Step 3) Identify the Expected Future Events: Listing of all the possible occurrence of future events is the third step for the decision making. It is very much possible to identify the happening of the events, but the difficulty is in the identification of the occurrence of any particular event. In decision theory, those events which are not under the control of decision maker are known as states of nature. Definitely, only one event will occur from the given list.

For example, a manufacturing company having greatest uncertainty about the demand for the product. The future events related to the demand may be:

i) High demand.
ii) Moderate demand.
iii) Low demand.
iv) No demand.

Step 4) Construct a Payoff Table: The decision maker constructs a payoff table for each possible combination of alternative course of action and state of nature (table comprising of profit, benefit, etc.)

Step 5) Select Optimum Decision Criterion: After the construction of the payoff table, the decision maker has to decide the choice of the best criterion, which yields largest payoff. It may be economic, quantitative or qualitative. **For example,** market share, profit, fragrance of a perfume, etc.

Step 6) Apply the Model and Make Decision: The selected model is finally applied by the decision maker, which facilitates in decision-making.

Example 2: A firm manufactures three types of products. The fixed and variable costs are given below:

Product	Fixed Cost (₹)	Variable Cost Per Unit (₹)
A	25,000	12
B	35,000	9
C	53,000	7

The likely demand (units) of the products is given below:
Poor Demand: 3,000
Moderate Demand: 35,000
High Demand: 11,000

If the sale price of each type of product is ₹25, then prepare the payoff matrix.

Solution: Let D_1, D_2 and D_3 be the poor, moderate and high demand, respectively.

The payoff is then given by:

Payoff = Sales Revenue – Cost

The calculations for payoff (in '000₹) for each pair of alternative demand (course of action) and the types of product (state of nature) are shown below:

$D_1 A = 3 \times 25 - 25 - 3 \times 12 = 14$
$D_1 B = 3 \times 25 - 35 - 3 \times 9 = 13$
$D_1 C = 3 \times 25 - 53 - 3 \times 7 = 1$
$D_2 A = 35 \times 25 - 25 - 35 \times 12 = 430$
$D_2 B = 35 \times 25 - 35 - 35 \times 9 = 525$
$D_2 C = 35 \times 25 - 53 - 35 \times 7 = 577$
$D_3 A = 11 \times 25 - 25 - 11 \times 12 = 118$
$D_3 B = 11 \times 25 - 35 - 11 \times 9 = 141$
$D_3 C = 11 \times 25 - 53 - 11 \times 7 = 145$

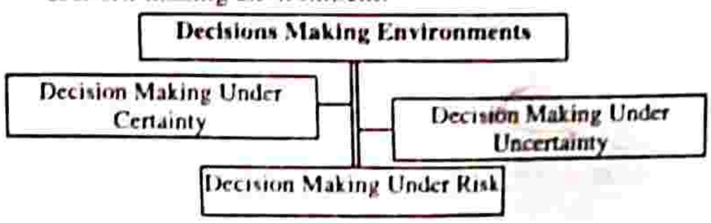
The payoff values are shown in **table 3.1:**

Table 3.1

Product Type	Alternative Demand (in '000₹)		
	D_1	D_2	D_3
A	14	430	118
B	13	525	141
C	1	577	145

3.1.4. Decisions Making Environments

Whenever a decision maker encounters several decision alternatives, decision theory is utilised to determine the best strategies. Following are mentioned three types of decision-making environment:



3.1.5. Decision-Making Under Certainty

There are certain problems wherein the decision maker is fully equipped with the information and is well aware of all the facts related to the states of nature. Besides, he has the knowledge of its occurrence as well. He is also aware of the consequences of that state of nature. The problem of decision making is simple in such situations. Thus, it is the selection of the strategy by the decision maker which will yield maximum payoff, with respect to utility under the state of nature whose occurrence is well known to him.

In this case, the easy selection of the course of action by the decision maker is a presumption, that only one state of nature is purposeful. To make decisions under certainty, is a situation when there is only one event and only one outcome for each action with each decision alternative is available.

For example, for a certain period, a person is willing to deposit Rs.10,000. The Unit Trust of India (UTI) offers 5.5% interest, Bank deposits giving 4.75% interest, and Government bond rate is 5% p.a. Despite all the investments being fully secured, investment in Unit Trust of India (UTI) is definitely the best choice.

For example, there is only one possible event for the two possible actions: "Do nothing" at a future cost of ₹3.00 per unit for 10,000 units, or "rearrange" a facility at a future cost of ₹2.80 for the same number of units. A decision matrix (or payoff table) would look as follows:

Actions	State of Nature(with probability of 1.0)
Do Nothing	₹30,000(10,000 units, ₹3.00)
Rearrange	₹28,000(10,000 units, ₹2.80)

It is to be noted down that there is only one State of Nature in the matrix because there is only one possible outcome for each action (with certainty). The decision is obviously to choose the action that will result in the most desirable outcome (least cost), that is to "rearrange".

For example, following are three pay-off measures for traveling by three different routes:

Routes	Time Saving (Hours)	Fuel Saving (Litres)	Enjoyment (Subjective Ratings)
A	4	3	1
B	3	7	3
C	0	0	10

If a person values time, he will go by route A and if he values saving fuel, he will go by route B and if he wants more enjoyment he might take to route C.

3.1.6. Decision Making Under Uncertainty

Situation which contains the imperfect and/or unknown information is known as **uncertainty**. Uncertainty is generally used in different ways in different fields like physics, statistics, economics, finance, psychology, sociology, engineering, metrology, philosophy and others.

Uncertainty can be used to predict the future events, to physical measurements which is observed or to the unknown. This situation arises generally due to indolence and/or ignorance as well as in partially observable and/or stochastic environments.

If there is incomplete information available for a decision environment, then the decisions taken under such an environment are known as **decision under uncertainty.** Then the information of the decision environment, in such a situation is unable to be explained in the form of probability distribution.

For example, let there is a game with two players and each one of them is having a set of alternatives to work, for each combination of the alternatives of the players, resulting in a combined outcome.

Both the players will decide their options simultaneously in enhancing the outcome of the game for their benefit. Hence, there is no possibility for describing any probability distribution to define the decision environment.

A decision under the uncertainty is when no objective information is available about the states of nature and their probabilities of occurrence. Therefore, due to unavailability of historical data and its relative frequency, that could have informed about the probability of happening of a particular state of nature.

For example, it is not known if Mr. Z will be the prime minister of the country after 15 years, this is improbable.

Process for making managerial decision is included in the following steps:

1) Recognizing such a situation that decides what action is to be taken.
2) Identify and develop the alternate course of action.
3) Evaluation of alternative.
4) Choosing one from alternatives.
5) Implementing the selected course of action.

Due to lack of knowledge, majority of the decisions that are taken by the managers are classified under the category of decisions under uncertainty, as what would be the outcome of alternate course of action.

The total range of possible outcomes is not possible under uncertainty and in real sense the probabilities of the possible outcome are unknown.

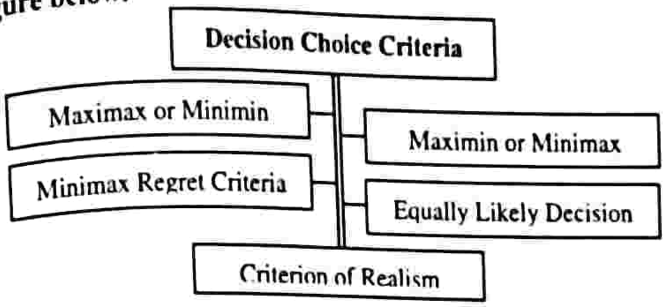
For example, while taking decisions related to new products, packaging, promotional activities, prices and channel of distribution, a marketing executive is challenged by extraordinary situations.

Some influential factors such as, product characteristics, price and the volume of sale effort can be structured by the manager, but they must compete with the uncertainties concerning factors like, economic conditions, taste of consumers and competitors etc. that are out of his control.

At the time of attempting a given course of action, how it will be affected by the various factors and how does a manager predicts will depend on his scarcity of information.

Decision Choice Criteria/ Quantitative Methods for Decision-Making under Uncertainty/ Approaches for Decision under Uncertainty

There are several rules and techniques to take decisions under uncertainty situation. Important ones are shown in figure below:



3.1.6.1. Maximax or Minimin (Criterion of Optimism)

According to this criterion, it is ensured by the decision maker to achieve maximum payoff or minimum cost without missing the opportunity. Thus, he chooses a different course of action represented by the maximum of the maxima or minimum of the minima payoffs. The summary of the working method is as follows:

1) Locating the payoff value (maximum or minimum) that corresponds to each and every course of action.
2) Selection of an alternative, having the best expected payoff value which maximise the profit and minimise the loss.

This is called an **optimistic decision criterion,** due to the selection of an alternative with highest/lowest available payoff value by the decision maker.

3.1.6.2. Maximin or Minimax (Criterion of Pessimism)

According to this criterion, it is ensured by the decision maker that his earnings are not less than the specified amount. Therefore his selection of alternative is represented by the maximum of the minima payoffs in case of profit, or minimum of the maxima in case of loss. Summary of the working method is as follows:

1) Locate the payoff values (both minimum in case of loss and maximum in case of profit) that correspond to each alternative.
2) Select an option having best expected payoff value (maximum for profit and minimum for loss or cost)

According to this criterion, the decision maker being conservative regarding future and he anticipates_worst possible outcome (minimum for profit or maximum for cost or loss). Hence, it is known to be pessimistic decision criterion. It is also called **Wald's criterion.**

3.1.6.3. Minimax Regret Criteria (Savage Principle)

This decision criterion was developed by **L.J. Savage.** According to this criterion he suggested, that due to the decision of the decision maker and on the occurrence of event i.e. states of nature, he may have the feeling of regret. Therefore, he should attempt to lessen the regret prior to the actual selection of a particular alternative.

The summary of the working method is as follows:

1) The amount of regret that is equivalent to each alternative for every state of nature is determined. The regret for j^{th} event that corresponds to the i^{th} alternative is given by the following formula:
   j^{th} regret = (maximum payoff – i^{th} payoff) for the i^{th} event.
2) Maximum regret amount for each alternative is to be determined.
3) Choosing the alternative that corresponds to the minimum of the above said maximum regrets.

3.1.6.4. Equally Likely Decision (Laplace Criterion)

On the assumption, that the occurrence of all states of nature having same probabilities as they have been assigned with equal probability is due to the reason that the probabilities of states of nature are unknown. Since these states of nature are mutually exclusive and collectively exhaustive therefore, each of these must have the probability equal to 1/(number of states of nature).

Following is the summary of the working method:

1) According to the formula: **1/number of states of nature,** assigns equal probability to each state of nature.
2) Expected (average) payoff for each alternative is to be computed either by adding all the payoffs and then dividing them by the number of possible states of nature or by the applying of the formula.
   (Probability of state of nature j) × (Payoff (P_{ij}) implies value for the combination of alternative i and state of nature j
3) Selection of the best expected payoff value (maximum for profit and minimum for loss/cost).

Generally the decision maker is being totally unaware regarding various states of nature and their probability of occurrence. Anyhow, this criterion does not have much practical utility.

3.1.6.5. Criterion of Realism (Hurwicz Criterion)

Due to the assumption of both the maximax and maximin criterion, the decision maker is neither optimistic nor pessimistic. According to the **Hurwicz Criterion,** the measure of the decision makers confidence as the decision payoffs are weighted by the coefficient of optimism defined as "α". It lies between 0 and 1 ($0 < \alpha < 1$):

1) If the value of $\alpha = 1.0$, the decision maker is totally optimistic;
2) If the value of $\alpha = 0$, the decision maker is totally pessimistic.

The Coefficient of Pessimism is **$(1 - \alpha)$.** This is an advantageous approach, as it allows the decision maker in building personal feelings regarding optimism and pessimism relatively.

The formula states:

H (Criterion of realism) = α (Maximum in column) + (1 –α) (Minimum in column)

The summary of the working method is as follows:

1) Decide the coefficient of optimism. α followed by coefficient of pessimism (1 – α).
2) After selecting the largest and lowest payoff value for each alternative, multiply it with the values of α and (1 – α) respectively. Calculation of weighted average, H is done by applying the above formula.
3) Selection of an option having best expected weighted average payoff value.

Example 3: A food products company is replacing its current product with a new product at much higher price (S_1) or a moderate change in the composition of the current product.

There is small increase in price (S_2) with new packaging or a small change in composition of current product (only the tag of the product is 'New') with approximately small increase in price (S_3).

There are three possible events or states of nature:

1) High increase in sales (N_1),
2) No change in sales (N_2) and
3) Decrease in sales (N_3).

The marketing department have the following payoffs in terms of annual net profits for the three states of nature or events:

Strategies	States of Nature		
	N_1	N_2	N_3
S_1	7,00,000	3,00,000	2,50,000
S_2	5,00,000	4,50,000	0
S_3	3,00,000	3,00,000	3,00,000

Find what strategy should be chosen by executive according to:

1) Maximin Criterion
2) Maximax Criterion
3) Minimax Regret Criterion
4) Laplace Criterion

Solution: The above payoff matrix can be written as below:

1) **Maximin Criterion:** In this criterion, that course of action is chosen which maximises the minimum payoffs as shown below:

Status of Nature	Strategies		
	S_1	S_2	S_3
N_1	7,00,000	5,00,000	3,00,000
N_2	3,00,000	4,50,000	3,00,000
N_3	2,50,000	0	3,00,000
Column Minimum	2,50,000	0	**3,00,000**

Hence, the company will adopt strategy S_3 because the maximum value of column minima is 3, 00,000.

2) **Maximax Criterion:** The maximum payoffs matrix in this criterion will be as follows, from where the strategy is to be selected:

Status of Nature	Strategies		
	S_1	S_2	S_3
N_1	7,00,000	5,00,000	3,00,000
N_2	3,00,000	4,50,000	3,00,000
N_3	2,50,000	0	3,00,000
Column Maximum	**7,00,000**	5,00,000	3,00,000

Hence, the company will select the strategy S_1 because the maximum of column maxima is 7, 00, 000.

3) **Minimax Regret Criterion:** In this criterion, we get the opportunity loss table as follows:

Status of Nature	Strategies		
	S_1	S_2	S_3
N_1	7,00,000 – 7,00,000 = 0	7,00,000– 5,00,000 = 2,00,000	7,00,000 – 3,00,000 = 4,00,000
N_2	4,50,000 – 3,00,000= 1,50,000	4,50,000 – 4,50,000 = 0	4,50,000 – 3,00,000 = 1,50,000
N_3	3,00,000 – 2,50,000 = 50,000	3,00,000 – 0 =3,00,000	3,00,000 – 3,00,000 = 0
Maximum Opportunity Loss	**1,50,000**	3,00,000	4,00,000

Hence the company will adopt strategy S_1 because the minimum opportunity loss is 1,50,000.

4) **Laplace Criterion:** Assume that the probabilities of states of nature are equal, hence each state of nature has 1/3 probability of occurrence. Thus, we get the following payoffs matrix:

Strategy	Expected Return (₹)	
S_1	1/3 (7,00,000 + 3,00,000 + 2,50,000)	= 416666.66
S_2	1/3 (5,00,000 + 4,50,000 + 0)	= 3,16,666.66
S_3	1/3 (3,00,000 + 3,00,000 + 3,00,000)	= 3,00,000

Hence, the marketing executive will choose the strategy S_1 because the maximum expected return is 416666.66 for this strategy.

Example 4: A Tata steel company is worried about the possibility of a strike. In order to acquire a suitable stockpile, it will cost ₹ 30,000 extra. The company management calculates the extra expense of ₹ 70,000, in case there is a strike and company has not stockpiled. Determine whether company stockpile or not in case it is using:

1) Minimin criterion
2) Minimax criterion
3) Minimax Regret criterion
4) Hurwicz criterion for α = 0.4
5) Laplace criterion

Solution: From the above data, the conditional cost table can be constructed as follows:

Conditional Cost Table (₹)

States of Nature	Alternatives	
	Stockpile (A_1)	Do not Stockpile (A_2)
Strike (S_1)	30,000	70,000
No Strike (S_2)	30,000	0

Thus we have,

1) **Minimin Criterion:** Here we will use minimin criterion as table shows the costs. The minimum cost is ₹ 30,000 for A_1 and ₹0 for A_2 respectively. Hence, the company will adopt the alternative A_2 which means that it is not stockpile and related cost is ₹0.

2) **Minimax Criterion:** Here, we will use the minimax criterion as the table again represents costs. The maximum value is ₹30,000 for alternative A_1 and ₹70,000 for alternative A_2. The company will adopt alternative A_1 which means that it is stockpile and related cost is ₹ 30,000.

3) **Minimax Regret Criterion:** First we have to make the conditional regret table. The regret for the Strike (S_1) will be the cost subtracted from the minimum cost of ₹30,000 and for No-Strike (S_2) will be the cost subtracted from the minimum cost of ₹0. The conditional regret table can be constructed as follows:

Conditional Regret Table (₹)

States of Nature	Alternatives	
	Stockpile (A_1)	Do not Stockpile (A_2)
Strike (S_1)	0	40,000
No Strike(S_2)	30,000	0

The maximum regret is ₹ 30,000 and ₹ 40,000 for alternative A_1 and A_2 respectively. Hence, company will select alternative A_1 having minimax regret cost is ₹30,000.

4) **Hurwicz Criterion (Weighted Average Criterion):** When $\alpha = 0.4$, we have:
   i) Cost related to alternative $A_1 = 30,000 \times 0.4 + 30,000 \times 0.6 =$ ₹ 30,000
   ii) Cost related to alternative $A_2 = 70,000 \times 0.4 + 0 \times 0.6 =$ ₹ 28,000

   Hence, the company will adopt alternative A_2 and the related cost is ₹ 28,000.

5) **Laplace Criterion (Equal Probability Criterion):** Assume equal probability cost for both alternatives. In such case, we have:

$$A_1 = ₹\frac{1}{2}(30,000 + 30,000) = ₹30,000,$$

$$A_2 = ₹\frac{1}{2}(70,000 + 0) = ₹35,000.$$

   Hence the company will adopt the alternative A_1 and related cost is ₹30,000.

Example 5: Consider the following data in daily not profit, find the best order size based on the:

i) Maximum criterion
ii) Savage minimax regret criterion and
iii) Hurwicz criterion Take ($\alpha = 0.4$)

		Demand				
		50	100	150	200	250
Order Size	75	950	1200	575	–675	–1425
	150	50	1700	2000	2250	1600
	225	–850	850	2550	3550	4525
	300	–1800	600	1800	2000	5000

Solution:

i) **Maximin Criterion:** The Daily Net Profit can be determined as illustrated in **table below:**

		Demand					
		50	100	150	200	250	Minimum
Order Size	75	950	1200	575	–675	– 1425	– 1425
	150	50	1700	2000	2250	1600	50*
	225	–850	850	2550	3550	4525	–850
	300	–1800	600	1800	2000	5000	–1800

In table above, the maximin value is for row 2. Hence, the action corresponding to row 2 is the best action. This shows that the optimal daily order size of the perishable item should be 150 kg.

ii) **Savage Minimax Regret Criterion: Table below** shows the regret values:

Order Size	Demand 50	100	150	200	250	Maximum
75	950 – 950 = 0	1700 – 1200 = 500	2550 – 575 = 1975	3550 – 675 = 2875	5000 – 1425 = 3575	3575
150	950 – 50 = 900	1700 –1700 = 0	2550 – 2000 =550	3550 – 2250 =1300	5000 – 1600 = 3400	3400
225	950 + 850 = 1800	1700 – 850 = 850	2550 – 2550 = 0	3550 – 3550 = 0	5000 – 4525 = 475	**1800***
300	950 + 1800 = 2750	1700 – 600 = 1100	2550 – 1800 = 750	3550 – 2000 = 1550	5000 – 5000 = 0	2750

From **above table**, it is shown that row 3 has the minimax value. Thus action taken under third row is the best action. This shows that optimal daily order size of the perishable item should be 225 kg.

iii) **Hurwicz Criterion:** When $\alpha = 0.4$, we get the following table:

Order Size	Demand 50	100	150	200	250	Maximum (i)	Minimum (ii)	Weighted Outcome $h = \alpha x\,(i) + (1-\alpha)\,(ii)$
75	950	1200	575	–675	– 1425	1200	– 1425	–375
150	50	1700	2000	2250	1600	2250	50	930
225	–850	850	2550	3550	4525	4525	–850	**1300***
300	–1800	600	1800	2000	5000	5000	–1800	920

From above table, it is shown that third row has the maximum value. The action taken under row third is the best action. This shows that the optimal daily order size of the perishable item should be 225 kg.

Example 6: A steel manufacturing company is concerned with the possibility of a strike. It will cost an extra ₹20,000 to acquire an adequate stockpile. If there is a strike and the company has not stockpiled, management estimates an additional expense of ₹60,000 on account of lost sales should the company stockpile or not if it is to use.

1) Optimistic criterion
2) Savage criterion
3) Hurwicz criterion for a = 0.4
4) Laplace criterion.

Solution: The Conditional cost table of the above problem is shown in table below:

Conditional Cost Table (in ₹)

States of Nature	Alternatives: Stockpile, A_1	Do not Stockpile, A_2
Strike, S_1	20,000	60,000
No strike, S_2	20,000	0

1) **Optimistic (Minimin) Criterion:** The above table shows costs; hence the minimin criterion will be applied. The minimum value of alternatives A_1 is ₹ 20,000 and the minimum value for the alternative A_2 is ₹ 0. Thus, company chooses alternative A_2 because it should not stockpile and associated cost is ₹ 0.

2) **Savage (Minimax Regret) Criterion:** First we make a conditional regret table as shown below:

Conditional Regret Table (in ₹)

States of Nature	Alternatives: Stockpile, A_1	Do not Stockpile, A_2
Strike, S_1	0	40,000
No strike, S_2	20,000	0

For states of nature S_1, the regret will be cost minus the minimum cost of ₹ 20,000; for states of nature S_2, it will be cost minus the minimum cost of ₹0.

The maximum value for A_1 is ₹ 20,000 and for A_2 is ₹ 40,000. Hence, company must select A_1 with minimax regret of ₹ 20,000.

3) **Hurwicz Criterion (Weighted Average Criterion):** For the value $\alpha = 0.4$, the cost associative with alternative A_1 = ₹ (20,000 × 0.4 + 20,000 × 0.6) = ₹ 20,000 and

Cost associated with alternative A_2 = ₹ (60,000 × 0.4 + 0 × 0.6) = ₹ 24,000.

So, the company should stockpile and associated cost is ₹ 20,000.

4) **Laplace Criterion (Equal Probability Criterion):** Equal probability cost for alternatives A_1 and A_2 can be calculated as follows:

$$A_1 = ₹\frac{1}{2}(20{,}000 + 20{,}000) = ₹\,20{,}000$$

$$A_2 = ₹\frac{1}{2}(60{,}000 + 0) = ₹\,30{,}000.$$

Hence, the company should stockpile and associated cost is ₹ 20,000.

Example 7: Mr. Senthil has ₹10,000 invest in one of the three options. A, B or C. The return on his investment depends on whether the economy experiences inflation, recession or no change at all. His possible returns under each economic condition are given below:

Strategy	State of Nature: Inflation	Recession	No change
A	2000	1200	1500
B	3000	800	1000
C	2500	1000	1800

What should he decide using the:

i) Maximax Criterion,
ii) Maximin Criterion,
iii) Regret Criterion,
iv) Hurwicz Criterion ($\alpha = 0.5$), and
v) Laplace Criterion?

Solution:

i) **Maximax Criterion:** The maximum payoffs matrix in this criterion will be as follows.

Strategy	Inflation	Recession	No change	Maximum Value
A	2,000	1,200	1,500	2,000
B	3,000	800	1,000	**3,000**
C	2,500	1,000	1,800	2,500

Maximum among the maximum pay-offs is 3,000. Hence strategy B is to be selected.

ii) **Maximin Criterion:** In this criterion, that course of action is chosen which maximises the minimum payoffs as shown below:

	Inflation	Recession	No change	Minimum Value
A	2,000	1,200	1,500	**1,200**
B	3,000	800	1,000	800
C	2,500	1,000	1,800	1,000

Maximum among the minimum pay-offs is 1,200. Thus, the corresponding option A is to be selected.

iii) **Regret Criterion:** First we have to make the conditional regret table:

	Inflation	Recession	No change	Regret Pay-Off			Maximum Regret
				Inflation	Recession	No change	
A	2,000	1,200	1,500	1,000	0	300	1,000
B	3,000	800	1,000	0	400	800	800
C	2,500	1,000	1,800	500	200	0	**500**
Max	3,000	1,200	1,800				

Among the maximum regrets, we find that the minimum value is 500. Hence the alternative C is to be selected.

iv) **Hurwicz Criterion (Weighted Average Criterion):** The Hurwicz criterion is given in the following table for $\alpha = 0.5$.

	A	B	C	Max (i)	Mini (ii)	$H = \alpha \times (i) + (1 - \alpha) \times (ii)$
Inflation	2000	3000	2500	3000	2000	**2500**
Recession	1200	800	1000	1200	800	1000
No Change	1500	1000	1800	1800	1000	1400

As the maximum value of H is 2500 which correspond to inflation, thus by Hurwicz Criterion, the recommended decision is inflation.

v) **Laplace Criterion:** The corresponding table is shown below:

	Inflation	Recession	No Change	Expected Pay-Off
A	2,000	1,200	1,500	1/3(2,000 + 1,200 + 1,500) = 1566
B	3,000	800	1,000	1/3(3,000 + 800 + 1,000) = 1600
C	2,500	1,000	1,800	1/3(2,500 + 1,000 + 1,800) = **1766**

Alternative C has the maximum expected pay-off. Thus alternative C is to be selected.

3.1.7. Decision Making Under Risk

Risk may be defined as the potential of losing something of value. Values can be anything such as physical health, social status, emotional well-being or financial wealth. Whenever someone takes risks resulting from a given action or activity then values can be gained or lost. In other words, risks can also be defined as the planned interaction with uncertainty.

Uncertainty is the unpredictable, uncontrollable and unmeasurable result. The risk is consequence of result taken in spite of uncertainty.

If there is no knowledge of the state of nature and assigning of probabilities to various state of nature is on the basis of objective or factual evidence then the decisions taken under such environment are known as decision making under risk.

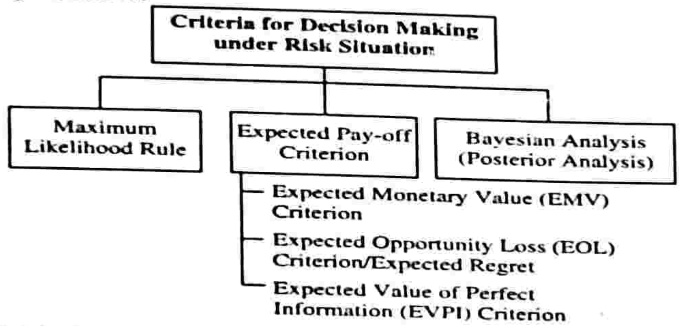
Problems that are linked with inventory level or spare parts etc. Assigning of the probabilities to different states of nature is possible on the basis of historical data and past experience. Thus, in these cases the payoff matrix helps in arriving at an excellent decision by assigning probabilities to various states of nature. Various states of nature can be counted under this condition as well as the long-run frequency of them is supposed to be known. Decision maker cannot predict about the occurrence of outcome due to the result of his selecting a particular course of action as the information regarding the states of nature is probabilistic.

Normally it is difficult to calculate the exact monetary payoffs or outcomes for several combinations of such courses of action and states of nature because each course of action results in more than one outcome.

The decision maker has often been enabled to assign the probability values to the probable occurrence of each state of nature in accordance to the past records or previous experiences. The best decision will be in selecting that course of action which having the largest expected pay-off value after knowing about the probability distribution of the states of nature.

Criteria for Decision Making under Risk Situation

The important ones, out of several rules and techniques for one stage decision under the situation of risk are shown in **figure below:**



3.1.7.1. Maximum Likelihood Rule

According to this criterion the decision maker selects that state of nature which have the highest probability of occurrence and chooses the decision alternative yielding the highest payoff in that state of nature.

For example, if the probability distribution of demand is as follows:

Demand (Units)	0	1	2	3	4	5	6
Probability	0.1	0.2	0.3	0.05	0.05	0.2	0.1

The most likely demand is the demand of 2 units, and if one has to place the order then he should place for 2 units as per the most likelihood rule. The disadvantage of this rule is that no consideration is given to less likely but it has more consequential results.

For example, if the throw of a dice costs ₹1 and if someone has to receive ₹1000 for throwing a six then he or she is more likely to lose ₹1 rather winning ₹1000. Thus, he/she would have made the wrong decision if he/she would not have accepted the offer of throwing a dice.

3.1.7.2. Expected Pay-off Criterion

Following are the techniques which are based on Expected Pay-off Criterion:

1) Expected Monetary Value(EMV)
2) Expected Opportunity Loss (EOL)/Expected Regret
3) Expected Value of Perfect Information (EVPI)

3.1.7.3. Expected Monetary Value(EMV)

The weighted average payoff for a given course of action is the Expected Monetary Value (EMV). It means the total of the payoffs for each course of action multiplied by the probabilities combined with each state of nature is known as EMV.

The mathematical description of the EMV is as follows:

$$EMV(S_j) = \sum_{i=1}^{m} P_{ij} P_i$$

where, m = Number of possible states of nature,
P_i = Probability of occurrence of i^{th} state of nature
P_{ij} = Payoff connected with state of nature N_i and course of action S_j

Steps for Calculating EMV

Step 1) Building of the payoff matrix by enlisting all the possible courses of action and states of nature. Enter all the possible combination of course of action and state of nature that are linked with limited payoff values as well as the probabilities of the happening of each state of nature.

Step 2) Calculate the value of EMV for each course of action by multiplying with the conditional payoffs of the combined probabilities which are added to the weighted values for each course of action.

Step 3) Selection of that course of action which yield the optimum EMV.

Example 8: A newspaper seller has the following probabilities of selling newspapers:

Sold Copies	Probability
10	0.10
11	0.15
12	0.20
13	0.25
14	0.30
Total	1.00

The selling price of a newspaper is 60 paisa and its cost is 40 paisa. The condition is that seller cannot return the copies of newspaper. Determine how many newspapers, the seller will order?

Solution: The seller has 10, 11, 12, 13 or 14 numbers of copies for purchases and sales. This means he/she buys more than 10 and less than 14.

Profit resulting from any combination of supply and demand is shown in following profit table (**table 3.2**). Irrespective of demand, the profit will be 200 paisa per day when there is a stock of 10 copies.

For example, if seller sells 11 copies while the demand is 14 copies, then the conditional profit will be 220 paisa. If he stocks 12 copies, his profit is 240 paisa when the buyer orders 12, 13 or 14 copies.

If he stocks 11 copies and the buyers buy 10 copies, then the profit decreases to 160 paisa (200 paisa profit on selling 10 copies minus 40 paisa of one unsold copy) because one copy of newspaper is unsold. When the seller stocks 12, 13 or 14 copies, then the same rule will be applicable.

Thus conditional profit can be given by as follows:

Payoff = 20 × No of Copies Sold – 40 × No of Copies not Sold.

Table 3.2: Conditional Profit Table (Paisa)

Possible Demand (No. of Copies)	Probability	Possible Stock Action				
		10 Copies	11 Copies	12 Copies	13 Copies	14 Copies
10	0.10	200	160	120	80	40
11	0.15	200	220	180	140	100
12	0.20	200	220	240	200	160
13	0.25	200	220	240	260	220
14	0.30	200	220	240	260	280

The average value of each alternative can be determined by multiplying the conditional profit by corresponding probability and then adding the resulting values as shown in table 3.3:

Table 3.3: Average Profit Table

Possible Demand	Probability	Average Profit from Stocking (Paisa)				
		10 Copies	11 Copies	12 Copies	13 Copies	14 Copies
10	0.10	20	16	12	8	4
11	0.15	30	33	27	21	15
12	0.20	40	44	48	40	32
13	0.25	50	55	60	65	55
14	0.30	60	66	72	78	84
Total Average Profit (Paisa)		**200**	**214**	**219**	**212**	**190**

Therefore the seller must order 12 copies in order to earn maximum profit as its daily average profit is 219 paisa. This stock will provide highest profit over a period of time. The next day profit of 219 paisa is not guaranteed. In case he stocks 12 copies per day, then the average profit is 219 paisa per day under given constraints. This is the best choice for him, as others stocks will not provide so much daily profit.

Example 9: Let consider the following payoff matrix:

State of Nature	Probability	Act		
		Do not Expand (₹)	Expand 200 Units (₹)	Expand 400 Units (₹)
High Demand	0.4	2,500	3,500	4,900
Medium Demand	0.4	2,500	3,500	2,500
Low Demand	0.2	2,500	1,500	1,000

Determine which act must be select using EMV criterion.

Solution: The various decisions have the following EMV:

Decision	EMV (₹)
Do not Expand	0.4 (2500) + 0.4 (2500) + 0.2(2500) = 2,500
Expand 200 Units	0.4 (3500) + 0.4 (3500) + 0.2 (1500) = 3,100
Expand 400 Units	0.4 (4900) + 0.4 (2500) + 0.2 (1000) = 3,160

Since the highest EMV value is 3,160, hence it favours expansion of 400 units.

Example 10: A producer of boats has estimated the following distribution of demand for a particular kind of boat:

No. of Demand	0	1	2	3	4	5	6
Probability	0.14	0.27	0.27	0.18	0.09	0.04	0.01

Each boat costs him ₹7,000 and he sells them for ₹10,000 each. Any boat that is left unsold at the end of season must be disposed off for ₹6,000 each. How many boats should be in stock so as to maximize his expected profit?

Solution: Given that,

Cost of boat = ₹7,000 and

Selling price = ₹10,000

∴ Profit if sold = ₹10,000 – ₹7,000 = ₹3,000

If it is not sold, then disposable price = ₹6,000

Using the above data, the conditional profit (pay-off) values for each act-event combination are given by:

Conditional profit = MP × Boats Sold – ML × Boats Unsold

= (10,000 – 7,000) × Boats Sold – (7,000 – 6,000) × Boats Unsold

$$= \begin{cases} 3{,}000S & = 3{,}000S \quad \text{if } P \le S \\ 3{,}000S + 1{,}000(S-P) & = 4{,}000S - 1{,}000P \quad \text{if } P > S \end{cases}$$

Here,

P = Number of boats produced

S = Number of boats sold in past

The resulting conditional pay-off is computed in the following table:

Table 3.4: Conditional Profit Values (Pay-Offs)

Event (Sales per Season)	Probability	Conditional Pay-Off Act (Production per Week)						
		0	1	2	3	4	5	6
0	0.14	0	–1,000	–2,000	–3,000	–4,000	–5,000	–6,000
1	0.27	0	3,000	2,000	1,000	0	–1,000	–2,000
2	0.27	0	3,000	6,000	5,000	4,000	3,000	2,000
3	0.18	0	3,000	6,000	9,000	8,000	7,000	6,000
4	0.09	0	3,000	6,000	9,000	12,000	11,000	10,000
5	0.04	0	3,000	6,000	9,000	12,000	15,000	14,000
6	0.01	0	3,000	6,000	9,000	12,000	15,000	18,000

The calculations for expected pay-offs and EMV for each act are shown in the following table:

Table 3.5: Computation of Expected Pay-Off and EMV

Event (Demand)	Probability (1)	Expected Pay-Off Act (Production per Week)						
		0	1	2	3	4	5	6
0	0.14	0	–140	–280	–420	–560	–700	–840
1	0.27	0	810	540	270	0	–270	–540
2	0.27	0	810	1,620	1,350	1,080	810	540
3	0.18	0	540	1,080	1,620	1,440	1,260	1,080
4	0.09	0	270	540	810	1,080	990	900
5	0.04	0	120	240	360	480	600	560
6	0.01	0	300	60	90	120	150	180
EMV (₹)			**2,440**	**3,800**	**4,080***	**3,640**	**2,840**	**1,880**

From the table, it is given that the maximum expected profit is 4,080 which occur when stock is 3 boats. Hence the order of 3 boats per season will maximise his expected profit.

Example 11: The probability of demand for lorries for hiring on any day in a given district is as follows:

No. of Lorries Demanded	0	1	2	3	4
Probability	0.1	0.2	0.3	0.2	0.2

Lorries have a fixed cost of ₹90 each day to keep the daily hire charges (net of variable costs of running) ₹200. If the lorry-hire company owns 4 lorries, what is its daily expectation? If the company is about to go into business and currently has no lorries, how many lorries should it buy?

Solution: Given that,
Fixed cost = ₹90
Variable cost = ₹200

Now, we have the following table:

No. of Lorries Demanded	0	1	2	3	4
Pay-Offs	0 – 90 × 4 = –360	200 – 90 × 4 = –160	400 – 90 × 4 = 40	600 – 90 × 4 = 240	800 – 90 × 4 = 440

Daily expectation = (–360 × 0.1) + (–160 × 0.2) + (40 × 0.3) + (240 × 0.2) + (440 × 0.2) = 80

Demand of Lorries	Probability	Conditional Pay-Off				
		0	1	2	3	4
0	0.1	0	–90	–180	–270	–360
1	0.2	0	110	20	–70	–160
2	0.3	0	110	220	130	40
3	0.2	0	110	220	330	240
4	0.2	0	110	220	330	440

Demand of Lorries	Probability	Conditional Pay-Off				
		0	1	2	3	4
0	0.1	0	–9	–18	–27	–36
1	0.2	0	22	4	–14	–32
2	0.3	0	33	66	39	12
3	0.2	0	22	44	66	48
4	0.2	0	22	44	66	88
	EMV→	0	90	140	130	80

As the EMV of course of action 2 is 140, which is higher than rest of all, the company should purchase two lorries.

3.1.7.4. Expected Opportunity Loss (EOL)/ Expected Regret

An alternative approach that refers to maximization of Expected Monetary Value (EMV) and minimization of Expected Opportunity Loss (EOL) is also known as **expected value of regret.**

The difference between the highest profit for a state of nature and the actual profit which is obtained for the specific course of action is defined as EOL.

Hence, the amount of payoff that is lost due to the rejection of a course of action, which is having the greatest payoff for the state of nature that has actually appeared, is referred as EOL. That course of action is recommended for which EOL is minimum.

Results obtained by EMV criterion and by EOL, which is an alternative decision criterion for making decision under risk area will always be the same.

Hence, only one of the two methods should be applied for reaching a decision. The mathematical description is as follows:

$$EOL(N_i) = \sum_{i=1}^{m} l_{ij} P_i$$

Where, l_{ij} = Opportunity loss due to state of nature, N_i and course of action, S_j

P_i = Probability of occurrence of state of nature, N_i

Steps for Calculating EOL

Step 1) Preparation of a conditional profit table that depicts each course of action, state of nature as well as associated probabilities.

Step 2) Calculate Conditional Opportunity Loss (COL) value for each state of nature by subtracted in each payoff from the maximum payoff of each event.

Step 3) Calculation of EOL for each course of action by multiplying probability of each state of nature with the value of COL and then adding it up.

Step 4) That course of action is selected for which the EOL value is minimum.

Example 12: A seller has the following probabilities of selling newspapers:

Copies Sold	10	11	12	13	14
Probability	0.10	0.15	0.20	0.25	0.30

The selling price is 60 paisa and its cost is 40 paisa. The condition is that seller cannot return the copies of newspaper. Using EOL criterion, determine how many newspapers, seller should order?

Solution: The conditional profit table is shown in **table 4.6**:

Table 3.6: Conditional Profit Table (Paisa)

Possible Demand (No. of Copies)	Probability	Possible Stock Action(Alternative)				
		10 Copies	11 Copies	12 Copies	13 Copies	14 Copies
10	0.10	200	160	120	80	40
11	0.15	200	220	180	140	100
12	0.20	200	220	240	200	160
13	0.25	200	220	240	260	220
14	0.30	200	220	240	260	280

When there is demand of 10 copies then the best alternative is that seller will order 10 copies and this will give a profit of 200 paisa. The conditional opportunity loss for every stock can be obtained by subtracting the respective conditional profits from 200 paisa. Similarly, each value of the rows subtracted from the maximum of that row will provide conditional payoff values for demand of 11, 12, 13 and 14 copies. Therefore, we get the following Conditional Opportunity Loss (COL) **table 3.7**:

Table 3.7: Conditional Loss Table (Paisa)

Possible Demand (no. of copies) (Event)	Probability	Possible Stock Action (Alternative)				
		10 Copies	11 Copies	12 Copies	13 Copies	14 Copies
10	0.10	0	40	80	120	160
11	0.15	20	0	40	80	120
12	0.20	40	20	0	40	80
13	0.25	60	40	20	0	40
14	0.30	80	60	40	20	0

Expected Opportunity Loss (EOL) can be determined by multiplying the probability of each state of nature with corresponding appropriate loss value and then adding them which provides resulting products. **For example,** for stocking 11 copies, we have:

Expected Opportunity Loss (EOL) = $0.10 \times 40 + 0.15 \times 0 + 0.20 \times 20 + 0.25 \times 40 + 0.30 \times 60$

$= 4 + 0 + 4 + 10 + 18 =$ 36 paisa

The EOL for different stocks can be calculated as shown in **table 3.8**:

Table 3.8: Expected Loss Table (Paisa)

Possible demand (no. of copies) (Event)	Probability	Possible Stock Action (Alternative)				
		10 Copies	11 Copies	12 Copies	13 Copies	14 Copies
10	0.10	0	4	8	12	16
11	0.15	3	0	6	12	18
12	0.20	8	4	0	8	16
13	0.25	15	10	5	0	10
14	0.30	24	18	12	6	0
EOL (Paisa)		50	36	31	38	60

Since minimum expected opportunity loss will represent the optimum stock action, hence the seller will stock 12 copies per day as the minimum expected loss is 31 paisa.

Example 13: Let consider the following payoff matrix:

Act	State of Nature	
	Cold Weather(₹)	Warm Weather(₹)
Sell Cold Drinks	40	90
Sell Ice Cream	70	40

From the previous experience, it is known that probability of happening of cold weather is 0.3.

1) Determine the opportunity loss table and then calculate the expected opportunity loss for each course of action.
2) Also illustrate that the EMV and EOL will give the decision.

Solution: The opportunity loss matrix is shown in table below:

Act	State of Nature	
	Cold Weather(₹)	Warm Weather(₹)
Sell Cold Drinks	70 – 40 = 30	90 – 90 = 0
Sell Ice Cream	70 – 70 = 0	90 – 40 = 50

For every alternative Couse of action, one can compute the EOL as shown below:

Act	State of Nature		Total
	Cold Weather(₹)	Warm Weather(₹)	
Sell Cold Drinks	$0.3 \times 30 = 9$	$0.7 \times 0 = 0$	$9 + 0 = 9$
Sell Ice Cream	$0.3 \times 0 = 0$	$0.7 \times 50 = 35$	$0 + 35 = 35$

Selling the cold drinks is the best act as EOL in such case is minimum.

The EML value of every course of action can be computed as follows:

Act	State of Nature		Total
	Cold Weather(₹)	Warm Weather(₹)	
Sell Cold Drinks	$0.3 \times 40 = 12$	$0.7 \times 90 = 63$	$12 + 63 = 75$
Sell Ice Cream	$0.3 \times 70 = 21$	$0.7 \times 40 = 28$	$21 + 28 = 49$

The selling cold drink is recommend as the EMV value is more, i.e., 75. Hence decision is same under EOL and EMV criteria.

Example 14: A newspaper boy has the probability of selling a magazine as shown in table:

No. of Copies Sold	9	10	11	12	13	14
Probability	0.05	0.1	0.15	0.3	0.25	0.15

The cost of a copy sold is ₹30 and the sale price of the magazine is ₹40. The unsold copies fetch a salvage value of ₹5 in the second sale market. How many copies should be ordered to maximise the gain? Use EOL criterion to solve the problem.

Solution: It is given that,

The cost of copy sold = ₹30

The sale of the magazine = ₹40

Thus unsold copies fetch a salvage value = ₹5
Profit = 40 – 30 = ₹10
Loss = 30 – 5 = ₹25

The conditional profitable is shown in **table 4.9**:

Table 3.9: Conditional Profit Table (₹)

Possible Demand (No. of Copies) (Event)	Probability	Possible Stock Action (Alternative)					
		9 Copies	10 Copies	11 Copies	12 Copies	13 Copies	14 Copies
9	0.05	90	65	40	15	–10	–35
10	0.1	90	100	75	50	25	0
11	0.15	90	100	110	85	60	35
12	0.30	90	100	110	120	95	70
13	0.25	90	100	110	120	130	105
14	0.15	90	100	110	120	130	140

When there is a demand of 9 copies and order of 9 copies, then best alternative gives optimal profit of ₹90. For this event, the conditional opportunity loss for every alternative is resulted by subtracting the respective conditional profit from optimal profit (i.e., ₹90). Similarly for demand of 10, 11, 12, 13, 14 copies, the conditional opportunity loss can be obtained by subtracting the conditional payoff values for every of these rows from maximum profit of that row. **Table 3.10** shows the resultant Conditional Opportunity Loss (COL):

Table 3.10: Conditional Loss Table (₹)

Possible Demand (No. of Copies) (Event)	Probability	Possible Stock Action (Alternative)					
		9 Copies	10 Copies	11 Copies	12 Copies	13 Copies	14 Copies
9	0.05	0	25	50	75	100	125
10	0.1	10	0	25	50	75	100
11	0.15	20	10	0	25	50	75
12	0.3	30	20	10	0	25	50
13	0.25	40	30	20	10	0	25
14	0.15	50	40	30	20	10	0

Now by multiplying the probability of every state of nature with appropriate loss value and adding the resultant product, one can calculate the Expected Opportunity Loss (EOL). **For example,** for holding a stock of 10 copies, we get the following result:

$EOL = 0.05 \times 0 + 0.1 \times 10 + 0.15 \times 20 + 0.30 \times 30 + 0.25 \times 40 + 0.15 \times 50$

$= 0 + 1 + 3 + 9 + 10 + 7.50 = ₹30.50$

Table 3.11 shows the EOL values for different stock actions:

Table 3.11: Expected Loss Table (₹)

Possible Demand (No. of Copies) (Event)	Probability	Possible Stock Action (Alternative)					
		9 Copies	10 Copies	11 Copies	12 Copies	13 Copies	14 Copies
9	0.05	0	1.25	2.5	3.75	5	6.25
10	0.1	1	0	2.5	5	7.5	10
11	0.15	3	1.5	0	3.75	7.5	11.25
12	0.3	9	6	3	0	7.5	15
13	0.25	10	7.5	5	2.5	0	6.25
14	0.15	7.5	6	4.5	3	1.5	0
EOL (₹)		30.5	22.25	**17.5**	18	29	48.75

The action which will minimise expected opportunity losses is the optimum stock action. Here this action is for stocking of 11 copies every day. The minimum expected loss will be ₹17.5 at this point.

Example 15: Your Company manufactures goods for a market in which the technology of the products is changing rapidly. The research and development department has produced a new product which appears to have potential for commercial exploitation. A further ₹60000 is required for development testing.

The Company has 100 customers and each customer might purchase, at the most, one unit of the product. Market research suggests a selling price of ₹6000 for each unit with total variable costs of manufacturing and selling estimated at ₹2000 for each unit. As a result of previous experience of this type of market, it has been possible to derive a probability distribution relating to the proportion of customers who will buy the product, as follows:

Table 3.12

Proportion of Customers	Probability
0.04	0.1
0.08	0.1
0.12	0.2
0.16	0.4
0.20	0.2

Determine the expected opportunity losses, given no further information than that stated above and state, whether or not, the company should develop the product.

Solution: Let p denotes the proportion of customers who purchase the new product, and now the conditional profit will be calculated as follows:

$CP = ₹(6000 - 2000)(p \times 100) - 60000$

$= ₹1000\,(400p - 60)$

The conditional profit table can now be constructed as follows:

Table 3.13: Conditional Profit Table

State of Nature (Proportion of Customers)	Probability	Alternative Actions	
		Do not Develop(A_1) ₹	Develop(A_2) ₹
0.04	0.1	0	– 44000
0.08	0.1	0	– 28000
0.12	0.2	0	– 12000
0.16	0.4	0	4000
0.20	0.2	0	20000

Table 3.14: Opportunity Loss Table

Proportion of Customers	Probability	Alternative Actions	
		Do not Develop(A_1) ₹	Develop(A_2) ₹
0.04	0.1	0	44000
0.08	0.1	0	28000
0.12	0.2	0	12000
0.16	0.4	4000	0
0.20	0.2	20000	0

$EOL\,(A_1) = 0.4\,(4000) + 0.2(20000) = 5600$

$EOL\,(A_2) = 0.1\,(44000) + 0.1(28000) + 0.2(12000)$
$= 9600$

Since A_1 gives the lower EOL of ₹5600, the best decision is not to develop the product.

3.1.7.5. Expected Value of Perfect Information (EVPI)

The criterion for decision making under risk for each state of nature is combined with its probability of occurrence and somehow, the decision maker is able to acquire perfect (complete and accurate) information regarding the occurrence of various states of nature.

Then he will be successful in selecting the course of action that yielding the expected payoff for whatever may be the state of nature that actually takes place. The maximum amount of money which the decision maker has to pay in acquiring additional information about the occurrence of various states of nature prior reaching to a decision is represented by Expected Value of Perfect Information (EVPI). Mathematical description is as follows:

EVPI = Expected profit (or value) with perfect information under certainty – Expected profit without perfect information

$$EVPI = EPPI - EMV^*$$

Where, EPPI = Expected profit (or value) with perfect information under certainty

EMV^* = Maximum expected monetary value.

Example 16: Let consider the three acts A, B and C and states of nature X, Y and Z. The payoffs are shown in table below:

Pay-off (in ₹)

		Acts		
		A	B	C
States of Nature	X	–20	– 50	200
	Y	200	–100	–50
	Z	400	600	300

The probabilities of X, Y and Z are 0.3, 0.5 and 0.2 respectively. Compute the EMV for above data and also find the best act. Determine the expected value of perfect information (EVPI) also.

Solution: The EMV for each act can be computed as following:

A = – 20 × 0.3 + 200 × 0.5 + 400 × 0.2 = – 6 + 100 + 80 = ₹ 174

B = –50 × 0.3 – 100 × 0.5 + 600 × 0.2 = – 15 – 50 + 120 = ₹55

C = 200 ×0.3 – 50 × 0.5 + 300 × 0.2 = 60 – 25 + 60 = ₹95

Hence act A should be chosen because EMV for this act is maximum.

Pay-off (in ₹)

		Acts				Max. for State of Nature	(Max. pay-off) x (prob.)
		Probability	A	B	C		
State of nature	X	0.3	–20	–50	200	200	200 × 0.3 = 60
	Y	0.5	200	–100	–50	200	200 × 0.5 = 100
	Z	0.2	400	600	300	600	600 × 0.2 = 120
						Total	280

Now EVPI = EPPI – EMV = 280 – 174 = ₹ 106.

Example 17: The selling of newspapers during off peak hours are allowed on buses under the employment promotion programmed. The vendor purchases the newspapers at the rate of 25 paisa per copy and sells them at the rate of 40 paisa per copy. Unsold copies of newspaper are dead loss. The numbers of copies demanded have the following probability as calculated by vendor.

Number of Copies	15	16	17	18	19	20
Probability	0.04	0.19	0.33	0.26	0.11	0.07

1) Find how many copies will the vendor order for attaining maximum expected profit?
2) Calculate EPPI.
3) The vendor wants to spend some money on small market survey in order to obtain extra information regarding demand levels. Determine how much money should he/she spend on this survey?

Solution: One can compute the conditional profit values for each action event combination using the above data. Let represent conditional profit as CP, stock as S and demand as D then we have:

$$CP = \begin{cases} (40-25)\ S = 15S, \text{ when } D \geq S \\ 40D - 25S, \text{ when } D < S. \end{cases}$$

The conditional profit matrix can be computed as shown in **table 3.15**:

Table 3.15: Conditional Profit Table

Demand (Event)	Probability (A)	Possible Stock Action (Alternative)					
		15 Copies (B)	16 Copies (C)	17 Copies (D)	18 Copies (E)	19 Copies (F)	20 Copies (G)
15	0.04	2.25	2.00	1.75	1.50	1.25	1.00
16	0.19	2.25	2.40	2.15	1.90	1.65	1.40
17	0.33	2.25	2.40	2.55	2.30	2.05	1.80
18	0.26	2.25	2.40	2.55	2.70	2.45	2.20
19	0.11	2.25	2.40	2.55	2.70	2.85	2.60
20	0.07	2.25	2.40	2.55	2.70	2.85	3.00

For each stock action, the expected payoffs and EMV can be determined as shown in **table 3.16**:

Table 3.16: Expected Profit Table

Demand (Event)	Probability (A)	Possible Stock Action (Alternative)					
		15 Copies (A)×(B) ₹	16 Copies (A)×(C) ₹	17 Copies (A)×(D) ₹	18 Copies (A)×(E) ₹	19 Copies (A)×(F) ₹	20 Copies (A)×(G) ₹
15	0.04	0.09	0.08	0.07	0.06	0.05	0.04
16	0.19	0.43	0.46	0.41	0.36	0.31	0.27
17	0.33	0.74	0.79	0.84	0.76	0.68	0.59
18	0.26	0.58	0.62	0.66	0.70	0.64	0.57
19	0.11	0.25	0.26	0.28	0.30	0.31	0.29
20	0.07	0.16	0.17	0.18	0.19	0.20	0.21
EMV (₹)		**2.25**	**2.38**	**2.44**	**2.37**	**2.19**	**1.97**

1) Since the maximum expected daily profit is ₹ 2.44, hence the vendor will order 17 copies.
2) EPPI can be determined as shown in **table 3.17**.

Table 3.17

Event	Probability	Payoff Under Perfect Information (₹)	Expected Payoff Under Perfect Information (₹)
15	0.04	2.25	0.09
16	0.19	2.40	0.46
17	0.33	2.55	0.84
18	0.26	2.70	0.70
19	0.11	2.85	0.31
20	0.07	3.00	0.21
		EPPI (₹)	2.61

Thus we have EPPI = ₹ 2.61.

3) EVPI can be calculated as following:
EVPI = EPPI – max EMV = ₹ (2.61 – 2.44) = ₹ 0.17.

Therefore the vendor should spend less than ₹ 0.17 on small market survey.

Example 18: A doctor purchases a particular vaccine each Monday. If the vaccine is not used within the week, it becomes useless. The vaccine costs ₹30 per dose and the doctor charges ₹60 for the same. The doses administered per week has the following distribution:

Doses Per Week	20	25	40	60
No. of Weeks	5	15	25	5

Draw a payoff matrix, obtain a regret matrix and determine the optimal number of doses the doctor should buy. Also find the value of EVPI.

Solution: Total number of weeks = 5 + 15 + 25 + 5 = 50

Profit per dose = 60 – 30 = 30

Loss per unused dose = 30

The Payoff and regret matrix is constructed as follows:

Table 3.18: Payoff /Regret Matrix

Doses Per Week	Probability	Act (Profit)			
		Stock 20 Doses	Stock 25 Doses	Stock 40 Doses	Stock 60 Doses
20	$\frac{5}{50}=0.1$	600	450	0	– 600
25	$\frac{15}{50}=0.3$	600	750	+300	– 300
40	$\frac{25}{50}=0.5$	600	750	1200	600
60	$\frac{5}{50}=0.1$	600	750	1200	1800

Expected Monetary Value (EMV) can now be calculated as follows:

EMV (20) = 600 × 0.1 + 600 × 0.3 + 600 × 0.5 + 600 × 0.1
= 60 + 180 + 300 + 60 = 600

EMV (25) = 450 × 0.1 + 750 × 0.3 + 750 × 0.5 + 750 × 0.1
= 45 + 225 + 375 + 75 = 720

EMV (40) = 0 × 0.1 + 300 × 0.3 + 1200 × 0.5 + 1200 × 0.1
= 90 + 600 + 120 = **810**

EMV (60) = –600 × 0.1 – 300 × 0.3 + 600 × 0.5 + 1800 × 0.1
= –60 – 90 + 300 + 180 = 230

As the purchase of 40 doses gives the highest EMV of ₹810, hence optimal act for the doctor would be to purchase 40 doses of the vaccine per week.

EVPI can be calculated as follows:

Best pay-off for the 1st state of nature S_1 = 600, $P(S_1) = 0.1$
Best pay-off for the 2nd state of nature S_2 = 750, $P(S_2) = 0.3$
Best pay-off for the 3rd state of nature S_3 = 1200, $P(S_3) = 0.5$
Best pay-off for the 4th state of nature S_4 = 1800, $P(S_4) = 0.1$

EPPI = 600 × 0.1 + 750 × 0.3 + 1200 × 0.5 + 1800 × 0.1
= 60 + 225 + 600 + 180 = 1065

EVPI = EPPI – EMV of Best Act
= 1065 – 810 = 255

Example 19: A doctor purchases a particular medicine on Sunday of each week. The medicine must be used within the week following, failing which it becomes useless. The cost of medicine is ₹2 per dose and the doctor charges ₹4 per dose. In the past 50 weeks, the records of use of medicine are as follows:

Dose Per Week	20	25	40	60
No. of Weeks	5	15	25	5

Calculate:

i) Expected monetary value
ii) Expected opportunity loss
iii) Expected values of perfect information

Solution: Total number of weeks = 50

Profit per dose = 4 – 2 = 2

Loss per unused dose = 2

Doses Per Week	Probability	Act (Profit)			
		Stock 20 Doses	Stock 25 Doses	Stock 40 Doses	Stock 60 Doses
20	$\frac{5}{50}=0.1$	40	30	0	–40
25	$\frac{15}{50}=0.3$	40	50	20	–20
40	$\frac{25}{50}=0.5$	40	50	80	40
60	$\frac{5}{50}=0.1$	40	50	80	120

i) **The Expected Monetary Value (EMV):** Calculation of EMV is as below:

EMV (20) = 40 × 0.1 + 40 × 0.3 + 40 × 0.5 + 40 × 0.1
= 4 + 12 + 20 + 4 = 40

EMV (25) = 30 × 0.1 + 50 × 0.3 + 50 × 0.5 + 50 × 0.1
= 3 + 15 + 25 + 5 = 48

EMV (40) = 0 × 0.1 + 20 × 0.3 + 80 × 0.5 + 80 × 0.1
= 6 + 40 + 8 = **54**

EMV (60) = – 40 × 0.1 – 20 × 0.3 + 40 × 0.5 + 120 × 0.1
= –4 – 6 + 20 + 12 = 22

Hence EMV (Max) = 54 for 40 doses.

ii) **Expected Opportunity Loss (EOL):** Calculation of EOL is as below:

Doses Per Week	Probability	Act (Loss)			
		Stock 20 Doses	Stock 25 Doses	Stock 40 Doses	Stock 60 Doses
20	$\frac{5}{50}=0.1$	0	10	40	80
25	$\frac{15}{50}=0.3$	10	0	30	70
40	$\frac{25}{50}=0.5$	40	30	0	40
60	$\frac{5}{50}=0.1$	80	70	40	0

The Expected opportunity Loss (EOL) can now be computed as:

EOL (20) = $0 \times 0.1 + 10 \times 0.3 + 40 \times 0.5 + 80 \times 0.1$
$= 3 + 20 + 8 = 31$

EOL (25) = $10 \times 0.1 + 0 \times 0.3 + 30 \times 0.5 + 70 \times 0.1$
$= 1 + 15 + 7 = 23$

EOL (40) = $40 \times 0.1 + 30 \times 0.3 + 0 \times 0.5 + 40 \times 0.1$
$= 4 + 9 + 4 =$ **17**

EOL (60) = $80 \times 0.1 + 70 \times 0.3 + 40 \times 0.5 + 0 \times 0.1$
$= 8 + 21 + 12 = 41$

Hence EOL (Min) = 17 for 40 doses.

iii) **Expected Values of Perfect Information (EVPI):** Calculation of EVPI is as below:

Best pay-off for the 1st state of nature $S_1 = 40$, $P(S_1) = 0.1$

Best pay-off for the 2nd state of nature $S_2 = 50$, P(S2) = 0.3

Best pay-off for the 3rd state of nature $S_3 = 80$, P(S3) = 0.5

Best pay-off for the 4th state of nature $S_4 = 120$, P(S4) = 0.1

EPPI = $40 \times 0.1 + 50 \times 0.3 + 80 \times 0.5 + 120 \times 0.1$
$= 4 + 15 + 40 + 12 = 71$

EVPI = EPPI – EMV of best act = 71 – 54 = 17

Loss per unused dose = 2

3.1.7.6. Bayesian Analysis

According to the Bayesian rule of decision theory, the decision maker selects a course of action with the help of rational basis by adopting personal evaluation of probability based on experience, past performance, judgment etc. Applying Baye's principle in the problem of Statistical decision probabilities are assigned to each state of nature by the decision maker. The strength of the decision maker is represented by these probabilities in his belief, that is a biased evaluation as regards to the possibility of the occurrence of the various states of nature. Baye's principle must be used in phases after the determination of probabilities. The three phases are:

1) **Prior Analysis:** Prior analysis is the method of utilising the former probabilities where a decision maker assigns probabilities to various events and a subjective evaluation of the possibility of occurrence of the various states are based on the experience of past performance.

2) **Pre-posterior Analysis:** Before the selection of the sample for additional information, the assessment of the expected value of sample information contrary to the expected value of present information is the procedure of pre-posterior analysis. The revision of probabilities using Baye's principle is included in this analysis. Hence, pre-posterior analysis includes a decision after the revision of the probabilities.

3) **Posterior Analysis:** If additional information is to be obtained and high EVPI is for prior analysis then these probabilities are revised on the basis of this additional information.

Application of Bayes' theorem of probability helps in computing revised probabilities. Thus, these probabilities are known as posterior probabilities.

Utilising these posterior probabilities in the further analysis of the problem yields new expected payoffs. The revision of the analysis of the problem is posterior analysis.

Likelihood of occurrence that is an event will occur when given that a related event has occurred earlier is measured by posterior probability. This is known as **prior probability,** which is the modified version of original probability or the probability without additional information. Bayes' theorem is used in calculating **Posterior probability.**

In finance, common application of posterior probability is financial modelling of stock portfolios. In the calculation of the posterior probability two dependent events under conditional probability are examined.

By using Bayes' theorem, posterior probability P (A|B) can be calculated. Let 'A' be the object event and P (A) be the prior probability. Second event 'B' being dependent or it is related to event 'A' thus its probability is P (B).

Suppose that the likelihood of occurrence of event 'B', given that 'A' occurs, then probability will be P (B|A).

Difference between Decision Making under Uncertainty and Risk

Table 3.19 shows the difference between Decision Making under Uncertainty and Risk:

Table 3.19: Difference between Decision Making under Uncertainty and Risk

Basis	Decision Under Risk	Decision Under Uncertainty
Meaning	The probability of winning or losing something worthy is known as risk.	Uncertainty implies a situation where the future events are not known.
Ascertainment	It can be measured	It cannot be measured.
Outcome	Chances of outcomes are known.	The outcome is unknown.
Control	Controllable	Uncontrollable
Minimization	Yes	No
Probabilities	Assigned	Not assigned

Example 20: A company is introducing a new product and considering it in existing product range. The decision is based on two levels of sales 'high' and 'low' and the chances of each market level with its cost and corresponding profits or losses are also estimated. The information regarding this is shown in table below:

States of Nature	Probability	Courses of Action	
		Market Product (₹'000)	Not to Market Product (₹'000)
High Sales	0.3	150	0
Low Sales	0.7	– 40	0

The marketing manager of a company proposes a market research survey in order to attain extra information over which all decisions are based. On past experience with a certain market research organisation, the marketing manager assesses its ability to give good information in the light of subsequent actual sales achievements as follows:

Market Research Survey Outcome	Actual Sales	
	Market 'High'	Market 'Low'
'High' Sales Forecast	0.5	0.1
Indecisive Survey Report	0.3	0.4
'Low' Sales Forecast	0.2	0.5

Given that to undertake the market research survey will cost ₹20,000. State whether or not there is a case for employing the market research organisation.

Solution: The Expected Monetary Value (EMV) for each course of action is given in **table 3.20**:

Table 3.20

States of Nature	Probability	Courses of Action		Expected Profit (₹'000)	
		Market Product	Not Market Product	Market Product	Not Market Product
High Sales	0.3	150	0	45	0
Low Sales	0.7	– 40	0	–28	0
			EMV	17	0

With no additional information, the company selects a course of action 'market product'. But, if the company had the perfect information about 'low sales', then it would not go ahead because the expected value is – ₹28,000. Thus, the value of perfect information is the expected value of low sales.

Let the states of nature be:

O_1 = high sales, O_2 = indecisive report, O_3 = low sales

Courses of action are:

S_1 = high market, S_2 = low market

The computations of prior probabilities of forecast are shown in **table 3.21**:

Table 3.21

States of Nature O_1	Sales Production	
	Market 'High' (S_1)	Market 'Low' (S_2)
High Sales (O_1)	$P(O_1/S_1) = 0.5$	$P(O_1/S_2) = 0.1$
Indecisive Report (O_2)	$P(O_2/S_1) = 0.3$	$P(O_2/S_2) = 0.4$
Low Sales (O_3)	$P(O_3/S_1) = 0.2$	$P(O_3/S_2) = 0.5$

With this additional information, the company can revise the prior outcome probabilities to determine posterior probabilities. These can also be used to re-calculate the EMV and to obtain the optimal course of action given the additional information.

Table 3.22: Calculation of Revised Probabilities given the Sales Forecast

States of Nature	Prior Probability	$P(O_1 \cap S_1)$	$P(O_2 \cap S_1)$	$P(O_3 \cap S_1)$
High Sales	$P(O_1/S_1) = 0.5$	0.15	–	–
	$P(O_2/S_1) = 0.3$	–	0.09	–
	$P(O_3/S_1) = 0.2$	–	–	0.06
Low Sales	$P(O_1/S_2) = 0.1$	0.07	–	–
	$P(O_2/S_2) = 0.4$	–	0.28	–
	$P(O_3/S_2) = 0.5$	–	–	0.35
Marginal Probability		0.22	0.37	0.41

The posterior probabilities of actual sales given the sales forecast are:

$$P(S_1/O_1) = \frac{P(S_1)\ P(O_1/S_1)}{P(O_1)} = \frac{0.3 \times 0.5}{0.22} = 0.68$$

Similarly, $P(S_1/O_2) = 0.243$, $P(S_1/O_3) = 0.146$, $P(S_2/O_1) = 0.318$, $P(S_2/O_2) = 0.756$, $P(S_2/O_3) = 0.853$.

Now for each outcome, the revised probabilities are used to determine the net expected value if the additional information is supplied by the outcome as shown in **table 3.23**:

Table 3.23

States of Nature Actual Sales	Revised Conditional Profit (₹)	Sales Forecast					
		High		Indecisive		Low	
		Prob.	EV (₹)	Prob.	EV (₹)	Prob.	EV (₹)
High	130	0.681	88.66	0.243	31.59	0.146	18.98
Low	–60	0.318	–19.08	0.756	–45.36	0.853	–51.18
Expected value of sale forecast			69.58		–13.77		–32.20
Probability of occurrence			0.22		0.37		0.41
Net expected value (Expected value × Prob.)			15.279		–5.095		13.22

Example 21: A Company receives shipments of certain items. It should decide whether to accept or reject the shipment, on the basis of inspection of a sample selected from the shipment. From the past experience, it is known that the percentage of defective items in a batch of shipment is either 0, 2 or 5, the probabilities for which are 0.5, 0.3 and 0.2 respectively.

The company can accept only those batches which have no defect. The cost of rejecting a good batch, i.e., batch with no defect is ₹ 200. The cost of accepting a defective batch is ₹ 600.

A sample of 10 items has been selected from the shipment and two items are found to be defective. The conditional probabilities of getting 2 defectives in a sample of 10 items from a batch of 0%, 2% and 5% defectives are calculated as 0.083, 0.185 and 0.265 respectively. Determine whether the shipment should be accepted.

Solution: The following table summarizes the given information:

State (Per cent defectives)	Prior Probability	Conditional Probability	Action (Cost in ₹)	
			Accept	Reject
0	0.5	0.083	0	200
2	0.3	0.185	600	0
5	0.2	0.265	600	0

First the posterior probabilities are calculated. The necessary computations are given in the table below:

State	Prior Probability (1)	Conditional Probability (2)	Joint Probability (1) × (2) = (3)	Posterior Probability (4) = (3) ÷ 0.15
0	0.5	0.083	0.0415	0.277
2	0.3	0.185	0.0555	0.370
5	0.2	0.265	0.0530	0.353
Total			0.1500	

$\therefore E_1$ (accept) $= 0.277 \times 0 + 0.370 \times 600 + 0.353 \times 600 = ₹433.8$

E_2 (reject) $= 0.277 \times 200 + 0.370 \times 0 + 0.353 \times 0 = ₹55.40$

Since E_2 results in lower cost, the company should reject the shipment.

3.2. DECISION TREES

3.2.1. Introduction

The graphic representation of the decision-making process that signifies decision alternatives, states of nature, probabilities assigned to the states of nature and provisional profits and losses is termed as **decision tree.**

It is one of the instruments that represent pictorial presentation of sequential and multi-dimensional form of a specific decision problem for analysing and evaluating it efficiently. The decision tree consists of nodes and branches. There are of two types of nodes, decision node and chance node. Course of action (or strategies) originate from the decision nodes as the main branches. There is a chance node at the closing of each main branch. Chance events in the form of sub branches emanates from the chance nodes. Sub branches display the various payoffs and the probabilities associated with alternative courses and the chance events. The expected value of the outcome is shown at the terminal of the sub branches.

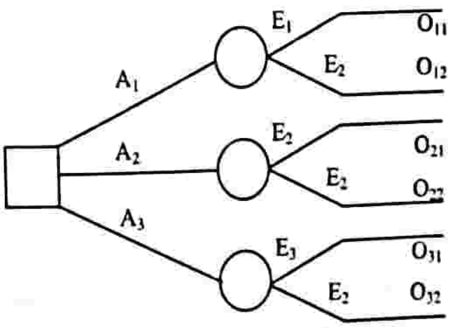


Figure 3.3 Decision Tree

For example, the **figure 3.3** represents the decision tree. Here A_1, A_2, A_3, A_4 represent strategies and E_1, E_2, E_3 represent events. Outcomes are represented by O_{11}, O_{12}, O_{21}, O_{22}, O_{31}, O_{32}

When there are multistage situations then a decision tree is extremely useful to the decision maker, because it involve a series of decisions which are dependent on the previous one. The general approach is applicable in the analysis of the decision tree which works in backward, through the tree from right to left by computing the expected value of each chance node. Then the particular branch is chosen by leaving a decision node, leading to the chance node having highest expected value. It is also known **as roll back** or **fold back process**.

3.2.2. Steps in Decision Tree Analysis

The steps involved in decision tree analysis are shown in **figure 3.4**.

Step 1) Systematic identification of the points for decision and the alternative course of action at each decision point.

Step 2) Determination of the probabilities and payoffs at each and every point that is associated with it.

Step 3) Computation of the expected payoffs (EMV) for each course of action starting from the extreme right end.

Step 4) Course of action yielding the best payoff for every particular decision should be selected.

Step 5) Proceed backwards to the next stage of decision points.

Step 6) Unless the first decision point has reached, above steps should not be repeated.

Step 7) Considering the whole situation, the final identification of the course of action is suitable from the beginning to the end for different possible outcomes.

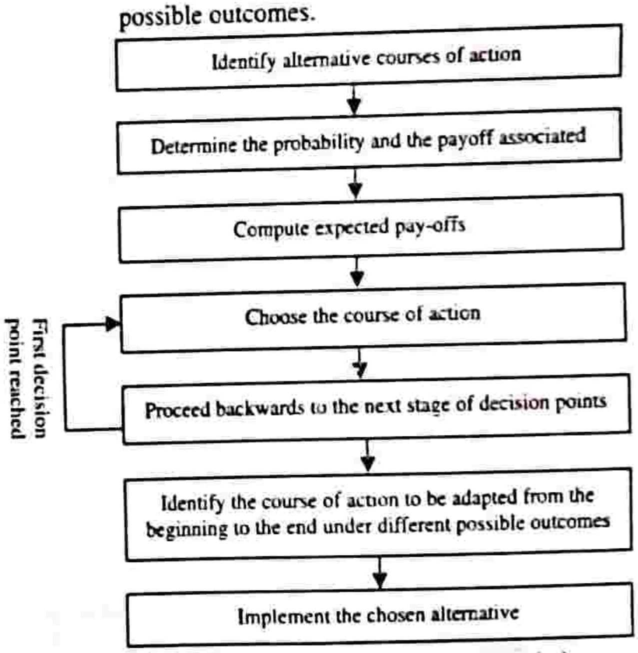


Figure 3.4 Steps in Decision Tree Analysis

3.2.3. Advantages of Decision Tree

1) Structuring the process of decision and helping to take decisions in an organised, systematic and sequential way.
2) The decision maker requires to scrutinise all possible outcomes either desirable or undesirable.
3) The decision making process is communicated to others in a simple and clear manner by demonstrating every assumption about the future.
4) Logical relationship between the parts of the complex decision is displayed, and identification of the time series under which various actions and successive events may occur is known.
5) It is very useful in the situations where successive decisions are affected by the initial decision and its outcome. It can be used in various fields like, introducing a new product, marketing, make or buy decisions, investment decisions etc.

3.2.4. Disadvantages of Decision Tree

1) Due to an increase in the number of decision alternatives and introduction of more variable, decision tree diagrams become more complex.
2) Due to the simultaneous presence of interdependent alternatives and dependent variables in the problem, it becomes highly complex.
3) It assumes that utility of money is linear.
4) The solution of an average value is yielded when the problem is analysed in terms of expected values.
5) Discrepancy is often found in assigning the probabilities for different events.

3.2.5. Applications of Decision Tree

The following are some general uses of tree-based analysis:

1) **Segmentation:** Recognise people who want to be members of a specific class.
2) **Stratification:** Assign cases into one of different categories, like high, medium, and low-risk groups.
3) **Prediction:** Creating rules and using them to predict future events.

   Prediction can also mean attempts to relate predictive attributes to values of a continuous variable.
4) **Data Reduction and Variable Screening:** Select a useful subset of predictors from a large set of variables for use in building a formal parametric model.
5) **Interaction Identification:** Identify relationships that relate only to particular subgroups and specify these in a formal parametric model.

Example 22: A firm owner is seriously considering of drilling a farm well. In the past, only 70% of wells drilled were successful at 200 feet of depth in the area. Moreover on finding no water at 200ft., some persons drilled it further upto 250ft but only 20% struck water at 250ft. The prevailing cost of drilling is ₹ 50 per foot. The farm owner has estimated that in case he does not get his own wells he will have to pay ₹ 15,000 over the next 10 years, in PV term, to buy water from the neighbor.

The following decisions can be optimal.

1) Do not drill any well
2) Drill up to 200ft and
3) If no water is found at 200ft. drill further up to 250ft.

Draw an appropriate decision tree and determine the farm owner's strategy under E.M.V. approach.

Solution: Figure 3.5 represents the decision tree diagram for the problem.

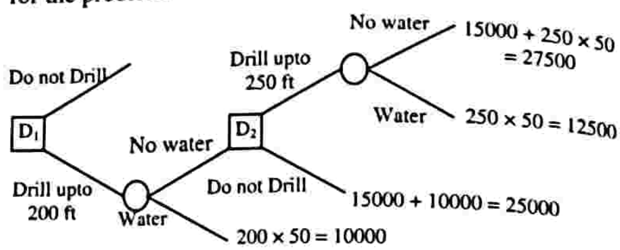


Figure 3.5: Decision Tree

At D_2 point

Decision:	a) drill upto 250feet	b) Do not drill
Event:	a) No water	b) Water

Probabilities are 0.2, 0.8

EMV for drill upto 250 feet = (12500 × 0.2) + (27500 × 0.8) = 24500

EMV for do not drill = 25000 (from the tree)

EMV is smaller for the act drill up to 250 feet. So it is optimal act.

At D_1 point

The decisions are drill upto 200 feet and do not drill. Events are same as those of D_2 point.

Probabilities are 0.7, 0.3.

EMV for drill upto 200 feet = (10000 × 0.7) + (24500 × 0.3) = 14350

EMV for do not drill = 15,000 from the tree.

The optimal decision is drill upto 200 feet (as the EMV is small).

Therefore Combining D_1 and D_2 the optimal strategy is to drill the well upto 200 feet and if no water is struck, then further drill it upto 250 feet.

Example 23: A large steel manufacturing company has three options with regard to production:

1) Produce commercially 2) Build pilot plant 3) Stop producing steel.

The management has estimated that their pilot plant, if built, has 0.8 chance of high yield and 0.2 chance of low yield. If the pilot plant does show a high yield, management assigns a probability of 0.75 that the commercial plant will also have a high yield. If the pilot plant shows a low yield, there is only a 0.1 chance that the commercial plant will show a high yield. Finally, management's best assessment of the yield on a commercial-size plant without building a pilot plant first has a 0.6 chance of high yield. A pilot plant will cost ₹3,00,000. The profits earned under high and low yield conditions are ₹1,20,00,000 and – ₹12,00,000 respectively. Find the optimum decision for the company.

Solution: Figure 3.6 represents the decision tree diagram for the problem.

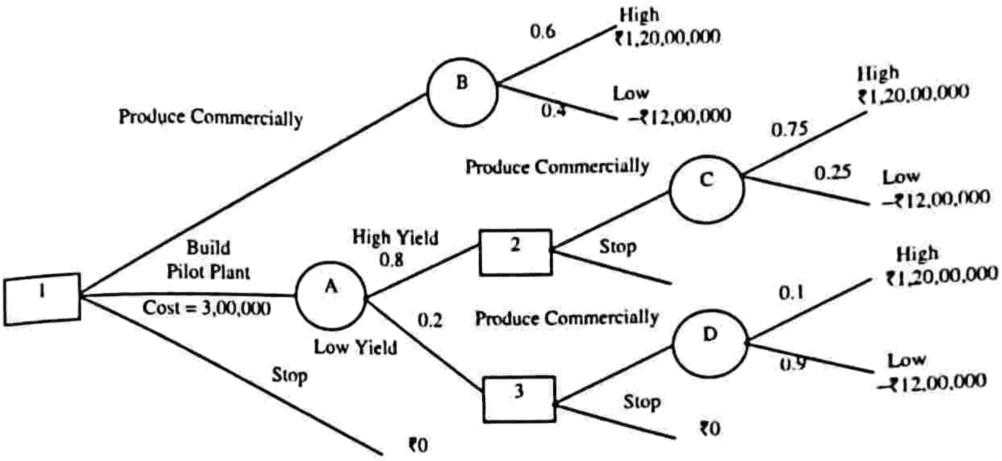


Figure 3.6: Decision Tree

EMV of chance node C = ₹ [0.75 × 1, 20, 00,000 – 0.25 × 12, 00,000] = ₹ [90, 00,000 – 3, 00,000] = ₹ 87, 00,000.

EMV of chance node D = ₹ [0.1 × 1, 20, 00,000 – 0.9 × 12, 00,000] = ₹ [12, 00,000 – 10, 80,000] = ₹ 1, 20,000.

EMV of decision node 2 = ₹ 87, 00,000.

EM V of decision node 3 = ₹ 1, 20,000.

EMV of chance node A = ₹ [0.8 × 87, 00,000 – 0.2 × 1, 20,000] = ₹ [69, 60,000 – 24,000] = ₹ 69, 36,000.

EMV of decision node 1 if pilot plant is built = ₹ 69, 36,000 – ₹3, 00,000 = ₹ 66, 36,000.

EMV of chance node B = ₹ [0.6 × 1, 20, 00,000 – 0.4 × 12, 00,000]= ₹ [72, 00,000 – 4, 80,000] = ₹ 67, 20,000.

EMV of decision node 1 for alternative 'produce commercially' = ₹ 67, 20,000.

∴ The company should not build the pilot plant but should produce commercially.

Example 24: Amar Company is currently working with processes, which after paying for materials, labour, etc., brings profit of ₹12,000.

The following alternatives are made available to the company:

1) The company can conduct research (R_1) which is expected to cost ₹10,000 having 90% changes of success. If it proves a success, the company gets a gross income of ₹25,000.
2) The company can conduct research (R_2), which is expected to cost ₹8,000 having a probability of 60% success, the gross income will be ₹25,000.
3) The company can pay ₹6,000 as royalty for a new process which will bring a gross income of ₹20,000.
4) The company continues the current process.

Because of limited resources, it is assumed that only one of the two types of research can be carried out at a time. Use decision tree analysis to locate the optimal strategy for the company.

Solution: As per the data given in the problem, we can draw the decision tree as shown in **figure 3.7**, to calculate EMV (Expected Monetary Value).

Decisions	Events	Probability	Income (Gross)	Income (Expected)
Conducting R_1	Success	0.9	25,000	22,500 – 10,000 = 12,500 (EMV)
	No success	0.1	0	
Conducting R_2	Success	0.6	25,000	15,000 – 8,000 = 7,000 (EMV)
	Failure	0.4	0	
Pay royalty	–	1.0	20,000	20,000 – 6,000 = 14,000 (EMV)
Continue	–	1.0	12,000	12,000 (EMV)

As the EMV of pay royalty is the highest (i.e., ₹14,000), this is the optional decision.

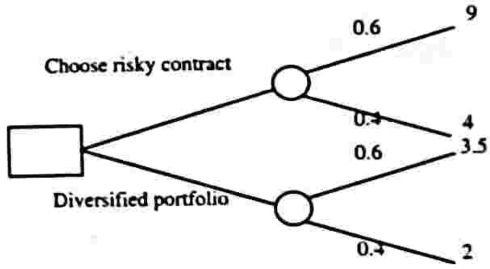


Figure 3.7

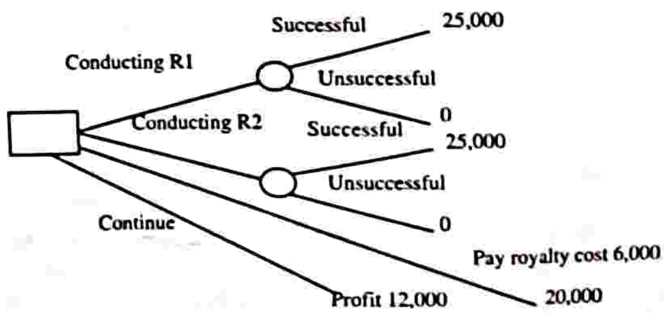


Figure 3.8

Example 25: A person has two independent investment – A and B available to him but he can undertake only one at a time due to certain constraints. He can choose A first and then stop or if A is successful, then take B or *vice versa*. The probability of success of A is 0.6 while for B it is 0.4. Both the investments require an initial capital outlay of ₹10,000 and both return nothing if the venture is unsuccessful. Successful completion A will return ₹20,000 (over cost) and successful completion of B will return ₹24,000 (over cost). Draw decision tree and determine the best strategy.

Solution: The decision tree for this problem is shown in **figure 3.9**:

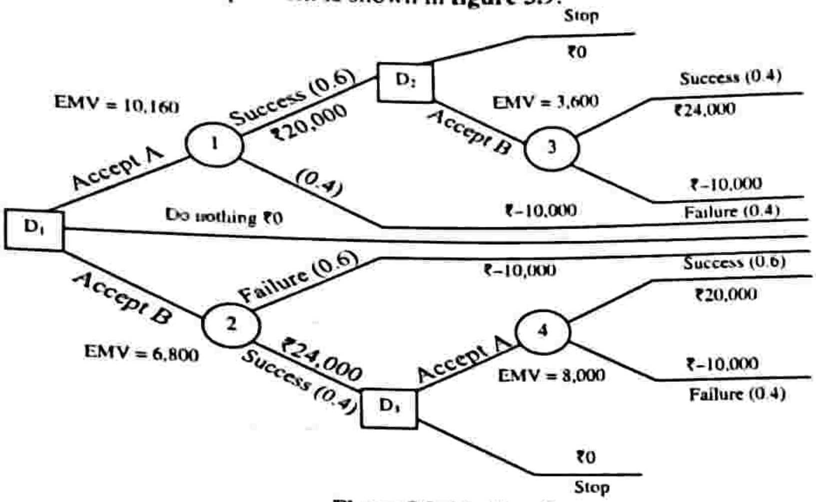


Figure 3.9: Decision Tree

Table 3.24 shows the determination of best strategy:

Table 3.24: Evaluation of Decision and Chance Nodes

	Decision Point	Outcome	Probability	Conditional Value (₹)	Expected Value
D_3	1) Accept A	Success	0.6	20,000	12,000
		Failure	0.4	−10,000	−4,000
					8,000
	2) Stop	–	–	–	0
D_2	1) Accept B	Success	0.5	24,000	9,600
		Failure	0.6	−10,000	−6,000
					3,600
	2) Stop	–	–	–	0
D_1	1) Accept A	Success	0.6	20,000 + 3,600	14,160
		Failure	0.4	−10,000	−4,000
					10,160
	2) Accept B	Success	0.4	24,000 + 8,000	12,800
		Failure	0.6	−10,000	−6,000
					6,800
	3) Do nothing	–	–	–	0

From above it is shown that the EMV is the largest at node 1, thus initially the course of action A is to be accepted by best strategy at node D_2 and in case A is successful, then it accept course of action B.

3.3. GAME THEORY

3.3.1. Concept

When two or more rational opponents are engaged under a situation of conflict and competition, then in this condition, the study of decision-making is known as Game Theory. In addition, one can say that a strategic decision-making is called as game theory. In other words, game theory is the "collaboration between intelligent rational decision-makers and mathematical model of conflicts". To find out the guidelines of rational behaviour in the game conditions is the main objective of the game theory. In this action, interdependent players define the outcomes. Generally, economics, political science, psychology, logic and biology use the concept of game theory.

3.3.2. Game

In a contest of game, there are more than one decision makers and every decision maker wishes to win. In a game theory, more than two persons perform an action on the basis of some set of instructions (rules) and at the end of the game every person gets few advantages or satisfaction or undergo with some losses. Under some pre-determined conditions when two or more parties are playing in a conflict situation, then it is characterised as a Game.

Characteristics of a Game

A competitive condition is known as Game when it satisfies the following properties:

1) **Finite:** There are a finite number of participants or competitors.
   i) In case of two competitors, it is known as two person game.
   ii) When number of competitors are more than two then it is known as n-person game.

2) **Series of Plans:** Every competitor has a finite number of possible courses of action (group of plans) which have various choices to perform his appropriate actions, yet every competitor's course of action is not the same.

3) **Conflict of Interests:** Between the participants there are conflict of interests.

4) **Well Known:** All participants know the principle rules. From the available list of courses, when players select a single course of action, then they can play the game.

5) **Choices:** Before selecting the course of action, no competitor knows the opponent's choice, because it is assumed that all the participants' choices are done simultaneously.

6) **Outcome:** The outcome of all specific set of choices by all the competitors is known in advance and defined numerically. The outcome of all combinations results in a gain to each player (positive, negative or zero). The negative gain is considered as a loss.

3.3.3. Competitive Situations

When two or more competitors (with different interests) have the dependent actions (such as one competitor action depends upon the other competitor action) then this condition is known as competitive situation. When competitors try to resolve the conflict of interest in their favour with a competitive situation and every competitor acts in a rational manner then this competitive situation is known as **competitive game**. Game theory deals with the decision-making under uncertainty (known as competitive situation).

There are three categories in which all the decision-making conditions exist:
1) Decision-making under certainty,
2) Decision-making under risk, and
3) Decision-making under uncertainty.

Figure 3.10 shows all the probabilities of decision-making under uncertainty:

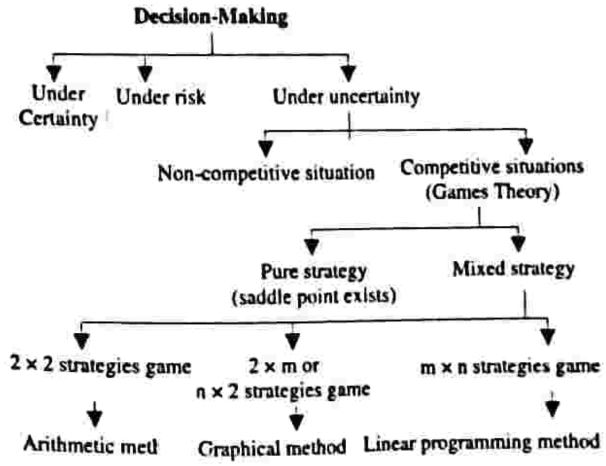


Figure 3.10

3.3.4. Terminology of Game

1) **Sum of Gains and Losses:** A game is said to be a zero-sum game if gains of one competitor is exactly equal to the losses of other competitors. In this case, gains and losses are equal to the zero. If the sum of gains and losses are not equal to zero then the game is known as non-sum zero game.

2) **Chance of Strategy:** When activities are measured by skills in a game then it is known as game of strategy. If the strategies are measured by chance then it is known as game of chance. So, one can say that a game strategy can be measured by chance as well as by skill.

3) **Number of Players:** In a game, decision is taken by a competitor also known as an agent. A game is known as two person game if there are two competitors, involved in the game, then it is known as 'n' person game (two or more than two competitors involved).

4) **Number of Activities:** Finite and infinite numbers of activities are involved.

5) **Number of Alternatives (choices) Available to each Person:** If finite numbers of activities (have finite number of alternatives) are involved in a game then the game is known as finite game else infinite.

6) **Payoff:** A payoff is a measure of satisfaction, which a competitor gets at the end of the game. In the game, it is an actual valued function of variables. Assume, if player P_i have v_i payoff, then:

$1 \le i \le n$, in an n-person game and

If $\sum_{i=1}^{n} v_i = 0$, then the game is known as a zero-sum game.

7) **Fair Game:** A game is known as fair game if maximum value = minimum value = 0.

8) **Saddle Point:** In a payoff matrix, a saddle point is the one which is the greatest value in its column and lowest value in its row.

3.3.5. Assumptions of Game

1) Every competitor has a limited set of possible course of approaches and actions.
2) Total number of participants is known before starting the competition.
3) Every participant wants to minimise the losses and maximise the gains.
4) In the absence of direct communication, individual participant take his decisions.
5) One competitor (say A) gains or another competitor (say B) losses and *vice versa*.
6) Competitions are rational and intelligent.
7) Every competitor takes individual decision.
8) Every competition's solution depends on the strategies, which are followed by the competitors, as it is mandatory that every competitor has the perfect knowledge of the game.
9) For every competitor, decision of the game can be positive, negative or zero.

10) For the different courses of the competitors pay-offs (payments to be made in settlement of the game) are predetermined (assumed).
11) Pay-off will be determined on the basis of some predetermined set of rules.
12) Every combination of course of action determines an outcome which results in a gain to each player.
13) Before starting the game, every competitor's gain and loss are fixed.

3.3.6. Saddle Point

A combination of strategies in which each player can find the highest possible payoff (assuming the best possible play by the opponent) is called saddle point. When each player is achieving the highest possible payoff then saddle point occurs and hence neither would benefit from changing strategies if the other did not changes.

An element of the matrix which is both the smallest element in its row and the largest element in its column is consider as saddle point. In game theory, saddle point is considered as an equilibrium point. A position in the payoff matrix is called saddle point where the maximum of row minima coincides with the minimum of the column maxima. The payoff at the saddle point is called the **value of the game**. If there is one saddle point then it means there is existence of pure strategies. On the basis of maximin and minimax concept, we have another classification of games:

1) If maximin value = minimax value = 0 then a game is said to be **fair.**
2) If maximin value = minimax value ≠ 0 then a game is said to be **strictly determinable.**

It is possible that there may be more than one saddle point or there might be none. If there is more than one saddle then it means there is absence of pure strategies. A player has the choice of adopting more than one strategies when there are more than one saddle points. **For example,** let consider the following table that shows the two-person-zero-sum game:

		Player B		
		B_1	B_2	B_3
Player A	A_1	5	9	3
	A_2	6	−12	−11
	A_3	7	12	9

The saddle point can be calculated as below:

	B_1	B_2	B_3	Row Min
A_1	5	9	3	3
A_2	6	−12	−11	−12
A_3	(7)	12	9	7
Column Max	7	12	9	

From the table, it is clear that saddle point is (A_3, B_1).

3.3.7. Value of Game

The amount which is obtained by each player with the use of his appropriate mix of strategies, i.e., plays according to his maximin mixed strategy is called value of the game. In two person zero sum game with payoff matrix $[a_{ij}]_{m \times n}$, the payoff which is determined with the use of maximin principle is called the lower value of the game and is represented by $\underline{v}$. The payoff which is determined with the use of minimax principle is called the upper value of the game and is represented by $\overline{v}$.

The value of game is the payoff which is determined by the best moves of the maximising player and minimising player.

The value of a game lies between the lower and upper values and it will be different from the lower value as well as from the upper value. If the game has the saddle point then pure strategies will be the best strategies which are associated with the corresponding saddle point. The amount at the point of intersection of pure strategies is the value of the game.

For example, let consider the following payoff matrix:

		Player B			
		B_1	B_2	B_3	B_4
Player A	A_1	1	2	3	1
	A_2	3	5	5	7
	A_3	0	3	4	2

The saddle point and value of game can be calculated as follows:

		Player B				
		B_1	B_2	B_3	B_4	Row Min
Player A	A_1	1	2	3	1	1
	A_2	(3)	5	5	7	3
	A_3	0	3	4	2	0
	Column Max	3	5	5	7	

Maximin = max {1, 3, 0} = 3,
minimax = min {3, 5, 5, 7} = 3.

Hence the value of the game = 3.

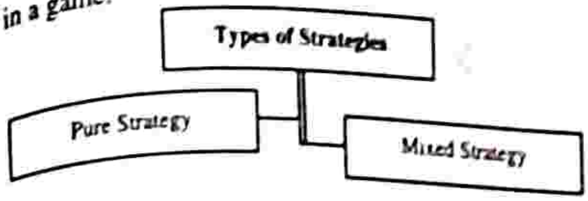
3.3.8. Strategies of Game

The strategy for a player is the list of all possible actions (or moves or courses of action) that he will take for every payoff (outcome) that might arise. It is assumed that the players know all the rules governing the choices in advance and it is represented in terms of numerical values (e.g. money, percent of market share or utility).

It is not necessary that players have definite information about each other strategies. The particular strategy (or complete plan) which is used by the player to optimise his gains or losses without knowing the competitor's strategies is called **optimal strategy**. The expected outcome which occurs when the player follows his/her optimal strategy is called **value of the game.**

Types of Strategies

Generally two types of strategies are taken by the players in a game:



3.3.8.1. Pure Strategy

This is a decision, which is used by the player to choose the particular course of action. Therefore, for fulfilling the objective of maximising gains as well as minimising losses, each player must select only one particular strategy in advance, out of other available strategies.

Pure strategy is such a strategy in which a player repeats the same strategy frequently. The main aim of pure strategy player is to maximise all gains and minimise all losses simultaneously.

3.3.8.2. Mixed Strategy

For winning the game if player is using more than one strategy instead of playing with one strategy then this strategy is known as mixed strategy. It is an activity which provides the probabilities to determine the decision of the player.

Mixed strategic game is defined as, the selection of particular course of action by both of the players with some fixed probability for particular game. Therefore it is such a probabilistic situation where the main objective of the player is to maximise gains and minimise losses by making a solution among pure strategies with fixed probabilities.

3.3.8.3. Difference between a Pure Strategy and a Mixed Strategy

The difference between a pure and mixed strategy is shown in **table 3.25**:

Table 3.25: Pure Strategy vs. Mixed Strategy

Pure Strategy	Mixed Strategy
It is the pre-determined course of action which is employed by the player.	In mixed strategy the player decides his course of action.
The opponent is sure of the course of action in the pure strategy.	The opponent cannot be sure of the course of action in the mixed strategy.
Objective of the players is to maximize gains or minimize losses.	Objective of the players is to maximize expected gains or to minimize expected losses by making a solution among pure strategies with fixed probabilities.

3.3.9. Solutions of Game

There may be games with saddle point or without saddle point and accordingly one has strategy for a game, i.e.,

1) For games with saddle point: **Pure Strategy**
2) For games without saddle point: **Mixed Strategy**

The steps to find the solution of game are shown in the **figure 3.11.**

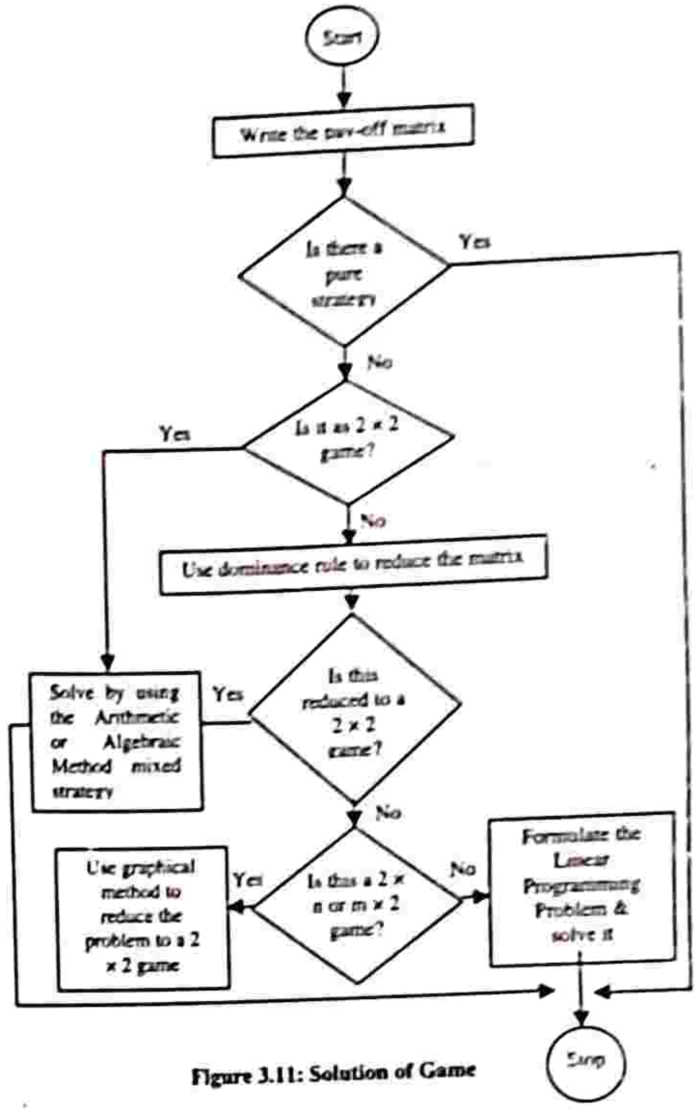


Figure 3.11: Solution of Game

3.3.10. Types of Game

Games can be of several types. Important ones are as follows:

1) **Two-Person Games and n-Person Games:** When there are two players or competitors involved in a game, it is called two-person game and when there are n persons involved in a game, the game is called n-person game.
2) **Zero Sums and Non-zero Sum Game:** A game where involvement of two players occur and a gain by one player must be matched by a loss by another player. Therefore, result is that the gain for one is always loss for other. In game theory, a non-zero sum game is a game where the aggregate of all gains and losses between the people involved in the game can be greater than or less than zero. It is necessarily not important that if there is a gain for one other will lose the game. Therefore, it can be said that in a game, each loss or gain of one is always related with its corresponding gain or loss of other, so that aggregate total always sums to zero.
3) **Games of Perfect Information and Games of Imperfect Information:** In a game theory if same strategy is accepted by either one of the player or by his competitor, then such games are known as game of perfect information. On the contrary, if in any game, neither player knows the whole situation nor he is guided about the real situation, such games are called games of imperfect information.

4) **Games with Finite, i.e., Limited Number of Moves (or Plays) and Games with Unlimited Number of Moves:** In games with limited number of moves, number of moves is limited to a fixed magnitude before game begins but in games with unlimited number of moves, it could be continued over an extended period of time and no limit is put on the number of moves.

5) **Constant-Sum Game:** A constant-sum game is the game where the sum of the payoffs across players is a constant, irrespective of the strategy chosen by the players. A constant-sum game is a special case of zero-sum game.

6) **2 × 2 Two-person Games and 2 × m and m × 2 Games:** The two person zero-sum game where only two choices are open for each player are called 2 × 2.

Player B

Player A		B_1	B_2
	A_1	2	5
	A_2	7	3

Two person games in which one of the player has more than two choices of rows and columns and other player has exactly two choices is referred to as m × 2 or 2 × m game respectively. **For example, table 3.25** represents 2 × 4 and **table 3.26** represents 4 × 2 game.

Table 3.25

Player B

Player A		B_1	B_2	B_3	B_4
	A_1	2	4	–1	0
	A_2	3	–2	2	5

Table 3.26

Player B

Player A		B_1	B_2
	A_1	4	–1
	A_2	3	4
	A_3	0	2
	A_4	2	–2

7) **3 × 3 and Larger Games:** The two person zero-sum game of size 3 × 3 or larger or we can say that the game in which players have three open choices are called 3 × 3 type game and simultaneously if players have more than three open choices to play then the game is known as larger size. 3 × 3 and larger games quite often present difficulties and in such situations linear programming as a solution method may allow us to solve for the optimum strategies.

8) **Negotiable (or Cooperative) and Non-negotiable (or Non-cooperative) Games:** There is no possibility of cooperation among the players in n-person and non-zero-sum games. According to this basis, anyone can divide such games into two parts-first in which players can negotiate and second in which negotiation among players is not permitted. The former types of games are known as negotiable games and the latter type of games are known as non-negotiable games.

3.3.11. Advantages of Game Theory

1) Game theory keeps deep insight to few less known aspects, which arise in situations of conflicting interests.
2) Game theory creates a structure for analysis of decision-making in various situations like interdependence of firms etc.
3) For arriving at optimal strategy, game theory develops a scientific quantitative technique for two person zero-sum games.

3.3.12. Disadvantages of Game Theory

1) The highly unrealistic assumption of game theory is that the firm has prior knowledge about its competitor's strategy and is able to construct the payoff matrix for possible solutions, which is not correct. The main fact is that any firm is not exactly aware of its competitor's strategy. He can only make guesses about its strategy.
2) The hypothesis of maximin and minimax clearly shows that players are not risk lover and have whole knowledge about the strategies but the fact is that it is not possible.
3) It is totally impractical to understand that the several strategies followed by the rival player against others lead to an endless chain.
4) Most economic problems occur in the game if many players are involved in comparison to two-person constant sum game, which is not easy to understand. For example, the number of sellers and buyers is quite large in monopolistic competition and the game theory does not provide any solution to it.
5) In real market situations, it is doubtful to find the use of mixed strategies for making non zero-sum games.

3.3.13. Applications of Game Theory

Some common applications of game theory are as follows:

1) **Political Science:** Application of game theory can be seen in various areas, where it plays an important role. It focuses on different areas of political science, i.e., fair division, political economy, public choice, positive political theory, and social choice theory. In above mentioned such areas researchers have created such game theoretic models where players are often voters, states, special interest groups and politicians.
2) **Economics and Business:** Game theory is especially used by economists for special analysis of various economic phenomena, i.e., auctions, bargaining, duopolies, fair division, oligopolies, social network formation, and voting systems.
3) **Biology:** In contrast to economics, in biology the payoffs for games are often interpreted as corresponding to fitness. Additionally the focus has been less on equilibrium that corresponds to a notion of rationality, instead of those that would be maintained by evolutionary forces. The best known equilibrium in biology is known as the Evolutionarily Stable Strategy or (ESS).
4) **Computer Science and Logic:** For computer science and logic, game theory plays a gradual and progressive role. Several logical theories have a basis in game semantics. In addition, computer scientists have used games to model interactive computations. A theoretical basis is provided by the game theory to the field of multi-agent systems.

   The first known use is to inform about how actual human populations behave. Some scholars believe that by finding the equilibriums of games they can predict how actual human populations will behave when confronted with situations analogous to the game being studied.

3.4. TWO PERSON ZERO-SUM GAME

3.4.1. Introduction

The situations when there are only two players in the game and the algebraic sum of gains and losses of all the players must be zero is called a **two person zero-sum game**. In other words, we can say that the gain of one player is exactly equal and balanced by the loss of other player.

For example, suppose that there are two firms A and B in an area, where both have been selling a competing product for a long period in the past and are now engaged in struggle for a larger share of the market

Now with the total market of a given size, any share of the market gained by one firm must be lost by the other and, therefore, the sum of the gains and losses is equal to zero.

3.4.2. Payoff Matrix

When players select their particular strategies then the payoffs (a quantitative measure of satisfaction a player gets at the end of the game) in terms of gains or losses can be represented in the form of a matrix and it is called payoff matrix.

The gain of one player is equal to the loss of other and *vice versa* because the game is zero-sum. In other words, one player's payoff table would contain the same amounts as in payoff table of other player with the sign changed. Thus it is sufficient to construct payoff only for one of the players.

A payoff matrix may be constructed using the following steps when a player A has m-courses of action and player B has n-courses:

1) Row designations for each matrix are the course of action available to A.
2) Column designations for each matrix are the course of action available to B.
3) With a two person zero sum game, the cell entries in B's payoff matrix will be the negative of the corresponding entries in A's payoff matrix and the matrices will be as shown below:

Table 3.27: A's Payoff Matrix

Player B (columns 1, 2, 3, ..., j, ..., n); Player A (rows 1, 2, ..., i, ..., m)

$$\begin{array}{c} 1\\2\\ \vdots \\ i \\ \vdots \\ m\end{array}\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n}\\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n}\\ \vdots & & & & & & \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in}\\ \vdots & & & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn}\end{bmatrix}$$

Table 3.27: B's Payoff Matrix

Player B (columns 1, 2, 3, ..., j, ..., n); Player A (rows 1, 2, ..., i, ..., m)

$$\begin{array}{c} 1\\2\\ \vdots \\ i \\ \vdots \\ m\end{array}\begin{bmatrix} -a_{11} & -a_{12} & -a_{13} & \cdots & -a_{1j} & \cdots & -a_{1n}\\ -a_{21} & -a_{22} & -a_{23} & \cdots & -a_{2j} & \cdots & -a_{2n}\\ \vdots & & & & & & \\ -a_{i1} & -a_{i2} & -a_{i3} & \cdots & -a_{ij} & \cdots & -a_{in}\\ \vdots & & & & & & \\ -a_{m1} & -a_{m2} & -a_{m3} & \cdots & -a_{mj} & \cdots & -a_{mn}\end{bmatrix}$$

For example, let consider the following pay-off matrix of player A:

		Player B I	Player B II
Player A	I	–2	–3
	II	–1	2
	III	1	3

From above pay-off matrix, it is clear that player A has three strategies and player B can have two strategies. If player A and B both opt for strategy I, then the pay-off to player A is – 2 (means loss of 2 units). In such case, player B gains of 2 units. Similarly if player A opts for strategy III and player B opts for strategy II then gains of player A is 3 units. The cell entries of player B's pay-off matrix will be negative of the corresponding cell entries of player A's pay-off matrix. The strategy can be determined on the basis of following:

$$\text{Max of A} = \max_i \{\min_j a_{ij}\}$$
$$\text{Min of B} = \min_j \{\max_i a_{ij}\}$$

Hence the pay-off matrix of player B will be as follows:

		Player A I	Player A II	Player A III
Player B	I	2	1	–1
	II	3	–2	–3

Note: The pay-off matrix will be of that player who exists on the left of the matrix.

3.4.3. Assumptions of Two Person Zero Sum Game

1) Each player has offered with him a finite number of possible courses of action. The list may not be same for each player.
2) Player A attempts to maximise gains and player B minimise losses.
3) The decisions of both players are made individually prior to the play with no communication between them.
4) The decisions are made simultaneously and also announced simultaneously so that neither player has an advantage resulting from direct knowledge of the other player's decision.
5) Both the players know not only possible payoffs to themselves but also that of each other.

3.4.4. Pure Strategy Game with Saddle Point: Minimax & Maximin Principles

The main problem for playing games is the selection of an optimal strategy by each player involved in the game, without knowing the strategy of its competitor's. Pay-off table of only one player is required to evaluate the decisions because pay-off for either player provides all the essential information. The pay-off table is constructed for the player whose strategies are represented by rows (say player A).

Now the objective of the study is to know how these players must elect their respective strategies so that they may optimize their pay-off. This decision making criterion is referred to as the **minimax-maximin principle**. Such principle in pure strategy games always lead to the best possible selection of a strategy for both players.

For example, if player A chooses his particular strategy then a minimum value in each row represents the least gain (payoff). These values are written in the matrix by row minima.

Player A must select the strategy that gives largest gain among the row minimum values. The choice which is selected by the player A is called maximin principle and the corresponding gain is called the maximin value of the game. If player B chooses his particular strategy then a maximum value in each column represents the maximum loss for player B. These values are written in the matrix by column maxima. Player B must select the strategy that gives minimum loss among the column maximum values. The choice which is selected by the player B is called minimax principle and the corresponding loss is called the minimax value of the game.

Saddle Point

A point in a payoff matrix where the maximum of row minima coincides with the minimum of the column maxima is referred as saddle point (or Equilibrium point). The payoff at the saddle point is called the value of the game and is obviously equal to the maximin and minimax values of the game.

Rule to Determine Saddle Point

1) From each row of the payoff matrix, select the minimum (lowest) element and write them under 'row minima' heading. Now select a largest element from these minimum values and enclose it in a rectangle.
2) From each column of the payoff matrix, select the maximum (largest) element and write them under 'column maxima' heading. Now select a lowest element from these maximum values and enclose it in a circle.
3) Find the element(s) which has the same value in the circle as well as rectangle. Mark the position of such element(s) in the matrix. This element represents the value of the game and is called the saddle (or equilibrium) point.

Example 26: Find optimum strategies for players A and B and also determine the value of the game for the game shown below:

	B_1	B_2	B_3	B_4
A_1	5	4	8	5
A_2	– 4	– 3	12	9
A_3	8	3	– 1	– 5
A_4	3	– 1	2	3

Solution: Applying minimax criteria on the game, we first choose the minimum element of rows and enclose it with circle. Next, select maximum element of columns and enclose it with the square as shown in table below:

	B_1	B_2	B_3	B_4	Row Minima
A_1	5	4	8	5	4
A_2	–4	–3	12	9	–4
A_3	8	3	–1	–5	–5
A_4	3	–1	2	3	–1
Column Maxima	8	4	12	9	

From above, it is shown that saddle point is (A_1, B_2). Hence player A chooses strategy A_1 while player B chooses the strategy B_2 and value of the game is 4.

Example 27: Find the saddle point for the following game:

		Player B		
		I	II	III
Player A	I	–2	14	–2
	II	–5	–6	–4
	III	–5	20	–8

Solution: For calculating the saddle point of the above game, first determine smallest element of each row and enclose it with circle, similarly determine the largest element of each column and enclose it with square as shown below:

		Player B			
		I	II	III	Row Minima
Player A	I	–2	14	–2	– 2
	II	–5	–6	–4	– 6
	III	–5	20	–8	– 8
	Column Maxima	– 2	20	– 2	

From above, it is clear that there are two saddle points exists in the cells (I, I) and (I, III). It means that player A uses strategy I and the player B uses two optimum strategies I and III. Hence the values of the game in both cases are –2.

Example 28: A company's management and the labour union are negotiating a new three year settlement. Each of these has 4 strategies:

1) Hard and Aggressive Bargaining
2) Reasoning and Logical Approach
3) Legalistic Strategy
4) Conciliatory Approach

The costs to the company are given for every pair of strategy choice.

		Company Strategies			
		I	II	III	IV
Union Strategy	I	20	15	12	35
	II	25	14	8	10
	III	40	2	10	5
	IV	–5	4	11	0

What strategy will be the two sides adopt? Also, determine the value of the game.

Solution: The two sides adopt minimax and maximin principles of game theory. Select the row minimum and enclose it by a rectangle. Then, select the column maximum and enclose it in a circle.

		Company Strategies I	II	III	IV	Row Minima
Union Strategies	I	20	15	12	35	12
	II	25	14	8	10	8
	III	40	2	10	5	2
	IV	-5	4	11	0	0
	Column Maxima	40	15	12	35	

Hence, the saddle point is (I, III) and the valve of the game is 12. Hence, the company will always adopt strategy III – Legalistic strategy and union will always adopt strategy I – Hard and aggressive bargaining.

Example 29: Find the value of the game $\begin{pmatrix} 2 & 6 \\ -2 & \mu \end{pmatrix}$ for any values of μ.

Solution: Consider the given pay-off matrix, find the maxmin and minmax values, by ignoring λ for the time being:

		Player B B_1	B_2	Row Minima
Player A	A_1	2	6	2
	A_2	−2	μ	−2
	Column Maxima	2	6	

Maxi (minimum) = Max (2, −2) = 2
Mini (maximum) = Min (2, 6) = 2
Maximin = Minimax = 2
So saddle point exists.

This implies that there is no restriction on μ. That is μ can take any value. Since, Maximin = Minimax = 2. Therefore the value of the game is 2.

Example 30: Find the Saddle point and hence solve the following game:

$$\text{Player B} \quad \begin{array}{c} \\ A1 \\ A2 \\ A3 \end{array} \begin{array}{c} \text{Player B} \\ \begin{array}{ccc} B1 & B2 & B3 \end{array} \\ \begin{pmatrix} 15 & 2 & 3 \\ 6 & 5 & 7 \\ -7 & 4 & 0 \end{pmatrix} \end{array}$$

Solution: Using Minimax criteria, we have following payoff matrix:

	B_1	B_2	B_3	Row Minima
A_1	15	2	3	2
A_2	6	5	7	5 ← Maximin
A_3	−7	4	0	−7
Column Maxima	15	5 ↑ Minimax	7	

Therefore A uses A_2 and B uses B_2, and value of game is 5.

3.4.5. Mixed Strategy Game without Saddle Points

Pure strategies are used as optimal strategies only for those games which have a saddle point. Mixed strategies are used for solving a game which does not have a saddle point.

3.4.5.1. Convex Linear Combination

A set $X \subset R^n$ is convex if the point $\lambda x + (1 - \lambda)y$ is an element of X for any $x, y \in X$ and $\lambda \in [0, 1]$. The point $\lambda x + (1 - \lambda)y$ is known as weighted average (or a convex combination) of x and y.

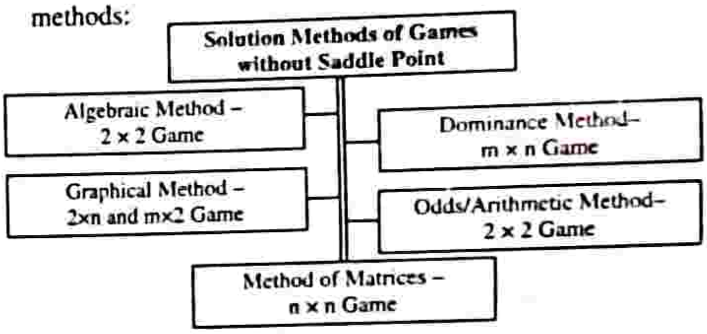
For example, the set [0, 1] is convex as the every point between the two points also belongs to the set. But X = [0, 1/4] ∪ [3/4, 0] is not convex as point $\lambda/4 + (1 - \lambda)\, 3/4 \notin X$ for every $\lambda \in (0, 1)$. If there are no holes in the resultant set then set shows the convexity. Let consider that we have numbers $y_1, y_2, \ldots, y_n$. Now a convex combination of these numbers can be represented as weighted sum of the form:

$$\lambda_1 y_1 + \lambda_2 y_2 + \ldots + \lambda_n y_n, \text{ where}$$

$$0 \le \lambda_i \le 1 \text{ for } i = 1, 2, \ldots, n; \sum_{i=1}^{n} \lambda_i = 1$$

3.4.5.2. Solution Methods of Games without Saddle Point

A mixed strategy game can be solved by the following methods:



3.4.5.3. Algebraic Method – 2 × 2 Game

Let us consider a 2 × 2 two-person zero sum game without any saddle point having the payoff matrix for player A.

$$\begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{array}$$

The optimum mixed strategies

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix} \text{ and } S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$$

Where,

$$p_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}, \; p_1 + p_2 = 1 \Rightarrow p_2 = 1 - p_1$$

$$q_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}, \; q_1 + q_2 = 1 \Rightarrow q_2 = 1 - q_1$$

The value of the game is $v = \dfrac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$

Example 31: Determine the optimum strategies and value of the game for the game whose payoff matrix is shown below:

$$A\overset{B}{\begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}}$$

Solution: There is not saddle point exist in this pay-off matrix. Hence the players use mixed strategies:

Let consider 2 × 2 game which does not contain saddle point and whose pay-off matrix is as below:

$$\begin{array}{c} \\ A_1 \\ A_2 \end{array}\begin{array}{c} B_1 \quad B_2 \\ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{array}$$

Now the optimum mixed strategies will be as follows:

$$S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix} \text{ and } S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}$$

Where,

$$p_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{4-3}{(5+4)-(2+3)} = \frac{1}{4}$$

$$p_2 = 1 - p_1 \Rightarrow p_2 = 1 - \frac{1}{4} = \frac{3}{4}$$

$$q_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{4-2}{(5+4)-(2+3)} = \frac{2}{4} = \frac{1}{2}$$

$$q_2 = 1 - q_1 \Rightarrow q_2 = 1 - \frac{1}{2} = \frac{1}{2}$$

Value of game

$$v = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{(5\times4)-(2\times3)}{(5+4)-(2+3)} = \left(\frac{20-6}{9-5}\right) = \frac{7}{2}$$

Hence the optimum mixed strategies are as follows:

$$S_A = \left(\frac{1}{4}, \frac{3}{4}\right); \quad S_B = \left(\frac{1}{2}, \frac{1}{2}\right)$$

$$\text{Value of game} = \frac{7}{2}$$

Example 32: Find the value of the following game:

		B 1	B 2
A	1	6	9
	2	8	4

Solution: There is not saddle point exist in this pay-off matrix. Hence the players use mixed strategies:

Let consider 2 × 2 game which does not contain saddle point and whose pay-off matrix is as below:

$$\begin{array}{c} \\ A_1 \\ A_2 \end{array}\begin{array}{c} B_1 \quad B_2 \\ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{array}$$

Now the optimum mixed strategies will be as follows:

$$p_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{4-8}{(6+4)-(9+8)} = \frac{-4}{10-17} = \frac{4}{7}$$

$$p_2 = 1 - p_1 = 1 - \frac{4}{7} = \frac{3}{7}$$

$$q_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{4-9}{(6+4)-(9+8)} = \frac{-5}{10-17} = \frac{5}{7}$$

$$q_2 = 1 - q_1 = 1 - \frac{5}{7} = \frac{2}{7}$$

Value of game

$$v = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{(6\times4)-(9\times8)}{(6+4)-(9+8)} = \frac{24-72}{10-17} = \frac{48}{7}$$

Example 33: In a game of matching coins player A win ₹2 if there are two heads, wins nothing if there are two tails and loses ₹1 when there are one head and one tail. Determine the pay-off matrix, best strategies for each player and value of game.

Or

Find the optimal strategies and game value of the following game.

		Player B H	Player B T
Player A	H	2	−1
	T	−1	0

Solution: According to the given data the pay-off matrix corresponding to the given game is as follows:

		Player B H	Player B T
Player A	**H**	2	−1
	T	−1	0

Since the above game has no saddle point we have to solve it for optimum mixed strategies.

Let $S_A = \begin{bmatrix} H & T \\ p_1 & p_2 \end{bmatrix}$, $S_B = \begin{bmatrix} H & T \\ q_1 & q_2 \end{bmatrix}$ be the optimum mixed strategies of player A and B respectively.

Let v be the value of the game.

$$\Rightarrow p_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{0-(-1)}{(2+0)-(-1-1)} = \frac{1}{4}$$

$$\Rightarrow p_2 = 1 - p_1 = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\Rightarrow q_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{0-(-1)}{(2+0)-(-1-1)} = \frac{1}{4}$$

$$\Rightarrow q_2 = 1 - q_1 = 1 - \frac{1}{4} = \frac{3}{4}$$

Value of the Game

$$(v)=\frac{a_{11}a_{22}-a_{12}a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})}=\frac{(2)(0)-(-1)(-1)}{(2+0)-(-1-1)}=\frac{-1}{4}$$

3.4.5.4. Dominance Rule – m × n Game

If a two person zero sum game has no saddle point then the dominance principle is used to reduce the size of the pay-off matrix. With the use of dominance rule one can obtain a 2 × 2 game and then solve it with the help of algebraic method.

Generally, it is possible to find an entire row (or column) which is avoided by the player when there is another row (or column) and this is better for him or her to play. In that case, avoided row (or column) is called the dominated row (or column):

1) If all the elements of a column (say i^{th} column) are less than or equal to the corresponding elements of any other column (say j^{th} column), then j^{th} column is dominated by the i^{th} column. The j^{th} column is thus removed from the pay-off table.
2) If all the elements of a row (say i^{th} row) are less than or equal to the corresponding elements of any other row (say j^{th} row), then j^{th} row dominates the i^{th} row. The i^{th} row is thus removed from the pay-off table.
3) A pure strategy of a player may also be dominated if it is inferior to some convex combination of two or more pure strategies in particular or inferior to the average of two or more pure strategies. In that case, one deletes that pure strategy.
4) It is also possible that a particular row (or column) dominates the average of two other rows (or columns). In that case, one will delete any one row (column), which was involved in finding the average.

This is to remember that the rows and columns which are once deleted will never be used in the determination of optimum strategy for both the players. For reducing the size of a game, dominance rule must be applied before evaluating it.

Example 34: Solve the following game using dominance rule:

		Player B	
		B_1	B_2
	A_1	1	3
	A_2	4	5
Player A	A_3	9	−7
	A_4	−3	−4
	A_5	2	1

Solution: The second row dominates on the first row, i.e., elements of second row are greater than first row. So, eliminating the first row, we get the following matrix:

	B_1	B_2
A_2	4	5
A_3	9	−7
A_4	−3	−4
A_5	2	1

Now, the first row dominates on the fourth row. Hence eliminating fourth row, we get the following matrix:

	B_1	B_2
A_2	4	5
A_3	9	−7
A_4	−3	−4

The first row dominates on the third row, so reduced matrix will be as follows:

	B_1	B_2
A_2	4	5
A_3	9	−7

Hence the reduced payoff matrix of order 2 × 2 is as follows:

Player B

$$\text{Player A}\begin{bmatrix}4 & 5\\ 9 & -7\end{bmatrix}$$

Let consider the optimum mixed strategies for player A and B is in the following form:

$$S_A=\begin{bmatrix}A_2 & A_3\\ p_1 & p_2\end{bmatrix}, p_1+p_2=1$$

$$\text{and } S_B=\begin{bmatrix}B_1 & B_2\\ q_1 & q_2\end{bmatrix}, q_1+q_2=1$$

Where,

$$p_1=\frac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})}, p_1+p_2=1\Rightarrow p_2=1-p_1$$

$$q_1=\frac{a_{22}-a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})}, q_1+q_2=1\Rightarrow q_2=1-q_1$$

$$p_1=\frac{-7-9}{(4-7)-(5+9)}=\frac{-16}{-17}=\frac{16}{17},\ p_2=1-\frac{16}{17}=\frac{1}{17},$$

$$q_1=\frac{-7-5}{(4-7)-(5+9)}=\frac{-12}{-17}=\frac{12}{17},\ q_2=1-\frac{12}{17}=\frac{5}{17}$$

The optimum strategy of the given payoff matrix is given by the following:

$$S_A=\begin{bmatrix}A_2 & A_3\\ \frac{16}{17} & \frac{1}{17}\end{bmatrix}, S_B=\begin{bmatrix}B_1 & B_2\\ \frac{12}{17} & \frac{5}{17}\end{bmatrix}$$

The value of the game (v)

$$=\frac{a_{11}a_{22}-a_{12}a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$$

$$=\frac{(4\times-7)-(5\times9)}{(4-7)-(5+9)}=\frac{-28-45}{-17}=\frac{73}{17}$$

Example 35: Reduce the following game with the help of dominance rule and then determine the value of the game:

		Player B			
		B_1	B_2	B_3	B_4
	A_1	3	2	4	0
Player A	A_2	3	4	2	4
	A_3	4	2	4	0
	A_4	0	4	0	8

Solution: Since this matrix does not contain any saddle point, hence we have to reduce the size of pay-off matrix using dominance rule. The third row dominates on the first row, so after eliminating first row, we get the following pay-off matrix:

Player B

Player A	B_1	B_2	B_3	B_4
A_2	3	4	2	4
A_3	4	2	4	0
A_4	0	4	0	8

Again, it is clear from the above matrix that the first column dominates on the third column, hence eliminating the first column, we get the following:

Player B

Player A	B_2	B_3	B_4
A_2	4	2	4
A_3	2	4	0
A_4	4	0	8

There is no row or column is dominated on others. On the other hand, the first column is dominated by the average of second and third columns:

$$\begin{bmatrix} \frac{2+4}{2} \\ \frac{4+0}{2} \\ \frac{0+8}{2} \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$

Now after eliminating the first column, we get the following pay-off matrix:

Player B

Player A	B_3	B_4
A_2	2	4
A_3	4	0
A_4	0	8

Similarly, average of second and third rows $\left(\frac{4+0}{2}, \frac{0+8}{2}\right) = (2, 4)$ dominated by the first row, hence eliminating the first row, we get the 2 × 2 game matrix as follows:

Player B

Player A	B_3	B_4
A_3	4	0
A_4	0	8

Let this matrix be of the following form:

$$\begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} B_1 \quad B_2 \\ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{array}$$

The optimum mixed strategies for player A and B is given by following matrices:

$$S_A = \begin{bmatrix} A_3 & A_4 \\ p_1 & p_2 \end{bmatrix}, p_1 + p_2 = 1$$

$$\text{and } S_B = \begin{bmatrix} B_3 & B_4 \\ q_1 & q_2 \end{bmatrix}, q_1 + q_2 = 1$$

Where,

$$p_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}, p_1 + p_2 = 1 \Rightarrow p_2 = 1 - p_1$$

$$q_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}, q_1 + q_2 = 1 \Rightarrow q_2 = 1 - q_1$$

$$p_1 = \frac{8-0}{(4+8)-(0+0)} = \frac{8}{12} = \frac{2}{3}, \quad p_2 = 1 - \frac{2}{3} = \frac{1}{3},$$

$$q_1 = \frac{8-0}{(4+8)-(0+0)} = \frac{8}{12} = \frac{2}{3}, \quad q_2 = 1 - \frac{2}{3} = \frac{1}{3}$$

The optimum strategy of the game is given by the following matrices:

$$S_A = \begin{bmatrix} A_3 & A_4 \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}, S_B = \begin{bmatrix} B_3 & B_4 \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

The value of the game (v)

$$= \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{(4\times8)-(0\times0)}{(4+8)-(0+0)} = \frac{32}{12} = \frac{8}{3}$$

Example 36: Solve the following game:

$$\text{Player A} \quad \overset{\text{Player B}}{\begin{pmatrix} 1 & 7 & 2 \\ 6 & 2 & 7 \\ 5 & 1 & 6 \end{pmatrix}}$$

Solution: As the all elements of second row is greater than or equal to third row, hence row II dominates over the row III. Delete the row III. The reduced payoff matrix is as follows:

$$\text{Player A} \overset{\text{Player B}}{\begin{bmatrix} 1 & 7 & 2 \\ 6 & 2 & 7 \end{bmatrix}}$$

The elements of first column is less than or equal to third column. That is, first column dominates over third column. Hence delete the dominated column 3. Now the reduced payoff matrix is given as follows:

$$\text{Player A} \overset{\text{Player B}}{\begin{bmatrix} 1 & 7 \\ 6 & 2 \end{bmatrix}}$$

The above reduced payoff matrix is now in the form of a 2×2 matrix.

For Player A and B, the optimum mixed strategies is given by:

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ p_1 & p_2 & 0 \end{pmatrix}, p_1 + p_2 = 1$$

$$\text{and } S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ q_1 & q_2 & 0 \end{pmatrix}, q_1 + q_2 = 1$$

Where,

$$p_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}, p_1 + p_2 = 1 \Rightarrow p_2 = 1 - p_1$$

$$q_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}, q_1 + q_2 = 1 \Rightarrow q_2 = 1 - q_1$$

$$p_1 = \frac{2-6}{(2+1)-(7+6)} = \frac{-4}{-10} = \frac{2}{5}$$

$$p_2 = 1 - \frac{2}{5} = \frac{3}{5}$$

$$q_1 = \frac{2-7}{2+1-(7+6)} = \frac{-5}{-10} = \frac{1}{2}$$

$$q_2 = 1 - q_1 = 1 - \frac{1}{2} = \frac{1}{2}$$

The value of the game

$$v = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{2\times1 - 7\times6}{2+1-(7+6)} = \frac{-40}{-10} = 4$$

The optimal strategy is given by $S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ \frac{2}{5} & \frac{3}{5} & 0 \end{pmatrix}$

and $S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$

Value of game is (v) = 4

3.4.5.5. Graphical Method – 2 × n and m × 2 Game

If in a game one of the players have two strategies only then graphical method is used to solve this type of game. If the game has no saddle point and has a pay-off matrix of type n × 2 or 2 × n then this method is used.

Algorithm for 2 × n Games

Step 1: With the help of dominance property, reduce the size of the pay-off matrix of Player A, if it is possible.

Step 2: Let us consider that x is the probability of selection of Alternative 1 by Player A and 1 – x be the probability of selection of Alternative 2 by Player A. Find the expected gain function of Player A with respect to each of the alternatives of Player B.

Step 3: Find the value of the gain from the gain functions which are derived in step 2, when x is equal to 0 as well as 1.

Step 4: The gain functions are plotted on a graph by assuming a suitable scale. Keep x on X-axis and the gain on Y-axis.

Step 5: Find the highest intersection point in the lower boundary of the graph because the Player A is a maximin player. Let it be the maximin point.

Step 6: If only two lines are passing through the maximin point then it forms a 2 × 2 pay-off matrix from the original problem by retaining only the columns corresponding to those two lines and go to step 8; otherwise go to step 7.

Step 7: Two lines are identified with opposite slopes passing through that point and form a 2 × 2 pay-off matrix from the original problem by retaining only the columns corresponding to those two lines which are having opposite slopes.

Step 8: Solve this 2 × 2 game with the help of algebraic method.

Algorithm for m × 2 Games

Step 1: With the help of dominance property reduce the size of the pay-off matrix of Player A, if it is possible.

Step 2: Let us consider that y is the probability of selection of Alternative 1 by Player B and 1 – y be the probability of selection of Alternative 2 by Player B. Find the expected gain function of Player B with respect to each of the alternatives of Player A.

Step 3: Find the value of the gain from the gain functions which are derived in step 2, when y is equal to 0 as well as 1.

Step 4: The gain functions are plotted on a graph by assuming a suitable scale. Keep y on X-axis and the gain on Y-axis.

Step 5: Find the lowest intersection point in the upper boundary of the graph because the Player B is a minimax player. Let it be the minimax point.

Step 6: If only two lines passing through the maximin point then it form a 2 × 2 pay-off matrix from the original problem by retaining only the rows corresponding to those two lines and go to step 8; otherwise go to step 7.

Step 7: Two lines are identified with opposite slopes passing through that point and form a 2 × 2 pay-off matrix from the original problem by retaining only the rows corresponding to those two lines which are having opposite slopes.

Step 8: Solve this 2 × 2 game with the help of algebraic method.

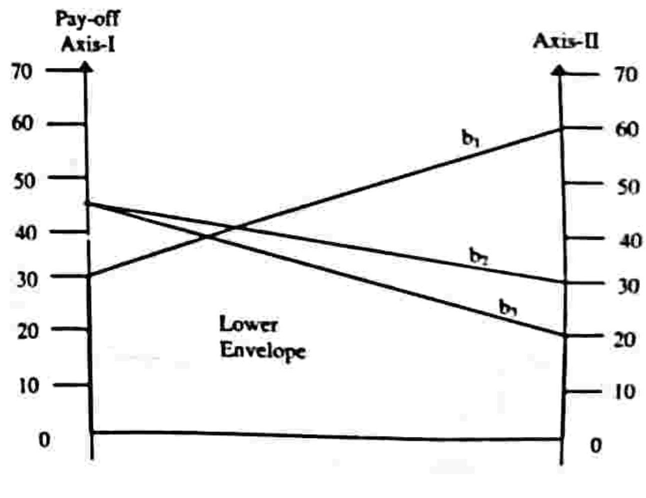
Example 37: Solve the following game for the payoff matrix shown below:

		Player B		
		B1	B2	B3
Player A	A1	20000	30000	60000
	A2	45000	45000	30000

Solution: Since no saddle point exists, we shall determine optimum mixed strategy.

B's Strategy

A's Strategy $\begin{matrix} & B_1 & \\ A_1 & 20,000 & 60,000 \\ A_2 & 45,000 & 30,000 \end{matrix}$



Correspondingly,

$$P_1 = \frac{a_{22} + a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{30,000 - 45,000}{(20,000 + 30,000) - (60,000 + 45,000)} = \frac{3}{11}$$

$$q_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{30,000 - 60,000}{(20,000 + 30,000) - (60,000 + 45,000)} = \frac{6}{11}$$

$$v = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{20,000 \times 30,000 - 60,000 \times 45,000}{(20,000 + 30,000) - (60,000 + 45,000)} = 38,182$$

Example 38: Consider the payoff matrix with respect to the player A as shown in table. Solve this game optimally using graphical method.

Table

	1	2	3	4	5
1	4	2	1	7	3
2	2	7	8	1	5

Solution: At first we check the saddle point. As the saddle point does not exist for the above problem, we follow the mixed strategies. Now we use the dominance rule to reduce the payoff matrix. Since matrix cannot be reduced using dominance rule, hence graphical method can be used to solve the game. The expected payoff of Player A corresponding to B's pure strategies is given below:

		Player B				
		1	2	3	4	5
Player A (x_1)	1	4	2	1	7	3
$(x_2 = 1 - x_1)$	2	2	7	8	1	5

B's Pure Strategies	A's Expected Payoffs
1	$4x_1 + 2(1 - x_1) = 2x_1 + 2$
2	$2x_1 + 7(1 - x_1) = -5x_1 + 7$
3	$1x_1 + 8(1 - x_1) = -7x_1 + 8$
4	$7x_1 + (1 - x_1) = 6x_1 + 1$
5	$3x_1 + 5(1 - x_1) = -2x_1 + 5$

The lines can be represented as function of x_1 as shows in **figure 3.12**:

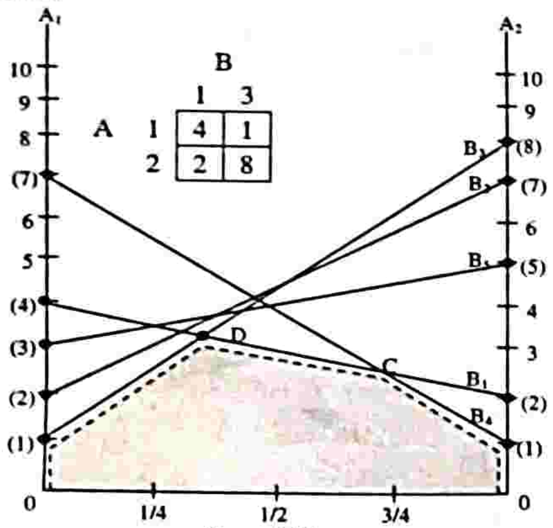


Figure 3.12

The figure is bounded from below. The two maximum points C and D shows the value of the game.

1) **Point C:** The player B selects the strategies B_1 and B_4. Then the 2×2 game is to be solved:

		B	
		1	4
A	1	a_{11} 4	a_{12} 7
	2	a_{21} 2	a_{22} 1

The optimum mixed strategies:

$$S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix} \text{ and } S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}$$

Where,

$$p_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{(1-2)}{(4+1)-(7+2)} = \frac{-1}{5-9} = \frac{1}{4}$$

$$p_2 = 1 - p_1 = 1 - \frac{1}{4} = \frac{3}{4}$$

$$q_1 = \frac{(a_{22} - a_{12})}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{(1-7)}{(4+1)-(7+2)} = \frac{-6}{-4} = \frac{3}{2}$$

$$q_2 = 1 - \frac{3}{2} = \frac{-1}{2}$$

Value of game

$$v = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{(4 \times 1 - 7 \times 2)}{(4+1)-(7+2)} = \frac{4-14}{5-9} = \frac{-10}{-4} = \frac{5}{2}$$

∴ The optimum mixed strategies:

$$S_A = \left(\frac{1}{4}, \frac{3}{4}\right) \qquad S_B = \left(\frac{3}{2}, \frac{-1}{2}\right)$$

Value of game $v = \frac{5}{2}$

2) **Point D:** The player B selects the strategies B_1 and B_3. Then the 2×2 game is to be solved:

The optimum mixed strategies:

$$S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix} \text{ and } S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}$$

$$\therefore \quad p_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{(8-2)}{(4+8)-(1+2)} = \frac{6}{9} = \frac{2}{3}$$

$$p_2 = \left(1 - \frac{2}{3}\right) = \frac{1}{3}$$

$$q_1 = \frac{(a_{22} - a_{12})}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{(8-1)}{(4+8)-(1+2)} = \frac{7}{(12-3)} = \left(\frac{7}{9}\right)$$

$$q_2 = \left(1 - \frac{7}{9}\right) = \left(\frac{2}{9}\right)$$

Value of game

$$v = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$= \frac{(4 \times 8 - 2 \times 1)}{(4 + 8) - (1 + 2)} = \frac{(32 - 2)}{9} = \frac{30}{9}$$

∴ The optimum mixed strategies:

$$S_A = \left(\frac{2}{3}, \frac{1}{3}\right) \qquad S_B = \left(\frac{7}{9}, \frac{2}{9}\right)$$

Value of the game $v = \frac{30}{9}$

Example 39: Consider the payoff matrix of player A as shown in table below and solve it optimally using the graphical method.

Player A \ Player B	1	2	3	4	5
1	3	6	8	4	4
2	−7	4	2	10	2

Solution: **Table 3.28** shows that the maximin value (3) is equal to the minimax value (3), hence this game problem has a saddle point. Thus player A and B both has a pure strategy.

Table 3.28: Payoff Matrix

Player A \ Player B	1	2	3	4	5	Minimum
1	3	6	8	4	4	3 (maximin)
2	−7	4	2	10	2	−7
Maximum	3 (minimax)	6	8	10	4	

The corresponding outcome is shown below:

A(1, 0), B(1, 0, 0, 0, 0) Value of the game, V = 3

But according to question, the game has to be solved using graphical method. In **table 3.28**, the every elements of column 2 is greater than column 5, i.e., column 2 is dominated by column 5. Thus after removing the column 2, we get the new resultant payoff matrix as shown in **table 3.29**.

Table 3.29: Payoff Matrix

Player A \ Player B	1	3	4	5
1	3	8	4	4
2	−7	2	10	2

Now let suppose that x represent the probability of selection of alternative 1 by Player A and then (1 − x) will show alternative 2 chosen by Player A. Hence the expected payoff of Player A corresponding to different alternatives of Player B is shown in **table 3.30**.

Table 3.30: Expected Payoff Functions of Player A

B's Alternative	A's Expected Payoff Function
1	$3x + (-7)(1 - x) = 10x - 7$
3	$8x + 2(1 - x) = 6x + 2$
4	$4x + 10(1 - x) = -6x + 10$
5	$4x + 2(1 - x) = 2x + 2$

With different conditions of x (0 or 1), the calculation of expected payoff of player A corresponding to each of the alternatives of Player B is shown in **table 3.31**.

Table 3.31: Expected Gain of Player of Player A

B's Alternative	A's Expected Payoff Function	A's Expected Gain	
		x = 0	x = 1
1	$10x - 7$	−7	3
3	$6x + 2$	2	8
4	$-6x + 10$	10	4
5	$2x + 2$	2	4

The expected gain functions of Player A corresponding to several alternatives of Player B are illustrated in **figure 3.13**.

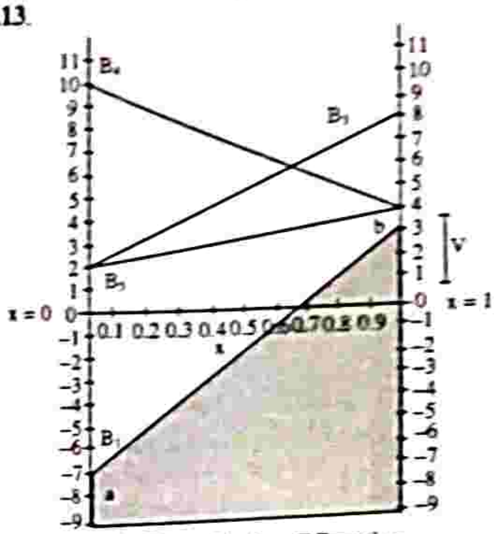


Figure 3.13: Player A's Payoff Function

We have to detect the highest intersection point in the lower boundary of the above graph because A is a maximin player. There are two intersection points 'a' and 'b' in the lower boundary of the graph which are located at highest level. Thus the solution obtained will be the optimal solution. The value of the game is 3 corresponding to intersection point 'b'.

The horizontal distance from left vertical line to the point 'b' shows the probability of selection of alternative 1 by Player A which is 1. It means that x= 1. Hence, the 1 − x = 0 which shows the probability of selection of alternative 2 by Player A.

The point 'b' is present at rightmost vertical line which is the maximin intersection point. The only line passing though this point is the line B_1 of the Player B. Thus, probability of selection of the alternative 1 by the Player B is 1 and alternative 2 by the Player B is 0.

The final results of the game are:

A(1, 0), B(1, 0, 0, 0, 0), Value of the game, V= 3

Example 40: Solve the following game by graphical method.

Player A \ Player B	1	2	3
1	6	4	3
2	2	4	8

Solution: For solving the above game problem with graphical method, we first draw two parallel lines at a distance of 1 unit and mark a scale on each. The two parallel lines show the strategies of player A. In case the player B chooses the strategy B1, then player A can have two wins for 6 or 2 units depending on the selection of strategies A.

The value 6 is plotted along the vertical axis under strategy A1 and the value 2 is plotted along the vertical axis under strategy A2.

A straight line joining the two points is then drawn.

Similarly, we can plot strategies B2, B3 also. The problem is graphed in the **figure below**.

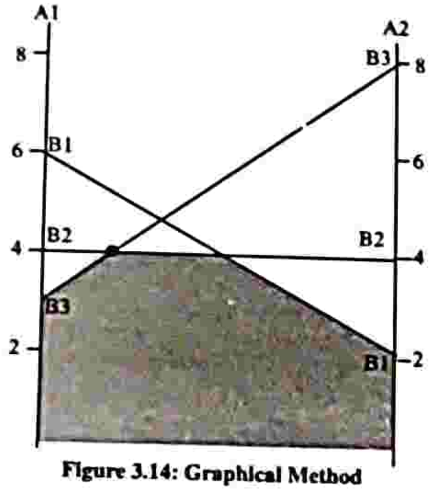


Figure 3.14: Graphical Method

The highest point V in the shaded region indicates the value of game. From the above **figure 3.14**, the value of the game is 4 units.

1) The point of optimal solution occurs at the intersection of two lines

$4p_1 + 4p_2$ ...(1)

$3p_1 + 8p_2$ ...(2)

Comparing the above two equations, we have

$4p_1 + 4p_2 = 3p_1 + 8p_2$

Substituting $p_2 = 1 - p_1$

$4p_1 + 4(1 - p_1) = 3p_1 + 8(1 - p_1)$

Solving $p_1 = \frac{4}{5}$

$p_2 = 1 - p_1 = 1 - \frac{4}{5} = \frac{1}{5}$

Substituting the values of p_1 and p_2 in equation (1)

$V = 4\left(\frac{4}{5}\right) + 4\left(\frac{1}{5}\right) = 4$

2) The point of optimal solution occurs at the intersection of two lines

$4q_1 + 3q_2$ ...(3)

$4q_1 + 8q_2$ ...(4)

Comparing the above two equations, we have

$4q_1 + 3q_2 = 4q_1 + 8q_2$

Substituting $q_2 = 1 - q_1$

$4q_1 + 3(1 - q_1) = 4q_1 + 8(1 - q_1)$

Solving $q_1 = 1$

$q_2 = 1 - q_1 = 1 - 1 = 0$

Substituting the values of q_1 and q_2 in equation (4),

$V = 4(1) + 3(0) = 4$

3.4.5.6. Odds Method/Arithmetic Method–2 × 2 Game

The arithmetic method which is also known as **short cut method** provides a simple method for obtaining the optimal strategies for each player in a payoff matrix of size 2 × 2 without saddle point.

Algorithm of Odds Method/Arithmetic Method

The steps of this method are as follows:

Step 1) Select the first row and find the difference between the two values. Put this value against the second row of the matrix without considering negative sign (if any).

Step 2) Select the second row and find the difference between the two values. Put this value against the first row of the matrix without considering negative sign (if any).

Step 3) Repeat Step 1 and 2 for the two columns.

The values which are obtained by 'swapping the differences' (oddments) represents the optimal relative frequencies of game for both player's strategies. They are divided by their sum to convert into probabilities. This method is also known as **"oddments method"**.

Example 41: Two companies A and B which are business rival are doing business in such a manner that company A's gain will provide company B's loss and *vice-versa.* The pay-off matrix of company A is as follows:

Company A	Company B		
	No Advertising	**Medium Advertising**	**Heavy Advertising**
No Advertising	10	5	–2
Medium Advertising	13	12	15
Heavy Advertising	16	14	10

Determine the optimal strategies for the both companies and also find out the net outcome.

Solution: Since this pay-off matrix does not contain saddle point, hence companies will use the mixed strategies. First apply the dominance rule. As the elements of first column are greater than the elements of second column (i.e., first column dominates on second column), hence eliminating first column, we get the following payoff matrix:

Company A	Company B	
	Medium Advertising	**Heavy Advertising**
No Advertising(A_1)	5	–2
Medium Advertising(A_2)	12	15
Heavy Advertising(A_3)	14	10

From the above matrix, it is clear that elements of second row is greater than the first row, hence eliminating the first row, we get the following 2 × 2 payoff matrix:

		Company B	
		Medium advt., B_2	**Heavy advt., B_3**
Company A	**Medium advt., A_2**	12	15
	Heavy advt., A_3	14	10

Now using arithmetic method, we get the following mixed strategies for both the companies:

Firm B

$$\text{Firm A} \begin{matrix} & B_2 & B_3 \\ A_2 & 12 & 15 \\ A_3 & 14 & 10 \end{matrix} \quad \begin{matrix} 14-10=4,\ p(A_2)=\frac{4}{4+3}=\frac{4}{7} \\ 15-12=3,\ p(A_3)=\frac{3}{4+3}=\frac{3}{7} \end{matrix}$$

$$15-10=5 \qquad 14-12=2$$

$$p(B_2)=\frac{5}{5+2}=\frac{5}{7} \quad p(B_3)=\frac{2}{5+2}=\frac{2}{7}$$

1) **For Company A:** Consider that p_2 and p_3 are the probabilities of choosing strategies A_2 and A_3 respectively. Now the expected gain to company A when company B uses B_2 and B_3 strategies, are given by following:

$$12p_2 + 14p_3 \text{ and } 15p_2 + 10p_3;\ p_2 + p_3 = 1.$$

For company A, p_2 and p_3 will be such that expected gains under both situations should be equal. Now we have,

$$12p_2 + 14p_3 = 15p_2 + 10p_3$$

$$\Rightarrow \quad 12p_2 + 14(1-p_2) = 15p_2 + 10(1-p_2)$$

$$\Rightarrow \quad 7p_2 = 4 \Rightarrow p_2 = \frac{4}{7} \text{ and } p_3 = 1 - p_2 = 1 - \frac{4}{7} = \frac{3}{7}$$

2) **For Company B:** Consider that q_2 and q_3 is the probabilities of choosing strategies B_2 and B_3 respectively. Now the expected loss to company B when company A uses B_2 and B_3 strategies, are given by following:

$$12q_2 + 15q_3 = 14q_2 + 10q_3;\ q_2 + q_3 = 1.$$

$$\Rightarrow \quad 12q_2 + 15(1-q_2) = 14q_2 + 10(1-q_2)$$

$$\Rightarrow \quad 7q_2 = 5 \Rightarrow q_2 = \frac{5}{7} \text{ and}$$

$$q_3 = 1 - q_2 = 1 - \frac{5}{7} = \frac{2}{7}$$

So that, company A should select the strategy A_2 (with 57% of time) and A_3 (43% of time). In similar manner, company B should select the strategy B_2 (with 71% of time) and B_3 (29% of time).

Hence the expected gain and loss for company A and B can be calculated as follows:

1) **Expected Gain of Company A**
   i) $12p_2 + 14p_3 = 12 \times (4/7) + 14 \times (3/7) = (90/7)$, company B accept B_2
   ii) $15p_2 + 10p_3 = 15 \times (4/7) + 10 \times (3/7) = (90/7)$, company B accept B_3
2) **Expected Loss of Firm B**
   i) $12q_2 + 15q_3 = 12 \times (5/7) + 15 \times (2/7) = (90/7)$, company A accept A_2
   ii) $14q_2 + 10q_3 = 14 \times (5/7) + 10 \times (2/7) = (90/7)$, company A accept A_3.

Example 42: There are two player X and Y that put a coin (with head or tail up) on the table in such a manner that it is not shown to other. If both coins represent head then player X wins ₹8, however in case of both coins represent tails then he/she wins ₹1. The player Y wins ₹3 if one coin represents head and other tail, i.e., both coins do not match. There are two options, i.e., being matching player X or non-matching player Y, find which one would you select and what would be the strategy.

Solution: As this problem does not contain saddle point, hence mixed strategy can be used to find out the optimal strategies. Player X's payoff matrix is given below:

Player Y

Player X	H	T		
H	8	−3	4	$p_1 = \frac{4}{11+4} = \frac{4}{15}$
T	−3	1	11	$p_2 = \frac{11}{11+4} = \frac{11}{15}$
	4	11		

$$q_1 = \frac{4}{11+4} = \frac{4}{15} \qquad q_2 = \frac{11}{11+4} = \frac{11}{15}$$

To find out the strategy and value of game, the steps are as follows:

Step 1: Initially subtract the elements located in column I and put the result under the column II. The result is $8 - (-3) = 11$.

Step 2: Next, subtract the elements located in the column II and put the result under the column I. The result is $(-3 - 1) = -4$ (take only magnitude).

Step 3: Similar process is applied on the two rows also. Thus, player X uses strategy H (with probability 4/15) and strategy T (with probability 11/15) for optimal gains. Player Y uses strategy H (with probability 4/15) and strategy T (with probability 11/15).

Step 4: The value of the game can be calculated using anyone of the following expressions:

1) **Using B's Oddments:** When player Y plays H, then the value of game (v)

$$= \frac{4\times8 + 11\times(-3)}{11+4} = \left(-\frac{1}{15}\right)$$

When Player Y plays T, then value of the game (v) =

$$\frac{4\times(-3) + 11\times1}{11+4} = \left(-\frac{1}{15}\right)$$

2) **Using A's Oddments:** When Player X plays H, then the value of game (v)

$$= \frac{4\times8 + 11\times(-3)}{4+11} = \left(-\frac{1}{15}\right)$$

When Player X plays T, then value of the game (v) =

$$\frac{4\times(-3) + 11\times1}{4+11} = \left(-\frac{1}{15}\right)$$

If the sums of vertical and horizontal oddments are equal, then the value of game v will be equal as shown above.

Hence the solutions of game are as following:

1) Optimum strategy for player X is (4/15, 11/15), and for player Y is (4/15, 11/15).
2) Value of the game for Player X is $v = (-1/15)$ and for player Y is 1/15, i.e., player X gains ₹ (− 1/15) and player Y loses ₹ (1/15).

Example 43: Solve the following game using Odds method:

	B	
A	−1	3
	2	−1

Solution: As the given game has no saddle point, therefore both the players will use mixed strategies.

Now, we use odds method to solve 2 × 2 game.

	B_1	B_2	Odds	Probability
A_1	−1	3	3	$p_1 = 3/7$
A_2	2	−1	4	$P_2 = 4/7$
Odds	4	3		
Probability	2/3	1/3		
	$q_1 = 4/7$	$q_2 = 3/7$		

Hence, firm A should adopt strategy A_1 and A_2 with 43% of time respectively.

Similarly, firm B should adopt strategy B_1 and B_2 with 57% of time and 43% of time respectively.

Expected gain of Firm A

1) $-1\times\frac{3}{7}+2\times\frac{4}{7}=\frac{5}{7}$, Firm B adopt B_1

2) $3\times\frac{3}{7}-1\times\frac{4}{7}=\frac{5}{7}$, Firm B adopt B_2

Expected gain of Firm B

1) $-1\times\frac{4}{7}+3\times\frac{3}{7}=\frac{5}{7}$, Firm A adopt A_1

2) $2\times\frac{4}{7}-1\times\frac{3}{7}=\frac{5}{7}$, Firm A adopt A_2

3.4.5.7. Method of Matrices – n × n Game

For solving n × n games this method provide a simple method for obtaining the optimal strategies for each player in a payoff matrix.

Step 1: Let $A = (a_{ij})$ be an n × n payoff matrix. A new matrix C is obtained in which the first column is obtained from A by subtracting its 2nd column from 1st; second column is obtained by subtracting A's 3rd column from 2nd and so on till the last column of A has been taken care of. Thus, C is an n × (n −1) matrix.

Step 2: A new matrix R is obtained from A in which rows are obtained by subtracting its successive rows from the preceding ones, in exactly the same manner as it was done for columns in step 1. Thus, R is an (n − 1) × n matrix.

Step 3: The magnitude of oddments are determined corresponding to each row and each column of A. The oddment corresponding to i^{th} row of A is defined as the determinant $|C_i|$, where C_i is obtained from C by deleting its i^{th} row. Similarly, oddment (j^{th} column of A) = $|R_j|$, where R_j is obtained from R by deleting its j^{th} column.

Step 4: Write the magnitude of oddments (after ignoring negative signs, if any, against their respective rows and columns).

Step 5: Check whether the sum of row oddments is equal to the sum of column oddments. If so, the oddments expressed as fractions of the grand total yield the optimum strategies. If not, the method fails.

Step 6: The expected value of game is calculated corresponding to the optimum mixed strategy determined above for the Row Player (against any move of the column player).

Example 44: Solve the following game using matrix oddment method.

$$A\overset{B}{\begin{pmatrix}0 & 1 & 2\\ 2 & 0 & 1\\ 1 & 2 & 0\end{pmatrix}}$$

Solution: Using the above pay-off matrix, let first calculate the matrices C and R and then obtain the oddments of row and column. To find matrix C, subtract the elements of first column from the second and second column from the third. Similarly to determine matrix R, subtract the elements of first row with second and second row from the third row. Thus, we get the following:

$$C=\begin{bmatrix}-1 & -1\\ 2 & -1\\ -1 & 2\end{bmatrix} \text{ and } R=\begin{bmatrix}-2 & 1 & 1\\ 1 & -2 & 1\end{bmatrix}$$

The oddments are:

$$C_1=\begin{vmatrix}2 & -1\\ -1 & 2\end{vmatrix}=3, C_2=\begin{vmatrix}-1 & -1\\ -1 & 2\end{vmatrix}=-3, C_3=\begin{vmatrix}-1 & -1\\ 2 & -1\end{vmatrix}=3 \text{ and}$$

$$R_1=\begin{vmatrix}1 & 1\\ -2 & 1\end{vmatrix}=3, R_2=\begin{vmatrix}-2 & 1\\ 1 & 1\end{vmatrix}=-3, R_3=\begin{vmatrix}-2 & 1\\ 1 & -2\end{vmatrix}=3$$

Now the augmented pay-off matrix will be as follows:

			|Row Oddments|
0	1	2	3
2	0	1	3
1	2	0	3
3	3	3	9

(last row: |Column Oddments|)

As sum of row oddments = sum of column oddments, hence the optimal strategies are as below:

For Player A $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ and for Player B $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$

Thus, the value of the game = $0\times\frac{1}{3}+2\times\frac{1}{3}+1\times\frac{1}{3}=1$

Example 45: Consider the following 2 × 2 game:

$$\text{Player A}\overset{\text{Player B}}{\begin{pmatrix}4 & 7\\ 6 & 5\end{pmatrix}}$$

1) Does it contain a saddle point?
2) Is it right to say that the value of game, v will be 5<v< 6?
3) Determine optimum strategies by using matrix oddment method and also calculate the value of game.

Solution:

1) We first find the row minima and column maxima, in order to calculate the saddle points as given below:

	B_1	B_2	Row Min
A_1	4	7	4
A_2	6	5	5
Column Max	6	7	

From the table above, it is clear that maximin value 5 is not equal to minimax value 6 hence this does not have saddle point.

2) The value of v should be $5 \le v \le 6$ as we have the formula maximin value $\le v \le$ minimax value.

3) We get the matrix C by subtracting the elements of second column from the first, as follows:

$$C=\begin{bmatrix}-3\\1\end{bmatrix}$$

This gives oddments $|C_1| = 1$ and $|C_2| = -3$

Similarly, matrix R can be obtained by subtracting the elements of second row from the first, as follows:

$$R=\begin{bmatrix}-2 & 2\end{bmatrix}$$

Thus we have oddments $|R_1| = 2$ and $|R_2| = -2$

Now the augmented pay-off matrix will be as follows:

			|Row Oddment|
	4	7	1
	6	5	3
|Column Oddment|	2	2	4

As sum of row oddments = sum of column oddments (4), hence the optimal strategies are as below:

For Player A $\left(\frac{1}{4},\frac{3}{4}\right)$ and

For Player B $\left(\frac{1}{2},\frac{1}{2}\right)$

Thus, value of the game $(v) = 4\times\frac{1}{4}+6\times\frac{3}{4}=\frac{11}{2}$

Example 46: Solve the following 3×3 game by the method of matrices:

	B		
A	1	0	2
	3	0	0
	0	2	1

Solution: We compute the matrices C (subtracting first column from second, and similarly for second and third column) and R (subtracting first row from second, and similarly second from third column) from the given pay-off matrix, and then obtain the row and column oddments.

Thus we have,

$$C=\begin{bmatrix}1 & -2\\3 & 0\\-2 & 1\end{bmatrix}, \quad R=\begin{bmatrix}-2 & 0 & 2\\3 & -2 & -1\end{bmatrix}$$

The oddments are:

$$C_1=\begin{vmatrix}3 & 0\\-2 & 1\end{vmatrix}=3 \qquad R_1=\begin{vmatrix}0 & 2\\-2 & -1\end{vmatrix}=4$$

$$C_2=\begin{vmatrix}1 & -2\\-2 & 1\end{vmatrix}=-3 \qquad R_2=\begin{vmatrix}-2 & 2\\3 & -1\end{vmatrix}=-4$$

$$C_3=\begin{vmatrix}1 & -2\\3 & 0\end{vmatrix}=6 \qquad R_3=\begin{vmatrix}-2 & 0\\3 & -2\end{vmatrix}=4$$

The augmented pay-off matrix is:

				Row Oddments
	1	0	2	4
	3	0	0	-4
	0	2	1	4
Column Oddments	3	-3	6	

Since the sum of row and column oddments is not equal, hence matrix method fails.

3.4.6. Linear Programming Solution to Game Theory

Since one can convert any two-person zero-sum game to corresponding Linear Programming Problem (LPP) and *vice versa*. Hence both are claimed to be equivalent. By changing the two person zero-sum game into LPP, the game can also be solved by using linear programming approach.

The main benefit of this approach is that it can be used to solve the mixed strategy of any size. It is very hard to solve a game of m × n pay-off matrix which has neither saddle point nor dominant row and column. Here m and n are greater than 2.

When all the players have three or more strategies then simplex method of linear programming is the basic method to solve the game of all types.

Let consider that payoff matrix corresponding to Player A is $[a_{ij}]$ as shown in **table 3.32:**

Table 3.32: Generalised Payoff Matrix of Player A

		Player B					
		1	2	...	j	...	n
Player A	1	a_{11}	a_{12}	...	a_{1j}	...	a_{1n}
	2	a_{21}	a_{22}	...	a_{2j}	...	a_{2n}
	⋮	⋮	⋮		⋮		⋮
	i	a_{i1}	a_{i2}	...	a_{ij}	...	a_{in}
	⋮	⋮	⋮		⋮		⋮
	m	a_{m1}	a_{m2}	...	a_{mj}	...	a_{mn}

Where,

m = Total number of alternatives for player A

n = Total number of alternatives for player B

a_{ij} = Payoff to Player A in case A selects the Alternative i and Player B selects the Alternative j

V = Value of the game

p_i = Probability of selection of Alternative i by Player A, where i = 1, 2, 3,...., m

q_j = Probability of selection of Alternative j by Player B, where j = 1, 2, 3,...., n

3.4.6.1. Development of Linear Programming Model with Respect to Player A

The expected gain to Player A corresponding to the selection of each of the alternatives of Player B is shown in **table 3.33**:

Table 3.33: Expected Gain Functions of Player A

Player B's Alternative	Expected Gain Function to Player A
1	$a_{11}p_1 + a_{21}p_2 + \ldots + a_{i1}p_i + \ldots + a_{m1}p_m = \sum_{i=1}^{m} a_{i1}p_i$
2	$a_{12}p_1 + a_{22}p_2 + \ldots + a_{i2}p_i + \ldots + a_{m2}p_m = \sum_{i=1}^{m} a_{i2}p_i$
⋮	⋮
j	$a_{1j}p_1 + a_{2j}p_2 + \ldots + a_{ij}p_i + \ldots + a_{mj}p_m = \sum_{i=1}^{m} a_{ij}p_i$
⋮	⋮
n	$a_{1n}p_1 + a_{2n}p_2 + \ldots + a_{in}p_i + \ldots + a_{mn}p_m = \sum_{i=1}^{m} a_{in}p_i$

Descriptive Model for Player A

In this model, the Player A is maximum player and wants to maximise the game value. This objective can be accomplished by maximising the minimum of the functions as shown by last column in the **table 3.33**.

$$\max\left[\min\left(\sum_{i=1}^{m} a_{i1}p_i, \sum_{i=1}^{m} a_{i2}p_i, \ldots, \sum_{i=1}^{m} a_{ij}p_i, \ldots, \sum_{i=1}^{m} a_{in}p_i\right)\right]$$

subject to $p_1 + p_2 + \cdots + p_i + \cdots + p_m = 1$

Above equation shows that total of the probabilities of selection of alternatives of Player A should be 1. And $p_1, p_2, \cdots, p_i, \cdots, p_m \geq 0$.

This model is known as a **descriptive model** as the objective function is not in linear form.

Linear Model for Player A

The above descriptive model is converted into the linear model as it is very difficult to work with the descriptive model. We have the following transformation:

$$V = \min\left(\sum_{i=1}^{m} a_{i1}p_i, \sum_{i=1}^{m} a_{i2}p_i, \ldots, \sum_{i=1}^{m} a_{ij}p_i, \ldots, \sum_{i=1}^{m} a_{in}p_i\right)$$

The linear model can be represented as follows:

Maximise $Z = V$

subject to

$$\sum_{i=1}^{m} a_{i1}p_i \geq V, \sum_{i=1}^{m} a_{i2}p_i \geq V \ldots \sum_{i=1}^{m} a_{ij}p_i \geq V \ldots \sum_{i=1}^{m} a_{in}p_i \geq V$$

$$p_1 + p_2 + \cdots + p_i + \cdots + p_m = 1, \; p_i \geq 0, \qquad i = 1, 2, \ldots, m$$

The above constraints can be simplified by dividing it by V. We get the following constraints:

$$\sum_{i=1}^{m} a_{i1}p_i / V \geq 1, \sum_{i=1}^{m} a_{i2}p_i / V \geq 1 \ldots$$

$$\sum_{i=1}^{m} a_{ij}p_i / V \geq 1, \ldots \sum_{i=1}^{m} a_{in}p_i / V \geq 1$$

$$\frac{p_1}{V} + \frac{p_2}{V} + \ldots + \frac{p_i}{V} + \ldots + \frac{p_m}{V} = \frac{1}{V},$$

$p_i \geq 0, \qquad i = 1, 2, \ldots, m$

The above model shows the modified model, in which the following problem occurs:

1) The type of every constraint will be changed if the value of the game is less than zero.
2) The terms of the constraints will become infinite if the value of the game is equal to zero.

For avoiding the above problems, a constant K is added to every entry in the payoff matrix. The constant K is equal to the absolute value of the maximum negative values of the payoff matrix plus one.

Whenever these problems have been resolved, next step is subtracting the constant K from the game value for obtaining the true value of game.

There is no need to add constant K to every entry of payoff matrix, in case every values of matrix are positive, i.e., greater than zero.

Let $p_i/V = X_i$, i = 1, 2,...., m. Therefore,

$$\max V = \min\frac{1}{V} = \min\left(\frac{p_1}{V} + \frac{p_2}{V} + \ldots + \frac{p_i}{V} + \ldots + \frac{p_m}{V}\right)$$

$$= \min (X_1 + X_2 + \ldots + X_i + \ldots + X_m)$$

We get the following revised model if the above function is substituted in the model: Minimise

$$Z_1 = X_1 + X_2 + \ldots + X_i + \ldots + X_m$$

subject to

$$\sum_{i=1}^{m} a_{i1}X_i \geq 1, \sum_{i=1}^{m} a_{i2}X_i \geq 1 \ldots$$

$$\sum_{i=1}^{m} a_{ij}X_i \geq 1 \ldots \sum_{i=1}^{m} a_{in}X_i \geq 1$$

$X_i \geq 0, \qquad i = 1, 2, \ldots, m$

The value for p_i, where i = 1, 2,..., m and V are obtained using the following formulae:

$$V = \frac{1}{Z_1}, \; p_i = VX_i, \qquad \text{where } i = 1, 2, \ldots, m$$

3.4.6.2. Development of Linear Programming Model with Respect to Player B

The expected loss or gain of Player B corresponding to the selection of every alternatives of Player A is shown in table 3.34.

Table 3.34: Expected Payoff Function of Player B

Player A's Alternative	Expected Loss (+)/Gain (–) Function to Player B
1	$a_{11}q_1 + a_{12}q_2 + \ldots + a_{1j}q_j + a_{1n}q_n = \sum_{j=1}^{n} a_{1j}q_j$
2	$a_{21}q_1 + a_{22}q_2 + \ldots + a_{2j}q_j + a_{2n}q_n = \sum_{j=1}^{n} a_{2j}q_j$
⋮	⋮ ⋮ ⋮ ⋮ ⋮
i	$a_{i1}q_1 + a_{i2}q_2 + \ldots + a_{ij}q_j + a_{in}q_n = \sum_{j=1}^{n} a_{ij}q_j$
⋮	⋮ ⋮ ⋮ ⋮ ⋮
m	$a_{m1}q_1 + a_{m2}q_2 + \ldots + a_{mj}q_j + a_{mn}q_n = \sum_{j=1}^{n} a_{mj}q_j$

Descriptive Model for Player B

The player B tries to minimise the game value because B is a minimax player. This objective can be achieved by minimising the maximum of the functions situated in the last column of above **table 3.34**. Thus we have,

$$\min\left[\max\left(\sum_{j=1}^{n} a_{1j}q_j, \sum_{j=1}^{n} a_{2j}q_j, \ldots, \sum_{j=1}^{n} a_{ij}q_j, \ldots, \sum_{j=1}^{n} a_{mj}q_j\right)\right]$$

Subject to constraints,

$q_1 + q_2 + \cdots + q_j + \cdots + q_n = 1$

There is one condition that should be satisfied, that is, sum of the probabilities of selection of alternatives of player B is equal to 1.

$q_1, q_2, \cdots, q_j, \cdots, q_n \geq 0$

This model is known as **descriptive model** because objective function is not linear in this model.

Linear Model for Player B

Since it is difficult to work with the descriptive model, it is transformed into linear model. Let,

$$V = \max\left(\sum_{j=1}^{n} a_{1j}q_j, \sum_{j=1}^{n} a_{2j}q_j, \ldots, \sum_{j=1}^{n} a_{ij}q_j, \ldots, \sum_{j=1}^{n} a_{mj}q_j\right)$$

Whenever above is substituted then the linear model is as follows:

Minimise $Z = V$

Subject to

$$\sum_{j=1}^{n} a_{1j}q_j \leq V, \sum_{j=1}^{n} a_{2j}q_j \leq V \ldots$$

$$\sum_{j=1}^{n} a_{ij}q_j \leq V \ldots \sum_{j=1}^{n} a_{mj}q_j \leq V$$

$q_1 + q_2 + \cdots + q_j + \cdots + q_n = 1, q_j \geq 0,$
where $j = 1, 2, \ldots, n$

By dividing with game value V, the system of constraints of the above model can be converted into the simplified form as follows:

$$\sum_{j=1}^{n} a_{1j}\frac{q_j}{V} \leq 1, \quad \sum_{j=1}^{n} a_{2j}\frac{q_j}{V} \leq 1 \ldots$$

$$\sum_{j=1}^{n} a_{ij}\frac{q_j}{V} \leq 1 \ldots \quad \sum_{j=1}^{n} a_{mj}\frac{q_j}{V} \leq 1$$

$$\frac{q_1}{V} + \frac{q_2}{V} + \ldots + \frac{q_j}{V} + \ldots + \frac{q_n}{V} = \frac{1}{V},$$

$q_j \geq 0$, where $j = 1, 2, \ldots, n$

The above model shows the modified model, in which the following problem occurs:

1) The type of every constraint will be changed if the value of the game is less than zero.
2) The terms of the constraints will become infinite if the value of the game is equal to zero.

For avoiding the above problems, a constant K is added to every entry in the payoff matrix. The constant K is equal to the absolute value of the maximum negative values of the payoff matrix plus one. Whenever these problems have been resolved, next step is subtracting the constant K from the game value for obtaining the true value of game. There is no need to add constant K to every entry of payoff matrix, in case every values of matrix are positive, i.e., greater than zero.

Let $q_j/V = Y_j$, where $j = 1, 2, 3, \ldots, n$. Then,

$$\min V = \max \frac{1}{V} = \max\left(\frac{q_1}{V} + \frac{q_2}{V} + \ldots + \frac{q_j}{V} + \ldots + \frac{q_n}{V}\right) = \max$$

$(Y_1 + Y_2 + \ldots + Y_j + \ldots + Y_n)$

We get the following revised model if the above function is substituted in the model:

Maximise $Z_2 = Y_1 + Y_2 + Y_3 + \cdots + Y_j + \cdots + Y_n$

Subject to,

$$\sum_{j=1}^{n} a_{1j}Y_1 \leq 1, \sum_{j=1}^{n} a_{2j}Y_2 \leq 1 \cdots$$

$$\sum_{j=1}^{n} a_{ij}Y_3 \leq 1 \cdots \sum_{j=1}^{n} a_{mj}Y_j \leq 1$$

$Y_1 \geq 0, \quad j = 1, 2, \ldots, n$

The values for q_j, $j = 1, 2, 3, \ldots, n$ and V are obtained using the following formulae:

$$V = \frac{1}{Z_2}, q_j = VY_j, \quad j = 1, 2, 3, \ldots, n$$

Example 47: The payoff matrix with respect to Player A is shown in **table 3.35** Using the linear programming method, solve the game.

Table 3.35: Payoff Matrix of Player A

		Player B		
		1	2	3
Player A	1	1	–1	–1
	2	–1	–1	3
	3	–1	2	–1

Solution: Using **table 3.35**, the maximin and minimax value can be calculated as follows:

Table 3.36: Payoff Matrix with Maximin and Minimax Values

	Player B			
	1	2	3	Row minimum
1	1	–1	–1	–1 (maximin)

Player A	2	–1	–1	3	–1 (maximin)
	3	–1	2	–1	–1 (maximin)
Column maximum		1	2	3	
		(minimax)			

From above **table 3.36**, it is shown that the minimax value (1) is not equal to the maximin value (–1). Thus saddle point does not exist. The game can be solved using mixed strategies. Here we cannot apply the dominance rule, because the payoff matrix cannot be reduced. Hence we use the linear programming method to solve the game. We have to add the absolute value of the most negative value plus (K = 1 +1) to every entries of the payoff matrix because payoff matrix has negative values. We get a new revised payoff matrix as shown in **table 3.37**.

Table 3.37: Payoff Matrix after Adding K to Each Entry

		Player B		
		1	2	3
Player A	1	3	1	1
	2	1	1	5
	3	1	4	1

We have to develop a linear programming model for Player B which will have only "≤" constraints. The problem can be solved by the following ways:

1) For obtaining the strategies of Player B and game value, solve the model.
2) Using concept of duality, obtain the strategies of Player A from the optimum table of Player B.
3) By subtracting constant K from value of the modified game, obtain the true value of the game.

Since K(2) is added to the every cell of the payoff matrix, hence the last step is needed. Let consider that we have the payoff a_{ij} in case player A chooses his alternative i and Player B chooses his alternative j respectively. Also consider that V is the value of game and p_i is the probability of selection of Alternative i (where, i = 1, 2, 3) by Player A. The q_j be the probability of selection of Alternative j(where, j = 1, 2 and 3) by Player B.

Development of Linear Programming Model with Respect to Player B

The following function shows the expected loss (+)/gain (–) of Player B corresponding to selection of every alternatives of Player A. The necessary condition is that $q_1 + q_2 + q_3 = 1$.

$3q_1 + q_2 + q_3 \le V$; $\quad q_1 + q_2 + 5q_3 \le V$

$q_1 + 4q_2 + q_3 \le V$; $\quad q_1 + q_2 + q_3 = 1$

Whenever we divide the above constraints by value of game, i.e., by V then we get the following:

$$3\frac{q_1}{V}+\frac{q_2}{V}+\frac{q_3}{V}\le 1, \qquad \frac{q_1}{V}+\frac{q_2}{V}+5\frac{q_3}{V}\le 1,$$

$$\frac{q_1}{V}+4\frac{q_2}{V}+\frac{q_3}{V}\le 1 \text{ and } \frac{q_1}{V}+\frac{q_2}{V}+\frac{q_3}{V}=\frac{1}{V}$$

Replacing q_j/V by Y_j (where j = 1, 2, 3) in the above constraints, we get the following:

$3Y_1 + Y_2 + Y_3 \le 1, \qquad Y_1 + Y_2 + 5Y_3 \le 1,$

$Y_1 + 4Y_2 + Y_3 \le 1$ and $Y_1 + Y_2 + Y_3 = \frac{1}{V}$

Converting the above model into linear programming model as per rule for Player B, we get the following linear programming model:

Maximise $\quad Z_2 = Y_1 + Y_2 + Y_3$

Subject to $\quad 3Y_1 + Y_2 + Y_3 \le 1$

$Y_1 + Y_2 + 5Y_3 \le 1$

$Y_1 + 4Y_2 + Y_3 \le 1$

Y_1, Y_2 and $Y_3 \ge 0$

Adding slack variables to the above generalised model, we get the following standard model:

Maximise $Z_2 = Y_1 + Y_2 + Y_3 + 0S_1 + 0S_2 + 0S_3$

Subject to,

$3Y_1 + Y_2 + Y_3 + S_1 = 1$

$Y_1 + Y_2 + 5Y_3 + S_2 = 1$

$Y_1 + 4Y_2 + Y_3 + S_3 = 1$

$Y_1, Y_2, Y_3, S_1, S_2, S_3 \ge 0$

Table 3.38 shows the initial table:

Table 3.38: Iteration 1

CB_i	Basis	1	1	1	0	0	0	Solution	Ratio
		Y_1	Y_2	Y_3	S_1	S_2	S_3		
0	S_1	3	1	1	1	0	0	1	1/3 = 0.33→
0	S_2	1	1	5	0	1	0	1	1/1 = 1
0	S_3	1	4	1	0	0	1	1	1/1 = 1
	Z_j	0	0	0	0	0	0	0	
	$C_j - Z_j$	1↑	1	1	0	0	0		

In the **table 3.39**, it is shown that S_1 is the outgoing variable and Y_1 is the incoming variable. The next iteration is shown below:

Table 3.39: Iteration 2

CB_i	Basis	1	1	1	0	0	0	Solution	Ratio
		Y_1	Y_2	Y_3	S_1	S_2	S_3		
1	Y_1	1	1/3	1/3	1/3	0	0	1/3	1
0	S_2	0	2/3	14/3	–1/3	1	0	2/3	1
0	S_3	0	11/3	2/3	–1/3	0	1	2/3	2/11
	Z_j	1	1/3	1/3	1/3	0	0	1/3	
	$C_j - Z_j$	0	2/3	2/3	–1/3	0	0		

The next iteration is shown in **table 3.40**. Here S_3 is the outgoing variable and Y_2 is the incoming variable.

Table 3.40: Iteration 3

CB_i	Basis	1	1	1	0	0	0	Solution	Ratio
		Y_1	Y_2	Y_3	S_1	S_2	S_3		
1	Y_1	1	0	3/11	4/11	0	–1/11	3/11	1
0	S_2	0	0	50/11	–3/11	1	–2/11	6/11	3/25
1	Y_2	0	1	2/11	–1/11	0	3/11	2/11	1
	Z_j	1	1	5/11	3/11	0	2/11	5/11	
	$C_j - Z_j$	0	0	6/11	–3/11	0	–2/11		

The next iteration is shown in **table 3.41**. Here S_2 is the outgoing variable and Y_3 is the incoming variable.

Table 3.41: Iteration 4

CB_i	Basis	1	1	1	0	0	0	Solution
		Y_1	Y_2	Y_3	S_1	S_2	S_3	
1	Y_1	1	0	0	19/50	–3/50	–2/25	6/25
1	Y_3	0	0	1	–3/50	11/50	–1/25	3/25
1	Y_2	0	1	0	–2/25	–1/25	7/25	4/25
	Z_j	1	1	1	6/25	3/25	4/25	13/25
	$C_j - Z_j$	0	0	0	–6/25	–3/25	–4/25	

Since every value of $C_j - Z_j \le 0$, hence we get the optimal solution. Now we have the following solution model:

$$Y_1 = \frac{6}{25},\ Y_2 = \frac{4}{25},\ Y_3 = \frac{3}{25} \text{ and } Z_2 = \frac{13}{25}$$

The following formula can be used to determine the value of V and, q_1, q_2 and q_3

$$V = \frac{1}{Z_2} \quad \text{and } q_j = VY_j,\ j = 1, 2, 3$$

$$\text{Therefore, } V = \frac{1}{Z_2} = \frac{1}{(13/25)} = \frac{25}{13}$$

$$\text{Thus the value of original game} = \frac{25}{13} - K = \frac{25}{13} - 2 = -\frac{1}{13}.$$

$$\text{and } q_1 = \frac{25}{13} \times \frac{6}{25} = \frac{6}{13},\ q_2 = \frac{25}{13} \times \frac{4}{25} = \frac{4}{13} \text{ and}$$

$$q_3 = \frac{25}{13} \times \frac{3}{25} = \frac{3}{13}$$

Using the duality concept, the value of X_1, X_2 and X_3 can be obtained as shown in **table 3.42**:

Table 3.42: Solution of Player A

Basic Variable in the Initial Table	S_1	S_2	S_3
Corresponding Dual Variable	X_1	X_2	X_3
$-(C_j - Z_j)$ from table 12.62	6/25	3/25	4/25

The solutions of player A is given below:

$$X_1 = \frac{6}{25},\ X_2 = \frac{3}{25},\ X_3 = \frac{4}{25},\ Z_1 = \frac{13}{25}$$

Using the following formula, one can compute the value of V and, p_1, p_2 and p_3:

$$V = \frac{1}{Z_1},\quad p_i = VX_j,\ i = 1, 2, 3$$

Therefore,

$$V = \frac{1}{Z_1} = \frac{1}{13/25} = \frac{25}{13}$$

$$\text{The value of the original game} = \frac{25}{13} - K = \frac{25}{13} - 2 = -\frac{1}{13}$$

$$p_1 = \frac{25}{13} \times \frac{6}{25} = \frac{6}{13}$$

$$p_2 = \frac{25}{13} \times \frac{3}{25} = \frac{3}{13} \text{ and } p_3 = \frac{25}{13} \times \frac{4}{25} = \frac{4}{13}$$

Now we get the strategies of Player A and Player B in summarised form as follows:

$$A\left(\frac{6}{13}, \frac{3}{13}, \frac{4}{13}\right) \text{ and } B\left(\frac{6}{13}, \frac{4}{13}, \frac{3}{13}\right)$$

Where value of the original game is –1/13.

3.5. EXERCISE

3.5.1. Short Answer Type Questions

1) What is decision theory?
2) Write the steps of decision making?
3) Explain the decision-tree approach?
4) What are the elements of decision making?
5) What are the uses of decision tree?
6) What is game theory?
7) Define saddle point.
8) What are two person zero-sum games?
9) Discuss the odds method for solving 2×2 game.
10) Define strategy and what are the strategies used by the players in a game.
11) What are the assumptions and limitations of the game?
12) Differentiate pure strategy and mixed strategy.
13) Define payoff matrix of a game.
14) Explain method of dominance in game theory.
15) Apply minimax regret criterion to solve the following decision problem:

	E_1	E_2	E_3
A_1	70	30	15
A_2	50	45	10
A_3	30	30	20

[Ans: Strategy A_1 is to be adopted.**]**

16) Apply minimax regret criterion to solve the following decision problem:

	E_1	E_2	E_3	E_4
A_1	40	–1	60	180
A_2	200	50	4	0
A_3	200	150	–20	10

[Ans: Optimal strategy A_1**]**

17) Apply EMV criterion to solve the following decision problem:

	A_1	A_2	A_3	Probability
E_1	25	–10	–125	0.1
E_2	400	440	400	0.7
E_3	650	740	750	0.2

[Ans: Optimal strategy A_2**]**

3.5.2. Long Answer Type Questions

1) Dr. Thomas has been thinking about starting his own independent nursing home. The problem is to decide how large the nursing home should be. The annual returns will depend on both the size of nursing home and a number of marketing factors. After a careful analysis, Dr. Thomas developed the following **table:**

Table

Size of Nursing Home	Good Market (₹)	Fair Market (₹)	Poor Market (₹)
Small (S)	50,000	20,000	–10,000
Medium (M)	70,000	35,000	–25,000
Large (L)	90,000	35,000	–45,000
Very Large (VL)	2,00,000	25,000	–1,20,000

i) What is the maximax decision?
ii) What is the maximin decision?
iii) What is equally likely decision?
iv) What is the criterion of realism decision? Use $\alpha = 0.8$.

[Ans: (i) VL (ii) S (iii) VL (iv) VL]

2) A food product company is contemplating the introduction of a revolutionary new product with new packaging to replace the existing product at a large increase in price (S_1) or a moderate change in composition of the existing product with a new packaging at a small increase in price (S_2) or a small change in the composition of the existing product with a negligible increase in price (S_3). The three states of nature are:

i) High increase in sales,
ii) No change in sales (N_2), and
iii) Decrease in sales (N3).

The marketing department of the company worked out the pay-offs in terms of yearly net profits for each course of action for these events. This is represented in the following table:

Table

States of Nature	Courses of Action		
	S_1	S_2	S_3
N_1	₹7,00,000	₹5,00,000	₹3,00,000
N_2	₹3,00,000	₹4,50,000	₹3,00,000
N_3	₹1,50,000	0	₹3,00,000

Which strategy should the company choose on the basis of:

i) Maximin criterion,
ii) Maximax criterion,
iii) Minimax regret criterion, and
iv) Laplace criterion

[Ans: (i) S_3 (ii) S_1 (iii) S_1 (iv) S_1]

3) A decision problem has been expressed in the following pay-off table:

Table

Action	Outcome		
	I	II	III
A	10	20	26
B	30	30	60
C	40	30	20

i) What is the minimax pay-off action?
ii) What is the minimum opportunity loss action?

[Ans: (i) B (ii) B]

4) A company is currently working with a process, which, after paying for materials, labour, etc. brings a profit of 12,000. The company has the following alternatives:

i) The company can conduct research R_1 which is expected to cost ₹ 10,000 and having 90% probability of success. If successful, the gross income will be ₹ 26,000.
ii) The company can conduct research R_2, expected to cost ₹ 6,000 and having a probability of 60% success. If successful, the gross income will be ₹ 24,000.
iii) The company can pay ₹ 5,000 as royalty of a new process which will bring a gross income of ₹ 20,000.
iv) The company may continue the current process.

Because of limited resources, only one of the two types of research can be carried out at a time. Draw the decision tree and find the optimal strategy for the company.

[Ans: Company pay ₹ 5,000 as royalty of the new process to earn maximum expected profit of ₹ 15,000].

5) The daily demand for the breads in the city can assume one of the following values: 3400, 3600, 3700 or 3800 breads with the probabilities 0.18, 0.12, 0.20 and 0.50. If the stockiest stocks more than the needs, he can return them at a discount price of ₹8 per bread.

Assuming that he pays ₹ 8.50 per bread and sells it for ₹ 9.50 per bread, find the optimum stock level by using a decision tree representation.

[Ans: Optimum stock level is 3,800 breads.]

6) A company is currently working with a process, which, after paying for materials, labour, etc. brings a profit of 12,000. The company has the following alternatives:

i) The company can conduct research R_1 which is expected to cost ₹10,000 and having 90% probability of success. If successful, the gross income will be ₹ 26,000.
ii) The company can conduct research R_2, expected to cost ₹ 6,000 and having a probability of 60% success. If successful, the gross income will be ₹24,000.
iii) The company can pay ₹ 5,000 as royalty of a new process which will bring a gross income of ₹20,000.
iv) The company may continue the current process.

Because of limited resources, only one of the two types of research can be carried out at a time. Draw the decision tree and find the optimal strategy for the company.

[Ans: Company pay ₹ 5,000 as royalty of the new process to earn maximum expected profit of ₹ 15,000].

7) What is the optimal strategy in the game described by the matrix.

$$\begin{pmatrix} -5 & 3 & 1 & 20 \\ 5 & 5 & 4 & 6 \\ -4 & -2 & 0 & -5 \end{pmatrix}$$

[Ans: Optimal Strategy for

$$A=\begin{bmatrix} A_1 & A_2 \\ -1/16 & 17/16 \end{bmatrix}, B=\begin{bmatrix} B_1 & B_2 \\ -7/8 & 15/8 \end{bmatrix}]$$

8) Reduce the following two person zero sum game to 2×2 order and obtain the optimal strategies for each player and the value of the game.

Player B

Player A		B_1	B_2	B_3	B_4
	A_1	3	2	4	0
	A_2	3	4	2	4
	A_3	4	2	4	0
	A_4	0	4	0	8

[Ans: Optimal Strategy for Player A = (0, 0, 2/3, 1/3)
Optimal Strategy for Player B = (0, 0, 2/3, 1/3)]

Value of the Game (for A) = $\frac{8\times4+0\times4}{8+4} = 8/3$

9) Solve the following game graphically.

Player B

Player A		
	-3	1
	5	3
	6	-1
	1	4
	2	2
	0	-5

[Ans: Value of the game $= \frac{17}{5}$]

10) Solve the following game:

B's Strategy

A's Strategy	B_1	B_2	B_3
A_1	3	4	-2
A_2	-3	0	1
A_3	-1	-4	2

[Ans: (Optimal Strategy of A: 21/52, 12/52, 19/52
Optimal Strategy of B: 2/13, 3/13, 8/13
and Game Value V = 2/13)]

11) Reduce the following two-person zero-sum game to 2×2 order, and obtain the optimal strategies for each player and the value of the game:

Player B

Player A	B_1	B_2	B_3	B_4
A_1	3	2	4	0
A_2	3	4	2	4
A_3	4	2	4	0
A_4	0	4	0	8

[Ans: The optimal strategy for A is (0, 0, 2/3, 1/3), for B it is (0, 0, 2/3, 1/3) and the game value V = 8/3]

12) Find the saddle point (or points) and hence solve the following game:

Player B

$$\text{Player A}\;\begin{matrix} A_1 \\ A_2 \\ A_3 \end{matrix}\begin{bmatrix} 15 & 2 & 3 \\ 6 & 5 & 7 \\ -7 & 4 & 0 \end{bmatrix}\quad (\text{columns } B_1\; B_2\; B_3)$$

[Ans: (A_2, B_2), v = 5]

13) Find the saddle point (or points) and hence solve the following game:

B

$$\begin{matrix} A_1 \\ AA_2 \\ A_3 \end{matrix}\begin{bmatrix} 1 & 7 & 3 & 4 \\ 5 & 6 & 4 & 5 \\ 7 & 2 & 0 & 3 \end{bmatrix}\quad (\text{columns } B_1\; B_2\; B_3\; B_4)$$

[Ans: (A_2, B_2), v = 4]

14) Find the saddle point (or points) and hence solve the following game:

B

$$\begin{matrix} I \\ A\,II \\ III \end{matrix}\begin{bmatrix} -5 & 2 & 1 & 20 \\ 5 & 5 & 4 & 6 \\ 4 & -2 & 0 & -5 \end{bmatrix}\quad (\text{columns I II III IV})$$

[Ans: (II, III), v = 4]

15) Solve the following game with the help of linear programming approach whose pay-off matrix is given by:

$$\begin{matrix} I \\ II \end{matrix}\begin{bmatrix} 6 & 8 & 6 \\ 4 & 12 & 2 \end{bmatrix}\quad (\text{columns I II III})$$

[Ans: (I, I), (I, III), v = 6]

16) Solve the following game with the help of linear programming approach whose pay-off matrix is given by:

Player B

$$\text{Player A}\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & -1 \\ 1 & 3 & -2 \end{bmatrix}$$

[Ans: (I, I) or (I, III), v = 1 for A, v = −1 for B]

Unit 4

Inventory and Replacement Models

4.1. INVENTORY

4.1.1. Introduction

Inventory denotes "stock of goods". Various authors have defined this word in their own way. In terms of accounting, the word may be used to refer to stock of finished goods only, while in a manufacturing concern this may include work in process, stores, raw materials, etc.

According to International Accounting Standard Committee (I.A.S.C) defines inventories as "Tangible property

1) Held for sale in the ordinary course of business,
2) In the process of production for such sale, or
3) To be consumed in the process of production of goods or services for sale".

According to The American Institute of Certified Public Accountants (AICPA) defines "Inventory in the sense of tangible goods, which are held for sale, in process of production and available for ready consumption".

According to Bolten S.E., "Inventory refers to stock-pile of product, a firm is offering for sale and components that make up the product".

4.1.2. Types of Inventory

Following are the main types of inventory:

1) **Movement Inventories:** This is the inventory which is in the transportation mode and cannot be used for production purpose during that time. This type of inventory is also known as transit or pipeline inventory.
2) **Buffer Inventories:** This kind of inventory is maintained to protect against fluctuations in demand and supply. While a firm may estimate its demand and supply based on past experience, it cannot predict it exactly. In order to keep the production process smooth during such uncertainties, the firm maintains buffer inventories. This type of inventory is also known as 'safety stocks'.
3) **Anticipation Inventories:** These inventories are held by the company to meet future requirements. The production is done for a specialised period. **For example,** a company may decide to hold inventory of umbrellas before the arrival of monsoon and water-heaters before the winter season.
4) **Decoupling Inventories:** This type of inventory is meant to decouple various manufacturing processes from each other. **For example,** a machine may use the product of another machine as its input. In such case, it is necessary to maintain intermediate inventory, so that both the machines can work simultaneously.
5) **Cycle Inventories:** This type of inventory is held on account of the difference in the exact requirements of the firm and the order placed by the firm.
6) **Independent Demand Inventory:** This type of inventory results from forecasted demand. It generally consists of finished products and is not related to any higher level item. **For example,** complete finished products, soap and automobile.
7) **Dependent Demand Inventory:** This inventory consists of items which have derived demand. **For example,** general parts, materials, assemblies and components used in a finished product.

4.1.3. Costs Involved in Inventory Problem

Following are the four major costs involved in inventory problem:

1) **Ordering Cost/Set-up Cost:** The cost suffered/experience when the inventory is reordered/replenished is known as **ordering cost.** These costs are related with the processing and chasing of the purchase order, transportation, inspection for quality, expediting overdue orders, etc. These are also known as procurement cost. The ordering cost incurred is similar to set-up cost i.e. when the units are made within the organization. It describes the cost which has occurred while framing the production plan, the usage of resources while making the production system ready, etc. It consists of the following:
   i) **Reorder Cost:** The cost of allocating and preparing an order by purchase and accounts department is known as reorder cost. It includes the cost incurred when the goods are purchased or if goods are produced by the firm or while organizing the production process.
   ii) **Purchasing Cost:** It is also known as manufacturing or variable cost. It basically depends on whether the firm is buying the goods from the supplier or producing itself.
   iii) **Transportation Cost:** The transportation cost is a cost which is not included in the price of purchased goods just for making it easy. The fixed transportation costs are included in the reorder cost and the variable costs are included in the purchasing cost.

2) **Carrying Cost:** Carrying cost refers to the cost which is incurred due to storing of an item in an inventory. It is also popular by the name of **holding cost or shortage cost**. The carrying cost is equal to the amount of inventory and the time period until which it is stored.

The elements of carrying cost include the following:

i) The opportunity cost of capital invested in the stock.

ii) The costs directly linked/related to storing goods like store-men's salary, rates, heating and lighting, racking and palletization, protective clothing, store's transport, etc.

iii) The obsolescence cost includes scrapping and possible rework.

iv) The deterioration costs and costs incurred in preventing deteriorations.

The carrying cost is generally exhibited in the form of rate per unit or as a percentage of inventory value. It is assumed to be fixed for each unit of certain product of inventory store for a unit time. It consists of the following:

i) **Opportunity Cost:** The opportunity cost refers to as the cost showing the return on investment the firm would earn if the money had been invested in better profit bearing economic activity like in stock market instead of inventory. The cost is usually based on the standard banking interest rate.

ii) **Warehousing Cost:** Warehousing cost refers to the sum paid in the form of fee for the storage of goods in a third party's warehouse. If the company has its own warehouse, it has to bear cost such as space and equipment costs, personnel wages, insurance on inventories, maintenance costs, energy costs and state taxes etc.

3) **Shortage/Stock out Cost:** The stock out cost refers to the cost related to not catering to the customers. Stock out indicates shortage. If there is internal stock-out, then this means that the production will stop and it will result into wastage of time of both workers and machines and it will also lead to delay in the work which will lead the firm to bear loss. If there is external stock out then there will be loss in sales due to loss of potential sales or loss of customer goodwill.

The shortage will arise because of the different kind of reactions from different customers due to which there will backorder or lost sales. In backorder, the sales will be delayed instead of loss in the sales. These are divided into the following:

i) **Lost Sales Costs:** When the customer buys an unavailable item from the competitor, it is known as lost sale. The lost sales costs consist of the profit that would have been earned when the sale was done and also the adverse impact on the future sale due to shortage.

ii) **Back Order Costs:** The delay in sales occurs when there is any kind of shortage if the goods are not easily exchangeable. This leads to adverse impact on the future sale and also leads to fines.

4) **Purchase Cost:** It is also known as **nominal cost** of inventory. This is the purchase price of the item which is purchased from external sources and the cost of the production if the items are made inside the organization. Purchase costs depend on the quantity of items purchased. Usually, there is a situation where it may be specified like the unit price of the product is ₹40 for an order upto 50 units and ₹49.50 if the order is more than 50 units.

4.1.4. Meaning of Inventory Management

Inventory costs are generally very high for business concerns. A firm generally invests a substantial amount of resources in its inventory. In many cases, inventory cost constitutes about 90 per cent of the total working capital.

It is important for a firm to properly manage its inventory. Inventory management implies proper planning related to purchasing, handling, storing and accounting of goods. Inventory management decides, "what to purchase, how much to purchase, where to purchase from, where to store, etc."

Inventory management also ensures that the firm maintains optimum level of inventory, minimising the chances of over or under-stocking. Over-stocking may cause cash crunch whereas under-stocking may cause interruption in production process.

4.2. INVENTORY MODEL

4.2.1. Introduction

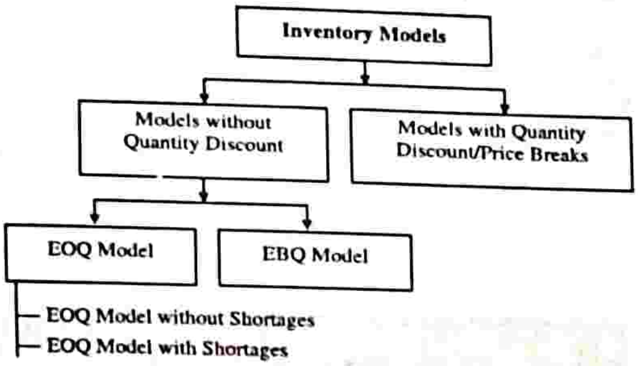
Inventory models are used for estimating the amount of inventory required with the firm by the help of financial equations. The basic problems in inventory models are the quantity and timing of orders. It helps in estimating the amount and time at which new inventory should be order. It is done for providing the timely delivery and accurate amount of product for the customers.

The level of uncertainty with lead-time or demand affects the inventory model of an organisation. The objective of inventory model is to estimate the accurate amount of inventory to be stored for minimizing the total inventory costs. It also helps in balancing the different types cost like purchase cost, ordering cost/set-up cost, carrying cost and stock-out cost, etc.

There are various models for inventory management, each of which attempts to reduce the overall inventory cost by striking a balance between various components of cost, viz. purchase cost, carrying / holding cost, ordering cost, stockout cost, etc. Such models are broadly categorized as independent demand type or dependent demand type. Under the **independent demand model**, the demand of a specific item of inventory does not depend upon the demand of other items of inventory, whereas under the **dependent demand model**, the demand of a specific item of inventory depends upon the demand of other items of inventory. An **example** of independent demand model is finished goods, the demand of which does not depend upon the demands of other items of inventory. Inventory of assembly – components is an **example** of dependent demand model.

Quantity discount model form of an Economic Order Quantity (EOQ) model that takes into account quantity discounts. **Quantity discounts** are price reductions designed to induce large orders. If quantity discounts are offered, the buyer must weigh the potential benefits of reduced purchase price and fewer orders against the increase in carrying costs caused by higher average inventories. Hence, the buyer's goal in this case is to select the order quantity that will minimise total costs, where total cost is the sum of carrying cost, ordering cost, and purchase cost.

The types of inventory model are as follows:



1) **Models without Quantity Discount:** It is also known as **independent demand models.** This model work on the principle that demand of one good is independent of other goods. These independent demand is of finished goods as its demand depends on some environmental factors like sales forecasts, consumer trends, etc. it takes the help of mathematical methods like forecast demand, order size and costs.

   The types of models without quantity discount are as follows:
   i) Basic EOQ Model,
      a) EOQ without shortage
      b) EOQ with shortage
   ii) EBQ Model.

2) **Models with Quantity Discount:** It is also known as **dependent demand models.** It work on the principle which states that the demand of an item is related to the present existing production plans or operating schedule like the number of keyboard to be produce by the company is related to the number of computers manufactured by them. The raw materials, components and assemblies which are part of the production of finished goods also have dependent demand.

4.2.2. EOQ (Economic Order Quantity) Model

Economic order quantity (EOQ) model is one of the oldest and most commonly known techniques. This model was first developed by **Ford Harris** and **R. Wilson** independently in 1915. The objective is to determine economic order quantity, Q, which minimizes the total cost of an inventory system when the demand occurs at constant rate. Ordering costs and Carrying costs are taken into consideration while determining Economic Order Quantity. Ordering cost is basically the costs associated with receiving an inventory while carrying costs includes handling warehousing and allied costs. Imbalance between these two costs can affect the profits adversely, so balance needs to be maintained besides keeping both of them at a minimum level. The point at which ordering cost is equal to carrying cost is called EOQ, which can also be determined from the graph shown. Assuming that inventory is allowed to fall to zero and then immediately replenished, the average inventory becomes EOQ/2. EOQ Model can be presented in **figure 4.1.**

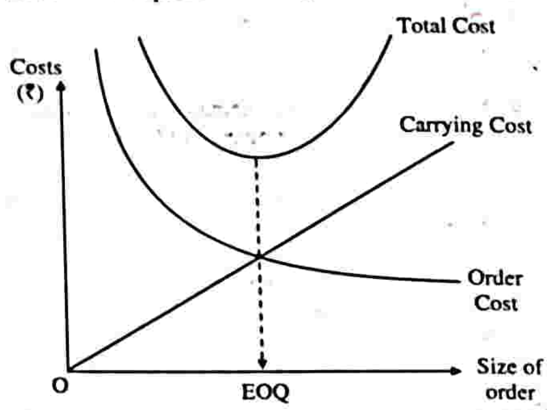


Figure 4.1: Graphical Presentation of EOQ

In the **figure 4.1**, it can be observed that when the size of the order increases, the ordering cost of an item decreases. This is result of large order size, the total number of orders will decrease and similarly the ordering cost. But at the same time, because of high inventory levels the carrying cost is bound to increase. The point of intersection results in EOQ and at this point both costs are equal and the Total Cost is at the lowest level.

EOQ may also be calculated with the help of the following formula:

$$EOQ(Q) = \sqrt{\frac{2DO}{h}}$$

Where, D = Demanded Annual quantity (in units)
O = Cost of ordering/placing (fixed cost)
h/ic = Cost of holding one unit/Annual carrying cost per unit.

Calculation of Number of Orders

$$\text{Number of Order per Year} = \frac{\text{Annual Demand}}{\text{EOQ or Q}}$$

Calculation of Total Inventory Cost = Cost of Material $(D \times P)$ + Ordering Cost per Annum $\left(\frac{D}{EOQ} \times O\right)$ + Carrying Cost per Annum $\left(\frac{EOQ}{2} \times h\right)$

Example 1: A company annually uses 24,000 units of raw material costing ₹2.5per unit. Considering each order costs ₹30 and the carrying costs are 15% per year per unit of the average inventory. Find the EOQ.

Solution: $EOQ(Q) = \sqrt{\frac{2DO}{h}}$

Here, Annual Consumption (D) = 24,000 units,
Ordering Costs (O) = ₹30 per unit
Inventory Carrying Costs (ic) = ₹2.5 per unit
Now, h = ic = 15% per year per unit of average inventory

$$= 0.15 \times 2.5 = 0.375$$

$$EOQ = \sqrt{\frac{2 \times 24000 \times 30}{0.375}} = 1960 \text{ Units}$$

Example 2: Find EOQ if annual demand 15000 units, ordering cost ₹125/order and carrying cost ₹15/unit/year.

Solution: $EOQ(Q) = \sqrt{\frac{2DO}{h}}$

Given, Annual Demand (D) = 15000 units
Ordering Cost (O) = ₹125
Carrying Cost (h) = ₹15

$$EOQ = \sqrt{\frac{2 \times 125 \times 15000}{15}}$$
$$= \sqrt{250000} = 500 \text{ units}$$

Example 3: From the following information, find out the EOQ and total variable cost associated with the ordering policy:
Annual Demand (D) = 2,500
Ordering Cost (O) = ₹10 per order
Inventory carrying cost (ic) = 20% of average inventory value

Solution: $EOQ \text{ (in rupees)} = \sqrt{\frac{2DO}{ic}}$

$$= \sqrt{\frac{2 \times 2{,}500 \times 10}{0.20}} = \sqrt{1{,}25{,}000} = ₹354$$

$$\text{Total Cost } T(Q) = \sqrt{2 \times 2{,}500 \times 10 \times 0.20} = ₹100$$

Example 4: Calculate EOQ from the following information:
Annual Usage, 20,000 units
Cost of placing and receiving one order ₹100
Cost of materials per unit ₹50
Annual carrying cost of one unit: 10% of inventory value.

Solution: $EOQ = \sqrt{\frac{2DO}{h}}$

Where, Annual consumption in units (D) = 20,000 units
Cost of placing an order (O) = ₹100 per order
Inventory carrying cost of one unit (h)
= ₹50 × 10% = 5 per unit

$$EOQ = \sqrt{\frac{2 \times 20{,}000 \times 100}{5}} = 894 \text{ units}$$

Example 5: Following information relating to a type of material is available.

1) Annual demand	4800 units
2) Unit price	₹2.40
3) Ordering cost per order	₹8.00
4) Storage cost	2% p.a.
5) Interest rate	10% per annum
6) Lead time	Half month

Calculate EOQ and total annual inventory cost from the above information.

Solution: $EOQ = \sqrt{\frac{2DO}{h}}$

Given, Annual Demand (D) = 4800 units
Ordering Cost (O) = 8 per order

$$\text{Carrying Cost (h)} = \left(2.40 \times \frac{2}{100}\right) + \left(2.40 \times \frac{10}{100}\right)$$
$$= 0.048 + 0.24 = 0.288$$

$$EOQ = \sqrt{\frac{2 \times 4800 \times 8}{0.288}} = 516 \text{ units}$$

$$\text{Number of Orders} = \frac{\text{Annual Demand}}{EOQ} = \frac{4800}{516} = 9.30$$

Total Inventory Cost = Cost of Material + Ordering Cost + Carrying Cost

$$= (4800 \times 2.40) + \left(\frac{4800}{516} \times 8\right) + \left(\frac{516}{2} \times 0.288\right)$$
$$= 11{,}520 + 74 + 74 = ₹11{,}668$$

Assumptions of EOQ

The model is based on the following basic assumptions:

1) Supply is available in the market and goods can be procured as and when required.
2) The quantity to be procured is pre-decided.
3) The prices of goods are constant.

Weaknesses of EOQ

EOQ has the following weaknesses:

1) **Erratic Usages:** The EOQ model assumes that the usage of materials can be predicted and evenly distributed throughout the year. So, even a slight deviation will make the formula of EOQ unfit and in reality they do not exist such situations. In order to cover such situations, some more formulas need to develop, which will make this more complex.

2) **Faulty Basic Information:** Two main components on which the calculations of EOQ are based are ordering cost and the carrying cost. So, EOQ can be ascertained only if correct figures of ordering cost and carrying cost are available. Practically this may not be possible as they cannot be calculated exactly all the time. This may vary from product to product and thus in many cases the formula will not give true results.

3) **Costly Calculations:** Since estimation of cost of ordering and cost of carrying is not easy, it requires services of professionals. Simple calculations are time consuming and complex formulae are rather expensive. In most of the cases the cost involved exceeds the benefits derived and value for money is not achieved.

4) **No Formula is a Substitute for Commonsense:** Sometimes the underlying conditions are such that no formula is as useful as the commonsense. Majority of the businesses today rely on their instincts and past experience for such a judgment.

5) **EOQ Ordering must be Tempered with Judgment:** Few corporate operating goals must be observed while preparing inventory control. These goals may sometimes contradict with the ordering but still emphasis should be given to these goals. Following might be included in EOQ restrictions:
   i) Those items that undergo constant changes like technological should not be put under the ambit of EOQ.
   ii) Items having a lesser shelf life should also be avoided under EOQ method.
   iii) Sales of items that is not constant throughout the year also does not fit under the concept of EOQ.
   iv) EOQ cannot be followed in case of items which are critical in nature and often experiences shortage.

4.2.2.1. EOQ Model without Shortages

EOQ for Simple Inventory Model with no Shortages and Instantaneous Replenishment

This model is characterized by uniformity in the demand and prompt production. It envisages placing of order of same size at periodical intervals. Supply of the items ordered is made promptly and they are consumed also at a constant rate. There is no change in purchase price per unit, notwithstanding the size of a specific order.

$$EOQ(Q) = \sqrt{\frac{2DO}{h \times P}}$$

Where,

D = Annual Demand in units
O = Ordering Cost per order/set-up cost
h = Carrying Cost per unit per year
P = Purchase Price per unit
Q = Order size

Assumptions of the Model

EOQ model without shortages is founded on certain underlying presumptions, which are as following:

1) There exists a definite demand for the items, which remains unchanged and unbroken over a period of time;
2) The **lead time** refers to the time in-between placing the order and having the goods in stock ready for use. It is fixed and known to all. It implies that if the lead time is zero, the supply of ordered items is immediate without any loss of time;
3) Per unit ordering costs and carrying costs remains unchanged irrespective of the quantity ordered, provided the ordered quantity is within the pre-decided (mutually agreed) range;
4) The purchase price of the items does not undergo any change; no discount is allowed even in the cases of bulk purchases; and
5) The supply is expeditious, whenever the level of stock holding reaches at zero. As a result, there are no instances of surplus or shortage of stock.

Inventory under the EOQ model without shortages may be displayed as under (with the presumptions indicated in the foregoing points):

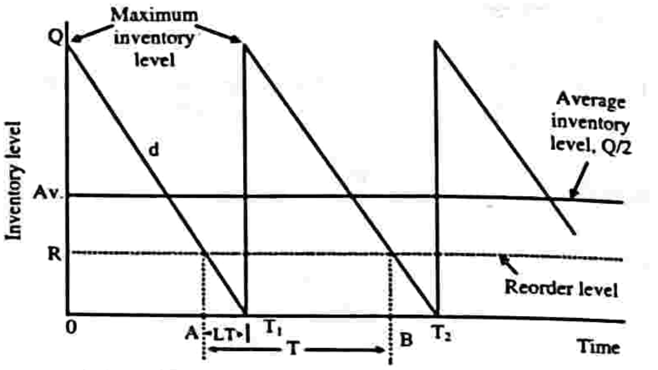


Figure 4.2: Inventory under the Classical EOQ Model

In the above figure, if we start with stock at 'Q' level on time at '0', the stock would be consumed at the rate 'd' unit per day. As the lead time is presumed to be zero, a new order would be required to be placed and supplies would be made at point 'T_1'. Next order would be required to be placed at point 'T_2'.

In case of a lead time 'L', the first order would be required to be placed at point 'A', when the stock level is at a point just sufficient to last during the lead period. By the time fresh supplies arrive, that stock would have been consumed. This is known as **reorder level or reorder point (R)**. In other words, The point at which the store-keeper makes a fresh request for the purchase is known as the reorder level. Generally, it lies between the maximum and the minimum level.

Reordering Level = Maximum Consumption × Maximum Reorder period

The next order needed to be placed at point 'B'.

The gap of time between two ordering points ('A' and 'B' or 'T_1' and 'T_2') is termed as **Inventory Cycle**. It is defined as the time taken in utilizing the whole lot of items received (Q). In the above figure, inventory cycle has been shown as 'T', maximum level of inventory held has been shown as 'Q', whereas the minimum level of inventory has been shown as 'O'. As such, the average inventory level would be Q / 2.

For taking a decision with regard to the level of economic order quantity (EOQ), two types of costs are taken into consideration; they are (i) the ordering/set-up costs, and (ii) holding / carrying costs. Therefore, for a period of one year, the cost model would be as under:

Total Cost (TC) = Total Ordering Cost (O) + Total Holding Cost (h)

$$\text{So, Total Annual Ordering Cost (O)} = \frac{D}{Q}O$$

$$\text{Total Annual Holding Cost (h)} = \frac{Q}{2}h$$

$$\text{Number of Order} = \frac{\text{Annual Demand (D)}}{\text{EOQ (Q)}}$$

It would be worthwhile to mention here that in the above formula, the figure of holding costs is indicated in absolute terms. However, in certain problems, the figure of holding costs is furnished as a proportion or percentage of the inventory value (instead of providing the same in absolute terms).

For example, in a problem, the figure of holding costs is stated to be 15% per annum of the value of an inventory item; if the value of an inventory item is given as ₹40, the holding costs would work out to be 15% of ₹40, or 15 × 40 / 100 i.e. ₹ 6 per unit per annum.

Therefore, if 'i' happens to be the proportion or percentage (also termed as holding rate), and 'c' is the value of inventory items, then the holding cost per unit (h) = ic.

An alternate method of **determining EOQ is graphic and tabulation method,** which involves a lot of complexity.

So, following formula is applied in this method:

$$\text{EOQ (Q)} = \sqrt{\frac{2DO}{h}}$$

(where, the holding cost is given as a proportion/ percentage).

$$\text{Or} \quad \text{EOQ (Q)} = \sqrt{\frac{2DO}{ic}}$$

1) **Annual Total Variable Inventory Cost:** For arriving at the minimum annual inventory cost, Q may be replaced by 'Q' in the cost equation as under:

$$T(Q) = \frac{D}{Q}O + \frac{Q}{2}h = \frac{DO}{\sqrt{\frac{2DO}{h}}} + \frac{\left(\sqrt{\frac{2DO}{h}}\right)h}{2} = \sqrt{2DOh}$$

If the holding cost is indicated in proportion or percentage terms, the formula would be as under;

$$T(Q) = \sqrt{2DOic}$$

2) If the demand ('D') and EOQ (Q*) happen to be uniform and constant, the optimal interval between the two successive orders may be ascertained with the help of the following formula without any difficulty:

The optimal interval between the two successive orders or $T^* = Q/D$.

(T^* is also termed as the inventory cycle time)

3) The optimum number of orders placed per annum (N),which is equal to the reciprocal of T, may also be calculated with the help of the following formula:

N = 1/T.

Example 6: An electronics company, engaged in the business of manufacturing refrigerators, produces 2000 refrigerators per annum. The cost for installing of cooling system is ₹10, for each refrigerator. The holding cost of each cooling system for a year is ₹2.40, ordering cost is ₹150 per order, irrespective of the order size. Find out the optimum number of orders and the order size with a view to reducing the cost function {T (Q)}.

Solution: Before an attempt is made to solve the actual problem, it would be in order to undertake an analysis of the entire prevailing situation.

Analysis: In case of an order for 2000 cooling system, only one order in a year would suffice. If the order size is 1000 cooling system, two orders would be sufficient in a year. Similarly, if the 500 units are ordered at one time, four orders would be required to be placed in a year. As the order cost is fixed for one order, irrespective of its size, the increasing number of orders would result in increasing ordering cost. However, increasing number of orders also mean order for smaller quantity, which leads to reduction in holding cost.

Therefore, there is a need to strike a balance between the ordering cost and holding cost in order to find out the order size with a view to reducing the cost function {T (Q)}. The solution of the above problem involves three distinct steps:

i) Total ordering cost per annum: O (Q) = N × O

Various options available for ascertaining the value are:

a) N=1 Q=2000 and O (Q) = 1×150 = ₹150
b) N=2 Q=1000 and O (Q) = 2×150 = ₹300
c) N=4 Q=500 and O (Q) = 4×150 = ₹600
d) N=5 Q=400 and O (Q) = 5×150 = ₹750

ii) Total Annual Holding Cost: $h(Q) = \frac{Q}{2}h$

Various options available for ascertaining the value are:

$$Q = 2000 \quad h(Q) = \frac{2000}{2} \times 2.40 = ₹2400$$

$$Q = 1000 \quad h(Q) = \frac{1000}{2} \times 2.40 = ₹1200$$

$$Q = 500 \quad h(Q) = \frac{500}{2} \times 2.40 = ₹600$$

$$Q = 400 \quad h(Q) = \frac{400}{2} \times 2.40 = ₹480$$

iii) Total (variable) Annual Inventory Cost:

Calculation of Total Cost				
Order Quantity	Number of orders per annum	Annual Ordering Cost	Annual Holding Cost	Total Annual Cost
100	20	3,000	120	3,120
200	10	1,500	240	1,740
400	5	750	480	1,230
500	4	600	600	1,200
1,000	2	300	1,200	1,500
2,000	1	150	2,400	2,550

From the above table, it may be seen that the total annual cost is lowest when the order size is 500 units. Thus, the EOQ or Q is equal to 500 units. It is interesting to note that at this level, the ordering cost and holding cost are equal (i.e. ₹600).

As mentioned earlier, the graphic and tabulation method of ascertaining the value of EOQ is rather complicated. Following formula is applied in this regard:

$$\text{EOQ (Q)} = \sqrt{\frac{2DO}{h}}$$

By putting various values given in the problem (O = ₹150 per order, h = ₹2.40 per unit per annum, and D = 2,000 units), we get the following:

$$Q = \sqrt{\frac{2\times 2000\times 150}{2.40}} = \sqrt{250000} = 500 \text{ units}$$

Example 7: From the following information, find out the EOQ and total variable cost associated with the ordering policy:

Annual Demand (D) = ₹50000
Ordering Cost ((O) = ₹200 per order
Inventory Carrying Cost (ic) = 40% of average inventory value

Solution: EOQ (in rupees) = $\sqrt{\frac{2DO}{ic}}$

$$= \sqrt{\frac{2\times 50000\times 200}{0.40}} = \sqrt{5{,}00{,}00{,}000} = ₹7{,}071$$

Total cost T (Q) $= \sqrt{2\times 50000\times 200\times 0.40} = ₹2{,}828$

Example 8: The annual requirement of one of the components used in the main product of Toy Industry is 5400 units. The ordering cost is ₹250 per order and the annual carrying / holding cost is ₹30 per unit.

Find out the (i) EOQ, (ii) number of orders per year, and (iii) the time gap between two successive orders (lead time).

Solution: Given, D = 5,400 units per year
O = ₹250 per order
h = ₹30 per unit per year

i) $$\text{EOQ}(Q) = \sqrt{\frac{2DO}{h}} = \sqrt{\frac{2\times 5400\times 250}{30}}$$
$$= \sqrt{90{,}000} = 300 \text{ units}$$

ii) Number of orders/ year $= \frac{D}{Q} = \frac{5400}{300} = 18$

iii) Time between successive orders (Lead Time)
$$= \frac{Q}{D} = \frac{300}{5400} = 0.0556 \text{ year}$$
$$= 0.6672 \text{ month} = 20 \text{ days (approx.)}$$

Example 9: Raw material requirements of a small unit firm are fulfilled by a supplier / vendor. The annual demand of the firm is 4500 units; ordering costs and carrying costs are ₹ 50 per order and 10% of the purchase price per unit per month respectively. If the purchase price is Re.1 per unit, find the following:

i) Economic Order Quantity (EOQ);
ii) Total cost w.r.t. EOQ;
iii) Number of orders placed in a year; and
iv) Time between two successive orders (lead time).

Solution: Given: D = 4500 units per year;
A = ₹50 per order;
p = Re. 1 per unit; and
h = Re.0.1 per unit per month = ₹1.2 per unit per year

i) The economic order quantity (EOQ)
$$\text{EOQ}(Q) = \sqrt{\frac{2DO}{h}} = \sqrt{\frac{2\times 4500\times 50}{1.2}} = \sqrt{3{,}75{,}000}$$
$$= 612.37 \text{per units(approx.)}$$

ii) Total cost $(TC) = \left(\frac{D}{Q}\right)O + \left(\frac{Q}{2}\right)h + pD$
$$= (4500/612)\times 50 + (612/2)\times 1.2 + 1\times 4500$$
$$= ₹5{,}308.37 \text{ per annum}$$

iii) Number of orders per year = D/Q
= 4500/612
= 7.35 order per year

iv) Time between two consecutive orders = Q/D
= 612/4500
= 0.136 year
= 1.632 month
= 48.96 days

Example 10: Compute the EOQ and the total variable cost for the following:
Annual Demand: 25 units
Unit Price: ₹2.50
Order Cost: ₹4.00
Storage Rate: 1% per year
Interest Rate: 12% per year
Obsolescence Rate: 7% per year.

Solution: Carrying Cost (h) = (Storage Rate + Interest Rate + Obsolescence Rate) × Unit Price
$$= \frac{1+12+7}{100}\times 2.50$$
$$= ₹0.50 \text{ per unit per year}$$

$$\text{EOQ} = \sqrt{\left(\frac{2DO}{h}\right)} = \sqrt{\left(\frac{2\times 25\times 4}{0.50}\right)} = 20 \text{ units}$$

$$\text{Total Variable Cost} = \sqrt{2DOh} = \sqrt{2\times 25\times 4\times 0.50} = ₹10$$

Example 11: From the following data compute the EOQ and total variable cost:

Annual Demand	5,000 units
Unit price	₹20.00
Order Cost	₹16.00
Storage Rate	2% per annum
Interest Rate	12% per annum
Obsolescence Rate	6% per annum

Solution: Carrying Cost (h) = (Storage Rate + Interest Rate + Obsolescence Rate) × Unit Price

$$= \frac{2+12+6}{100} \times 20$$

= ₹4 per unit per annum

$$EOQ = \sqrt{\left(\frac{2DO}{h}\right)} = \sqrt{\left(\frac{2\times5{,}000\times16}{4}\right)} = 200 \text{ units}$$

$$\text{Total Variable Cost} = \sqrt{2DOh}$$

$$= \sqrt{2\times5{,}000\times16\times4} = ₹800$$

Example 12: A manufacturing company has determined from an analysis of its accounting and production data for a certain part that:

1) Its demand is 9000 units per annum and is uniformly distributed over the year,
2) Its cost price ₹2 per unit,
3) Its ordering cost is ₹40 per order, and
4) The inventory carrying charge is 9 per cent of the inventory value.

Further its known that lead time is uniform and equals 8 working days, and that the total working days in a year are 300.

Calculate:

1) Economic order quantity (EOQ);
2) Optimum number of orders per annum;
3) Total variable cost;
4) Length of inventory cycle;
5) Amount of savings that would be possible by switching to the policy of ordering EOQ of 3,000 units from the present policy of ordering the requirements of this part thrice a year.

Solution: Given,
Annual Demand (D) = 9,000 units per year,
Ordering Cost (O) = ₹40 per order,
Inventory Carrying Charge (i) = 9% or 0.09,
Cost Price (c) = ₹2/ unit,
Therefore, Holding Cost (h) = i × c = 0.09 × 2 = 0.18.

1) $EOQ(Q) = \sqrt{\frac{2DO}{h}} = \sqrt{\frac{2\times9{,}000\times40}{0.18}} = 2{,}000 \text{ units}$

2) Optimum number of orders per year (N) = D/Q
= 9,000/2,000 = 4.5

3) Total Variable Cost, $T(Q) = \sqrt{2DOh}$
$= \sqrt{2\times40\times9000\times0.18}$
= ₹360

4) Length of Inventory Cycle (T) = Q/D
= 2,000/9,000
= 0.222 year or 0.222 × 300
= 66.7 days
Alternatively, T (in days) = Q/demand per day
= 2,000/30 = 66.7days

5) For the present policy of an order quantity = 3,000 units,
Ordering Cost (O) = 40 × 3 = ₹120
Holding Cost (h) = (3,000/2) × 0.18 = ₹270
T (3,000) = 120 + 270 = ₹390
Thus, Saving in Cost = ₹390 – ₹360 = ₹30 per year.

Example 13: A company currently replenishes its stock of a certain item by ordering enough supply to cover one month demand. The annual demand of the item 1500 units. It is estimated that it costs ₹20 every time an order is placed. The holding cost per unit per month is ₹2 and not shortage is allowed. Determine the optimal order quantity and the time between orders. Also find the difference in annual inventory costs between the optimal policy and the current policy of ordering one month supply 12 times a year.

Solution: Given,
D = 1500 units per year = 125 units per month
O = ₹20 per order
h = ₹2 per unit per month

$$\text{Optimum Order Quantity } (Q) = \sqrt{\frac{2DO}{h}} = \sqrt{\frac{2\times125\times20}{2}}$$
$$= 50 \text{ units}$$

Time between two consecutive orders = Q^*/D
= 50/125 = 0.4 month = 12 days.

Total Cost for Optimum Order Quantity

$$(TC) = \left(\frac{D}{Q}\right)O + \left(\frac{Q}{2}\right)h + pD$$
$$= \left(\frac{125}{50}\right)20 + \left(\frac{50}{2}\right)2$$
$$= 50 + 25 = 75 \text{ per month}$$

Optimal Policy $T(Q) = \sqrt{2DOic}$
$= \sqrt{2\times125\times20\times2} = 100$

Optimal Yearly Cost = 100 × 12 = ₹1,200

Current Policy (Q) = 125 monthly demand

Monthly Cost T(125) = Ordering Cost + Holding Cost
$$= O + \left(\frac{Q}{2}\right)h$$
$$= 20 + \frac{125}{2}\times2 = 145 \text{ per month}$$

Yearly Policy Cost = 145 × 12 = ₹1,740

Difference between Annual Inventory Cost
= Optimal Yearly Cost – Yearly Policy Cost
= 1,740 – 1,200 = ₹540

4.2.2.2. EOQ Model with Shortages

(Economic lot size system with instantaneous replenishment)

This model envisages (i) an expeditious supply of the ordered items, (ii) consumption of the received items at a constant rate, and (iii) unchanged purchase price per unit, notwithstanding the order size. The striking feature of this model is the condition that if there is no stock of the items at the time of ordering, the supply would be made at a later date with a penalty. This feature is known as **'Backordering'**.

The model may be expressed in the graphic form as under **(figure 4.3)**:

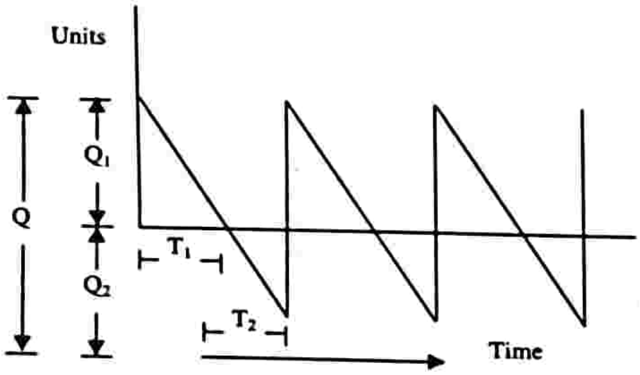


Figure 4.3: Inventory Model with Shortages

Following variables are used in this model:

- D = Demand/period
- h = Carrying cost/unit/period
- O = Ordering cost/order
- C_s = Shortage cost/unit/period
- Q = Order size
- Q_1 = Maximum inventory
- Q_2 = Maximum stockout
- T_1 = Period of positive stock
- T_2 = Period of shortage
- T = Cycle time ($T_1 + T_2$)

Following are the optimal values of the above variables:

$$EOQ(Q) = \sqrt{\frac{2DO}{h}\frac{C_s + h}{C_s}}$$

$$Q_1 = \sqrt{\frac{2DO}{h}\frac{C_s}{C_s + h}}$$

$$Q_2 = Q - Q_1$$

$$T = \frac{Q}{D}, \qquad T_1 = \frac{Q_1}{D}, \ T_2 = \frac{Q_2}{D}$$

Where, Number of orders/period $= \frac{D}{Q}$.

Example 14: The demand for one of the components used by a manufacturing concern is 9000 units per year. The holding / carrying cost of each unit is ₹ 500 per year, the ordering cost is ₹1000 per order and the shortage cost is ₹1500 per unit per year. Find the (i) optimal values of economic order quantity (EOQ), (ii) maximum inventory, (iii) maximum shortage quantity, (iv) cycle time (t), (v) inventory period (T_1), and (vi) shortage period (T_2).

Solution: Given,

- D = 9000units/year
- h = ₹ 500 unit/year
- O = ₹1000/order
- C_s = ₹1500/unit/year

By applying the appropriate formulae,

i) Economic Order Quantity $(Q) = \sqrt{\frac{2DO}{h}\frac{C_s + h}{C_s}}$

$$= \sqrt{\frac{2 \times 9000 \times 1000}{500} \times \frac{1500 + 500}{1500}} = 220 \text{ units (approx.)}$$

ii) Maximum Inventory $(Q_1) = \sqrt{\frac{2DO}{h}\frac{C_s}{C_s + h}}$

$$= \sqrt{\frac{2 \times 9000 \times 1000}{500} \times \frac{1500}{1500 + 500}} = 164 \text{ units (approx.)}$$

iii) Maximum Shortage Quantity $Q_2 = Q - Q_1$

$$= 220 - 164$$
$$= 56 \text{ units}$$

iv) Cycle time $(T) = \frac{Q}{D} = \frac{220}{9000} \times 365 = 9 \text{ days (approx.)}$

v) Period of positive stock $(T_1) = \frac{Q_1}{D}$

$$= \frac{164}{9000} \times 365 = 6.65 \text{ or } 7 \text{ days (approx.)}$$

vi) Period of shortage $(T_2) = T - T_1 = 9 - 7 = 2$ days

Number of orders per year $(N) = \frac{D}{Q} = \frac{9000}{220} = 40.9$

Example 15: The demand of a computer monitor cable is 1,050 cables per month and shortages are allowed. If the cost of making one purchase is ₹700, holding cost of one cable is ₹3 per year and the cost of one shortage is ₹50 per year, determine optimum purchase quantity and optimum number of shortages.

Solution: Given,

- D = 1050 units/month = 12,600 units/year
- h = ₹3/unit/year
- O = ₹700/order
- C_s = ₹50/unit/year

Therefore,

Optimum Purchase Quantity $(Q) = \sqrt{\frac{2DO}{h} \times \frac{C_s + h}{C_s}}$

$$= \sqrt{\frac{2 \times 12600 \times 700}{3} \times \frac{50 + 3}{50}} = 2497 \text{ units (approx.)}$$

Maximum Inventory $(Q_1) = \sqrt{\frac{2DO}{h} \times \frac{C_s}{C_s + h}}$

$$= \sqrt{\frac{2\times12600\times700}{3}\times\frac{50}{(50+3)}} = 2355 \text{ units (approx.)}$$

Optimum number of Shortage Units (S)

$$= Q - Q_1 = 2497 - 2355 = 142 \text{units}$$

4.2.3. EBQ (Economic Batch Quantity) Model

It is used for calculating the quantity of unit which are produced at the lowest average costs in a certain batch or production run. Economic Production Quantity (EPQ) Model is the advanced version of Economic Order Quantity model. Economic batch quantity (EBQ) is also known as **Optimal Batch Quantity** or **Economic Production Quantity.**

In the manufacturing industry, large amount of useful time was lost because of set-uptime i.e., change in the components to the machined. These changes require the restarting of machine for the new components. This setting time is the important component of cost of different manufacturing process like machining, forging, casting of ferrous and non-ferrous materials; and rubber and plastic moulding, etc. The manufacturing is done in large quantity of components for reducing the machine setting costs. **Economic Batch Quantity** refers to the batch quantity which is economically produced.

Thus, the economic batch quantity is no longer in use because of lowering the setting time for machining by the use of various technologies like group technology, computer integrated machining technology and methods like single minute exchange of dies, etc. for making the small lot sizes and fast delivery possible.

Assumptions of EBQ Model

The various assumptions of EBQ model are as follows:

1) The demand (D) is known from before and is same for a particular time period.
2) The per unit cost of the inventory item is same.
3) The holding cost per annum (h) is same.
4) The set-up cost per batch (C) is same.
5) The production time (tp) is known from before and will not change.
6) It is assumed that there is only one product produced.
7) There is no interaction with other products.
8) The set-up plays an important role rather than the aspect of time.
9) The set-up cost is same and will not act upon the batch quantity.

Limitations of EBQ Model

The Economic Batch Quantity is used in the production planning for the industries which produce large amount of components and they are also stored for sometimes before using them in the assembly work. Inspite of this, there are some limitations also which are as follows:

For example,

1) The rise in the batch frequency might not rise the set-up costs in the same manner as from the regular setting up which reduces the setting and tool wear more. If there is decrease in the batch quantity than there will be decrease in the number of failure in the machines and *vice versa*
2) The set-up costs is same for all the batch and there is no way for controlling and reducing of these costs done.
3) The different costs elements will work in different way. So, the plan that the Economic Batch Quantity will always reduce the cost may fail.

4.2.3.1. EBQ Model without Shortages

The manufacturing model refers to the model in which inventory is needed according to the requirement of main product. There is shortage allowed in this model. It is based on the principle that the rate of consumptions of item is same over the year. Both the production and consumption of the product is done altogether for the cycle time. In the remaining cycle time the consumption of the item will be done and the cost of the production per unit is same and will not consider production lot size.

Let us suppose, D = Annual demand in units

k = Production rate of the item (total number of units produced/year) r

O = Ordering cost/Cost per set-up

h = Carrying cost/unit/year

p = Cost of production/unit

t_1 = Period of production as well as consumption of the item

t_2 = Period of consumption only

t = Cycle time (i.e. $t = t_1 + t_2$)

The operation manufacturing model without strategies can be seen from the **figure 4.4:**

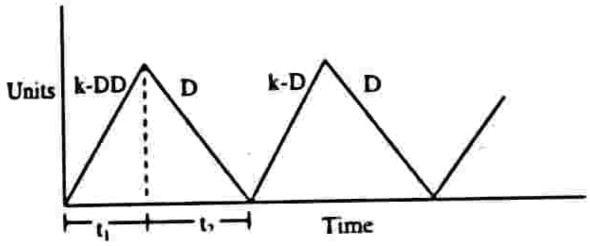


Figure 4.4: Manufacturing model without shortages

In the period t_1 the item is manufactured at the rate of k units per time and along with it is used at the rate of r units per period. At this time the inventory is made at the rate of k – D units per period. At the time of t_2 the production of item is stop but the consumption of the same item is still there. Thus, the inventory is lowered down at the rate of r units per period at the time of t_2. The formulas for this condition is are as follows:

$$\text{Economic batch quantity (EBQ or } Q^*) = \sqrt{\frac{2DO}{h[1-(D/k)]}}$$

$$\text{Period of production as well as consumption, } t_1^* = \frac{Q^*}{k}$$

Period of consumption only,

$$t_2^* = \frac{Q^*[1-(D/k)]}{D} = \frac{(k-D)t_1}{D}$$

Cycle time $t = t_1^* + t_2^*$

Number of set-ups per year $= \frac{D}{Q^*}$

4.2.3.2. EBQ Model with Shortages

In EBQ model with shortages an item is manufactured and is used at the same time for a part of cycle time and there is only consumption done of the item in the leftover time period in cycle time. The cost of production is same as the production lot size. The backordering is done in this model as the stock-out units will be provided from the units which will be manufactured at the future date with a fine. The stock-out is allowed in this model. The operation of this model is shown in the following **figure 4.5**:

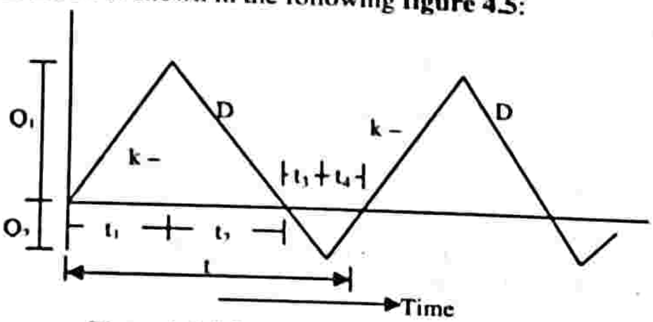


Figure 4.5: Manufacturing Model with Shortages

The variables which are used in this model are given below:

- D = Demand of an item/period
- k = Production rate of the item (number of units produced/period)
- O = Cost/set-up
- h = Carrying cost/unit/period
- C_s = Shortage cost/unit/period
- t = Total cycle time
- p = Cost of production/unit
- t_1 = Period of production as well as consumption of the item satisfying period's requirement
- t_2 = Period of consumption only
- t_3 = Period of shortage
- t_4 = Period of production as well as consumption of the item satisfying back order

The formula for the optimal values of the above variables is presented below:

$$\text{Economic Batch Quantity } Q^* \sqrt{\frac{2O}{h}\frac{kD}{k-D}\frac{h+C_s}{C_s}}$$

$$\text{Maximum Inventory } Q_1^* \sqrt{\frac{2O}{h}\frac{D(k-D)}{k}\frac{C_s}{h+C_s}}$$

$$\text{Maximum Stock Out } Q_2^* = \sqrt{\frac{2Oh}{C_s(h+C_s)}\frac{D(k-D)}{k}}$$

$$Q_1^* = \frac{k-D}{k}Q^* - Q_2^*$$

$$t^* = \frac{Q^*}{D},\ t_1^* = \frac{Q_1^*}{k-D},\ t_2^* = \frac{Q_1^*}{D},$$

$$t_3^* = \frac{Q_2^*}{D},\ t_4^* = \frac{Q_2^*}{k-D}$$

Example 16: The demand for an item is 6000 units per year. Its production rate is 1000 units per month. The carrying cost is ₹50/unit/year and the set-up cost is ₹2000 per set-up. The shortage cost is ₹1000 per unit per year. Find economic batch quantity, maximum inventory, maximum stock out and the period of shortage.

Solution: Given,

D = 6000 units/year

k = 1000 × 12 = 12,000 units/year

O = ₹ 2,000/set-up

h = ₹ 50/unit/year

C_s = ₹ 1000/unit/year

Therefore,

Economic Batch Quantity

$$(\text{EBQ or } Q^*) = \sqrt{\frac{2O}{h}\frac{kD}{k-D}\frac{h+C_s}{C_s}} = 1004 \text{ units (approx.)}$$

$$\text{Maximum Inventory } Q_1^* = \sqrt{\frac{2A}{h}\frac{D(k-D)}{k}\frac{C_s}{h+C_s}}$$

$$= \sqrt{\frac{2\times2000}{50}\times\frac{6000(12{,}000-6000)}{12{,}000}\times\frac{1000}{50+1000}}$$

$$= \sqrt{80\times3000\times0.9523}$$

$$= \sqrt{228552} = 478 \text{ units}$$

$$\text{Maximum Stock Out } Q_2^* = \sqrt{\frac{2Oh}{C_s(h+C_s)}\frac{D(k-D)}{k}}$$

$$= \sqrt{\frac{2\times2000\times50}{1000(50+1000)}\frac{6000(12{,}000-6000)}{12{,}000}}$$

= 24 units (approx.)

Period of Shortage

$$t_3^* = \frac{Q_2^*}{r}\times365 = \frac{24}{6000}\times365 = 1.5 \text{ days (approx.)}$$

Total Cycle Time

$$t^* = \frac{Q^*}{r}\times365 = \frac{1004}{6000}\times365 = 61 \text{ days (approx.)}$$

Period of Production as well as Consumption of the Item satisfying Period's Requirement

$$t_1^* = \frac{Q_1^*}{k-D}\times365 = \frac{478}{12{,}000-6000}\times365 = 29 \text{ days (approx.)}$$

Period of Consumption

$$t_2^* = \frac{Q_1^*}{D} \times 365 = \frac{478}{6000} \times 365 = 29\text{days(approx.)}$$

Period of Production as well as Consumption of the Item satisfying Back Order

$$t_4^* = \frac{Q_2^*}{k-D} \times 365 = \frac{24}{12{,}000-6000} \times 365 = 1.5\text{days(approx.)}$$

4.2.4. Quantity Discount Models/ EOQ Model with Quantity Discounts

The standard EOQ model is based on the assumption that the unit cost remains constant. However, in real life scenario, discounts are based on the order quantity. **For example,** an order with higher units is likely to receive higher discount, lowering cost per unit. There may also be several price breaks. In such cases, the discounts should be taken into account for determining the quantity ordered. In case of constant unit cost price, the purchasing cost does not affect the order. However, due to price breaks, the change in cost price necessitates its consideration for determining its order size. The cost model would also consider the holding costs, the ordering costs and the price of allied materials.

There are several advantages and drawbacks of bulk purchasing. The **advantages** attach to discount model is to lower unit cost, lower chances of stock outs and decrease in transportation costs. At the very same time, the **drawbacks** increases carrying costs of inventory, ties up capital and increases the chance of obsolescence and spoilage.

4.2.4.1. EOQ Model with Single Price Break

Ascertaining the optimum or economic order quantity 'Q' minimizes the total cost which is determined by using the following assumptions:

1) P = Price of an item,
2) O = Ordering Cost per order/set-up cost
3) h or C_h = cost of carrying one rupee worth of inventory for one year,
4) D = Annual demand (i.e., demand rate),
5) Q = Quantity ordered (Optimum order quantity),
6) No Shortages.

The model includes the purchasing price of the inventory in the total cost which needs to be minimized.

$$TC(Q_1^*) = DP_1 + \frac{D}{Q_1^*}O + \frac{Q_1^*}{2}h \times P_1$$

$$= DP_1 + \sqrt{2DA(hP_1)}$$

The process for deciding the quantity to order (Q*) in cases where discounts are permissible and the minimum total cost or TC* is contingent upon consecutive evaluation of several values Q.

The purchase inventory model involving single discount is stated as follows:

Order Quantity	Unit Price (₹)
$1 \le Q_1 < b$	P_1
$b \le Q_2$	P_2

Where, b denotes the quantity at and beyond the threshold where the quantity discount is applied and $P_2 < P_1$

The process for finding an optimal purchase quantity is summarized below:

Step 1: Determine the Optimal Order Quantity or EOQ for the lowest price i.e. highest discount).

$$Q_2^*(P_2) = \sqrt{\frac{2DO}{h \times P_2}}$$

and compare the values of Q_2^* and quantity b which is mandatory for the discount.

If $Q_2^* \ge b$, then orders for Q_2^* quantity should be placed and discount should be obtained. Otherwise step 2 should be taken

Step 2: If $Q_2^* < b$, the order cannot be placed at the reduced price P_2, then find Q_1^* for price P_1. Compare TC(Q*1) and TC (b). The TC (Q_1^*) and TC(b) are determined as shown below:

$$TC(Q_1^*) = DP_1 + \frac{D}{Q_1^*}O + \frac{Q_1^*}{2}h \times P_1 = DP_1 + \sqrt{2DO(hP_1)}$$

$$TC(b) = DP_2 + \frac{D}{b}O + \frac{b}{2}h \times P_2 = DP_2 + \sqrt{2DO(hP_2)}$$

If TC (Q_1^*) > TC (b), then orders with quantity size b should be placed to acquire the discount.

Example 17: Calculate the optimal economic order quantity using following data:

Annual demand = 7200 units
Ordering cost = ₹100
Cost of storage = 40% of the unit cost

Price Break

Quantity	Unit Cost (₹)
$0 \le Q_1 \le 200$	40
$200 \le Q_2$	36

Solution: Given,
Annual demand (D) = 7200 units
Ordering cost (O) = ₹100
Cost of storage (h) = 40% of the unit cost
Unit cost (P) = ₹40; (P_2) = ₹36

Step 1: The maximum discount offered is ₹36. Thus, computing Q_2^* corresponding to that amount, i.e.,

$$Q_2^* = \sqrt{\frac{2DO}{h \times P_2}} = \sqrt{\frac{2 \times 7200 \times 100}{0.40 \times 36}} = 316.23 \text{ units/order}$$

As Q2* is higher than b (i.e., 316.23 > 200), therefore, the optimum purchase quantity is

$Q_2^* = 316.23$ units/order

Example 18: By using the following data calculate the optimum order quantity:

Quantity	Unit Cost (₹)
$0 \le Q_1 < 1000$	20.00
$1000 \le Q_2$	18.50

The monthly demand is expected to be 400 units, while the cost of storage is calculated to be 4% of the unit cost. The cost of ordering is ₹700.00

Solution: Given,

Demand (D) = 400 units

Ordering cost (O) = 700 units

Holding cost (h) = ic = i = 4% or .04

$$EOQ(Q) = \sqrt{\frac{2DO}{h \times P}}$$

Step 1: The highest amount of discount available is ₹18.50. Thus, we can calculate EOQ (Q_2) by using cost as ₹18.50.

$$\therefore \quad EOQ(Q_2) = \sqrt{\frac{2 \times 700 \times 400}{18.50 \times 0.04}} \cong 870 \text{ units}$$

EOQ (Q_2) is 870 while b is 1000 showing that $Q_2 < b$, i.e., EOQ (Q_2) is contained in the range of $EOQ(Q_2) < 1000$.

So, the optimum purchase quantity is EOQ (Q_2) = 870 units

Example 19: A company manufacture tyres in large numbers and are kept in warehouse to avail discount from local vendors. Local vendor offer discount for tyres are given below:

Order Quantity	Unit Price (₹)
Upto 1399	20.00
1400 and above	18.50

The typical yearly replacement rate is 4,800 pallets. The carrying costs amounts to 24% of the average inventory while ordering cost per order is ₹200.

Solution: Given,

Yearly demand (D) = 4,800 tyres

Ordering cost (O) = ₹200 per order

Carrying cost (h) = 24% of the unit cost

Unit cost (P_1) = ₹20.00; (P_2) = ₹18.50

Step 1: The highest discount available is ₹18.50. Thus computing Q_2* corresponding to that quantity, i.e.,

$$Q_2^* = \sqrt{\frac{2DO}{h \times P_2}} = \sqrt{\frac{2 \times 4800 \times 200}{0.24 \times 18.50}} = 657.59 \text{ tyres}$$

Since, $Q_2^* < b$ (i.e., 657.59 < 1400), Q_2^* is not feasible.

Step 2: Calculate

$$Q_1^* = \sqrt{\frac{2DO}{h \times P_1}} = \sqrt{\frac{2 \times 4800 \times 200}{0.24 \times 20}} = 632.45 \text{ tyres}$$

$$TC(Q_1^*) = TC(632.45) = DP_1 + \frac{D}{Q^*_1}O + \frac{Q_1^*}{2}h \times P_1$$

$$= 4800 \times 20 + \frac{4800}{632.45} \times 100 + \frac{632.45}{2} \times 0.24 \times 20$$

$$= 48{,}000 + 759 + 1518$$

$$= ₹98{,}277$$

$$TC(b) = TC(1400) = DP_2 + \frac{D}{b}O + \frac{b}{2}h \times P_2$$

$$= 4800 \times 18.50 + \frac{4800}{1400} \times 100 + \frac{1400}{2} \times 0.24 \times 18.50$$

$$= 88{,}800 + 342.85 + 3108$$

$$= ₹92{,}250.85$$

As TC (b) > TC(Q_1*) and therefore the optimal order quantity is 632.45 units.

Example 20: Annual demand for an item is 6000 units, Ordering cost is ₹600 per order. Inventory carrying cost is 18% of the purchase price/unit/year. The price break-ups are as shown below:

Quantity	Price (in ₹) per Unit
$0 \le Q_1 \le 2000$	20
$2000 \le Q_2 < 4000$	15
$4000 \le Q_3$	9

Find the optional order size.

Solution: Given,

Annual Demand (D) = ₹6,000 units/year,

Ordering Cost (O) = 600,

Carrying Cost (h) = 0.18,

Unit Cost (P_1) = 20, (P_2) = 15, (P_3) = 9

Step 1: The highest discount available is ₹9.

Thus computing Q_3* corresponding to that quantity, i.e.,

$$Q_3^* = \sqrt{\frac{2DO}{P_3 \times h}} = \sqrt{\frac{2 \times 6{,}000 \times 600}{9 \times 0.18}} = 2{,}108 \text{ units}$$

Since $Q_3^* < b_2 = (4{,}000)$, proceed to next step.

Step 2: Compute Q_2^*. Here, the highest discount available is $P_2 = 15$. Now,

$$Q_2^* = \sqrt{\frac{2DO}{P_2 \times h}} = \sqrt{\frac{2 \times 6{,}000 \times 600}{15 \times 0.18}} = 1{,}633 \text{ units}$$

Here, $Q_2^* < b_1 = (2{,}000)$, so go to next step.

Step 3: Compute Q_1^*. Here, $P_1 = 20$. Now,

$$Q_1^* = \sqrt{\frac{2DO}{P_1 \times h}} = \sqrt{\frac{2 \times 6{,}000 \times 600}{20 \times 0.18}} = 1{,}414 \text{ units}$$

Since $Q_1^* < b_1$, compare TC(Q_1^*) with TC(b_1) and TC(b_2) to get the optimum purchase quantity.

$$TC(Q_1^*) = (6{,}000 \times 20) + \left(\frac{60 \times 6{,}000}{1{,}414}\right) + \frac{(0.18) \times (20) \times (1{,}414)}{2} = ₹1{,}22{,}800$$

$$TC(b_1) = (6{,}000 \times 15) + \left(\frac{600 \times 6{,}000}{2{,}000}\right) + \frac{(0.18) \times (15) \times (2000)}{2} = ₹94{,}500$$

$$TC(b_2) = (6{,}000 \times 9) + \left(\frac{600 \times 6{,}000}{4{,}000}\right) + \frac{(0.18) \times (9) \times (4000)}{2} = ₹58{,}140$$

The least cost is ₹58,140, hence optimal order quantity is b_2 = 4000 units.

4.2.4.2. EOQ Model with Multiple Price Breaks

Every now and then, multiple price breaks may be available. The process in that case is an extension of the regular single price break condition. Following is the example price schedule given by the supplier:

Quantity (Q) in Units	Price per Unit
Less than 200	20.00
$200 \le Q < 500$	19.80
$500 \le Q < 1000$	19.40
$1000 \le Q < 1500$	19.00
$Q \le 1500$	18.80

The first break is given at a quantity of 200 units, and it is referred to as Q_1. Similarly, Q_2, Q_3, and Q_4 would be 500, 1000, and 1500 units.

Example 21: A company sells electronic goods. Following is the information for one of its products:

Expected annual sales	= 16,000 units
Ordering cost	= ₹360 per order
Holding cost	= 20% of the average inventory value

The article can be purchased as per the following schedule:

Lot Size	Unit Price (₹)
1-999	₹44.00
1000-1499	₹40.00
1500-1999	₹38.00
2000 and above	₹37.00

Calculate the best order size.

Solution: $EOQ(Q) = \sqrt{\frac{2DO}{h \times P}}$

Where,

h = 20% of the unit price. Taking into account the lowest unit price of ₹37.00, we can calculate

$$EOQ(Q_3) = \sqrt{\frac{2 \times 360 \times 16000}{0.20 \times 37.00}} = 1{,}248 \text{ units}$$

However, the unit price of ₹37 cannot be availed for an order size of 1,248 units. So, with the unit price of ₹38, we can get.

$$EOQ(Q_2) = \sqrt{\frac{2 \times 360 \times 16000}{0.20 \times 38}} = 1{,}231 \text{ units}$$

This is also not feasible. So using the unit price of ₹40, we can calculate:

$$EOQ = \sqrt{\frac{2 \times 360 \times 16000}{0.20 \times 40}} = 1{,}200 \text{ units}$$

This order quantity is feasible.

So, the total cost related to 1,200 units:

$$TC(Q_1^*) = DP_1 + \frac{D}{Q_1^*}O + \frac{Q_1^*}{2} h \times P_1 = DP_1 + \sqrt{2DA(hP_1)}$$

$$TC(1200) = 16000 \times 40 + \frac{16000}{1200} \times 360 + \frac{20}{100} \times \frac{1200}{2} \times 40$$

$$= 6{,}40{,}000 + 4{,}800 + 4{,}800 = ₹6{,}49{,}600$$

The total cost at cut off points of 1,500 and 2,000 are calculated below

$$TC(1500) = 16000 \times 38 + \frac{16000}{1500} \times 360 + \frac{20}{100} \times \frac{1500}{2} \times 38$$

$$= 6{,}08{,}000 + 3{,}840 + 5{,}700 = ₹6{,}17{,}540$$

$$TC(2000) = 16000 \times 37 + \frac{16000}{2000} \times 360 + \frac{20}{100} \times \frac{2000}{2} \times 37$$

$$= 5{,}92{,}000 + 2{,}880 + 7{,}400 = ₹6{,}02{,}280$$

Example 22: A shop has an average demand of 400 units per month. The units are bought at the rate of ₹20.00 per unit while procurement cost is ₹200 each time. The stock holding cost per month stands at 4% of the cost of item. The price breaks are:

Quantity	Discount
1000 - 2000	10% i.e. ₹18
Above 2000	20% i.e. ₹16

Calculate the optimum order quantity?

Solution: $EOQ(Q) = \sqrt{\frac{2DO}{h \times P}}$

$$EOQ(Q_3) = \sqrt{\frac{2 \times 200 \times 400}{0.04 \times 16}} = 500 \text{ units}$$

Since EOQ < b_2, i.e., (500 < 2,000), so calculate

$$EOQ(Q_2) = \sqrt{\frac{2 \times 200 \times 400}{0.04 \times 18}} = 471 \text{ approx.}$$

Since EOQ < b_1, i.e., (500 < 1000), calculate

$$EOQ(Q_1) = \sqrt{\frac{2 \times 200 \times 400}{0.04 \times 20}} = 447 \text{ approx.}$$

Now, compare three costs TC (447) TC(1000) and TC(2,000)

$$TC(Q_1^*) = DP_1 + \frac{D}{Q_1^*}O + \frac{Q_1^*}{2} h \times P_1$$

$$TC(447) = 400 \times 20 + \frac{400}{447} \times 200 + \frac{447}{2} \times 0.40 \times 20$$

$$= 8000 + 179 + 1788 = 9967$$

$$TC(1000) = 400 \times 18 + \frac{400}{1000} \times 200 + \frac{1000}{2} \times 0.40 \times 18$$

$$= 7200 + 80 + 3600 = 10{,}880$$

$$TC(2000) = 400 \times 16 + \frac{400}{2000} \times 200 + \frac{2000}{2} \times 0.40 \times 16$$

$$= 6400 + 40 + 6400 = 12{,}840$$

Hence the offer of 20% discount should be accepted. The saving per month is ₹2,513

Example 23: Calculate the optimum order quantity using the following information:

Quantity	Purchasing cost per unit (₹)
$0 \le Q_1 < 200$	40
$200 \le Q_2 < 400$	36
$400 \le Q_3$	32

The average monthly demand is expected to be 800 units. The storage cost is likely to be 40% of the unit cost of the product. The ordering cost is ₹50.00 per month.

Solution: $EOQ(Q) = \sqrt{\frac{2DO}{h \times P}}$

Where, h = 40% of the unit cost.

Taking the lowest unit price of 32, we can calculate the

$$EOQ\,(Q_3) = \sqrt{\frac{2 \times 50 \times 800}{32 \times 0.40}} = 79 \text{ units}$$

However, the unit price of ₹32 cannot be availed for an order size of 79 units and hence it is not feasible.

Using unit price of ₹36, we get

$$EOQ(Q_2) = \sqrt{\frac{2 \times 50 \times 800}{36 \times 0.40}} = 75 \text{ units}$$

This quantity is not feasible either.

Using the unit price of ₹40 we get

$$EOQ(Q_1) = \sqrt{\frac{2 \times 50 \times 800}{40 \times 0.40}} = 71 \text{ units}$$

This order quantity is feasible.

Now, the total cost corresponds to units:

Now,
$$TC\,(71) = 50 \times \frac{800}{71} + 800 \times 40 + 40\,(0.40\,) \times \frac{71}{2}$$
$$= ₹33{,}131$$

We shall determine the total cost at cut off points 200 and 400

$$TC\,(b_1) = 50 \times \frac{800}{200} + 800 \times 36 + 36(0.40) \times \frac{200}{2} = ₹30{,}440$$

And,

$$TC(b_2) = 50 \times \frac{800}{400} + 800 \times 32 + 32(0.40) \times \frac{400}{2} = ₹28{,}262$$

The total cost is minimum at quantity of 400, so it is the optimal order quantity.

Example 24: Find the optimum order quantity for a product for which the price break is given below:

Quantity	Unit Cost (in ₹)
$0 \le q1 < 100$	20 per unit
$100 \le q2 < 200$	18 per unit
$200 \le q3$	16 per unit

The monthly demand for the product is 400 units. The storage cost is 20% of the unit cost of the product and the cost of ordering is ₹25.

Solution: $EOQ(Q) = \sqrt{\frac{2DO}{h \times P}}$

Given,

D = 400 and O = 25.

Where, h = 20% of the unit cost. Taking the lowest unit price of 16

So, P = ₹16

h = 20% of 16 = 3.2

$$EOQ\,(Q_3) = \sqrt{\frac{2 \times 400 \times 25}{16 \times 0.20}} = 79 \text{ units}$$

However, the unit price of ₹16 cannot be availed for an order size of 79 units and hence it is not feasible.

Using unit price of ₹18, we get

$$EOQ(Q_2) = \sqrt{\frac{2 \times 400 \times 25}{18 \times 0.20}} = 75 \text{ units}$$

This quantity is not feasible either.

Using the unit price of ₹20 we get

$$EOQ(Q_1) = \sqrt{\frac{2 \times 400 \times 25}{20 \times 0.20}} = 71 \text{ units}$$

This order quantity is feasible.

Now, the total cost corresponds to units:

Now,

$$TC\,(71) = \left(25 \times \frac{400}{71}\right) + (400 \times 20) + 20(0.20) \times \frac{71}{2}$$
$$= ₹8{,}283$$

We shall determine the total cost at cut off points 100 and 200

$$TC\,(b_1) = \left(20 \times \frac{400}{100}\right) + (400 \times 18) + 18(0.20) \times \frac{100}{2} = ₹7{,}460$$

And,

$$TC(b_2) = \left(25 \times \frac{400}{200}\right) + (400 \times 16) + 16(0.20) \times \frac{200}{2} = ₹6{,}770$$

The total cost is minimum at quantity of 200, so it is the optimal order quantity.

Example 25: Find the optimum order quantity of an item for which the price breaks are as follows. The monthly demand for the item is 400 units, the cost of storage is 20% of the unit cost and ordering cost is Rupees 50 per order.

Quantity	Purchasing Cost
0 – 100	200
101 – 200	180
Above 200	160

Solution: $EOQ(Q) = \sqrt{\frac{2DO}{h \times P}}$

Given, D = 400 and O = 50

Where,

h = 20% of the unit cost. Taking the lowest unit price of 160

So, P = ₹160

h = 20% of 160 = 32

$$EOQ\,(Q_3) = \sqrt{\frac{2 \times 400 \times 50}{160 \times 0.20}} = \sqrt{\frac{40{,}000}{32}} = 35.56 \text{ or } 36 \text{ units}$$

However, the unit price of ₹160 cannot be availed for an order size of 33 units and hence it is not feasible.

Using unit price of ₹180, we get

$$EOQ(Q_2) = \sqrt{\frac{2\times400\times50}{180\times0.20}} = \sqrt{\frac{40{,}000}{36}} = 33.33 \text{ or } 33 \text{ units}$$

This quantity is not feasible either.

Using the unit price of ₹200 we get

$$EOQ(Q_1) = \sqrt{\frac{2\times400\times50}{200\times0.20}} = \sqrt{\frac{40{,}000}{40}} = 31.62 \text{ or } 32 \text{ units}$$

This order quantity is feasible.

So, the total cost related to 32 units:

Total Cost = Purchasing Cost + Ordering Cost + Carrying Cost

$$TC(Q_1^*) = DP_1 + \frac{D}{Q_1^*}O + \frac{Q_1^*}{2}h\times P_1$$

Now,

$$TC(32) = (400\times200) + \left(\frac{400}{32}\times50\right) + \left(\frac{32}{2}\times200\times0.20\right)$$

$$= 625 + 80{,}000 + 640 = ₹81{,}265$$

The total cost at cut off points of 101 and 200 are calculated below:

$$TC(101) = (400\times180) + \left(\frac{400}{101}\times50\right) + \left(\frac{101}{2}\times180\times0.20\right)$$

$$= 72{,}000 + 198.02 + 1{,}818 = ₹74{,}016.02$$

$$\text{And, } TC(200) = (400\times160) + \left(\frac{400}{200}\times50\right) + \left(\frac{200}{2}\times160\times0.20\right)$$

$$= 64{,}000 + 100 + 3{,}200 = ₹67{,}300$$

The total cost is minimum at quantity of 200 units at price 160, so it is the optimal order quantity.

Example 26: Find the optimal order quantity for a product for which the price breaks are as follows:

Quantity (q)	Unit Cost (₹)
0 < q < 500	10
500 ≤ q < 750	9.25
750 ≤ q	8.75

The monthly demand for the product is 200 units. Storage costs are 2% of the unit cost and cost of ordering is ₹100.

Solution: $EOQ(Q) = \sqrt{\frac{2DO}{h\times P}}$

Given, D = 200 and O = 100

Where, h = 2% of the unit cost. Taking the lowest unit price of 8.75

So, P = ₹8.75

h = 2% of 8.75 = 0.175

$$EOQ(Q_3) = \sqrt{\frac{2\times200\times100}{8.75\times0.02}} = \sqrt{\frac{40{,}000}{0.175}} = 478.09 \text{ or } 478 \text{ units}$$

However, the unit price of ₹8.75 cannot be availed for an order size of 478 units and hence it is not feasible.

Using unit price of ₹9.25, we get

$$EOQ(Q_2) = \sqrt{\frac{2\times200\times100}{9.25\times0.02}} = \sqrt{\frac{40{,}000}{0.185}} = 464.99 \text{ or } 465 \text{ units}$$

This quantity is not feasible either.

Using the unit price of ₹10 we get

$$EOQ(Q_1) = \sqrt{\frac{2\times200\times100}{10\times0.02}} = \sqrt{\frac{40{,}000}{0.2}} = 447.21 \text{ or } 447 \text{ units}$$

This order quantity is feasible.

So, the total cost related to 447 units:

Total Cost = Purchasing Cost + Ordering Cost + Carrying Cost

$$TC(Q_1^*) = DP_1 + \frac{D}{Q_1^*}O + \frac{Q_1^*}{2}h\times P_1$$

Now,

$$TC(447) = (200\times10) + \left(\frac{200}{447}\times100\right) + \left(\frac{447}{2}\times10\times0.02\right)$$

$$= 2{,}000 + 45 + 45 = ₹2{,}090$$

The total cost at cut off points of 500 and 750 are calculated below:

$$TC(500) = (200\times9.25) + \left(\frac{200}{500}\times100\right) + \left(\frac{500}{2}\times9.25\times0.02\right)$$

$$= 1{,}850 + 40 + 46.25 = ₹1{,}936.25$$

And,

$$TC(750) = (200\times8.75) + \left(\frac{200}{750}\times100\right) + \left(\frac{750}{2}\times8.75\times0.02\right)$$

$$= 1{,}750 + 26.67 + 65.63 = ₹1{,}842.30$$

The total cost is minimum at quantity of 750 units at price 8.75, so it is the optimal order quantity.

Example 27: A company is considering three different sources of supply. The first supplier can deliver any quantity of units at the price of ₹300 each. The second supplier may provide units in lots of 150 or more at the price of ₹250 per units. The third supplier can offer units in lots of 250 or more at the price of ₹200 each. The company uses 3,000 units a year. Carrying costs stand at 40 percent, while ordering costs are ₹800. Determine the supplier with whom the orders should be placed and corresponding total annual costs.

Solution: Given,

Annual demand (D) = 3,000 units,

Ordering cost (O) = ₹800/per order

Carrying cost (h) = 40% of unit price.

Unit Price

Supplier	Lot size	Price per barrel
1	Any quantity	₹300
2	150 or more	₹250
3	250 or more	₹200

We first calculate the EOQ for each of the suppliers.

Supplier 1: $EOQ = \sqrt{\frac{2 \times 3{,}000 \times 800}{0.40 \times 300}} = 200$

Supplier 2: $EOQ = \sqrt{\frac{2 \times 3{,}000 \times 800}{0.40 \times 250}} = 219$

Supplier 3: $EOQ = \sqrt{\frac{2 \times 3{,}000 \times 800}{0.40 \times 200}} = 245$

Only the EOQ for supplier 2 is within the given range. Therefore, we will calculate total cost for order size of 219 units. For supplier 3, the EOQ is less than the minimum limit of the range of 250 or more units, the total cost at the price break level of 250 should be calculated. As a feasible EOQ is available at a lower price level of ₹250, supplier 1 with price of ₹300 per unit need not be considered.

Total Cost = Purchasing Cost + Ordering Cost + Carrying Cost

Supplier 4:

$$TC(150\,units) = 3{,}000 \times 250 + \frac{3{,}000}{150} \times 800 + \frac{150}{2} \times 250 \times 0.40$$

$$= 7{,}50{,}000 + 16{,}000 + 7500 = ₹7{,}73{,}500$$

(Due to rounding, there is difference between ordering and carrying costs at EOQ)

Supplier 5:

TC (250 units)

$$= 3{,}000 \times 200 + \frac{1{,}500}{250} \times 800 + \frac{250}{2} \times 200 \times 0.40$$

$$= 6{,}00{,}000 + 4{,}800 + 10{,}000 = \text{`}6{,}14{,}800$$

The total cost is minimum for supplier 3, the order should be placed with supplier 3 with the order quantity of 250.

Example 28: Annual demand for an item is 6000 units. Ordering cost is ₹600 per order. Inventory carrying cost is 18% of the purchase price/unit/year. The price break-ups are as shown below:

Quantity	Price (in ₹) per Unit
$0 \le Q_1 \le 2000$	20
$2000 \le Q_2 < 4000$	15
$4000 \le Q_3$	9

Find the optional order size.

Solution: Given,
Annual Demand (D) = ₹6,000 units/year,
Ordering Cost (O) = 600,
Carrying Cost (h) = 0.18,
Unit Cost $(P_1) = 20$, $(P_2) = 15$, $(P_3) = 9$

Step 1: The highest discount available is ₹9. Thus computing Q_3* corresponding to that quantity, i.e.,

$$Q_3^* = \sqrt{\frac{2DO}{P_3 \times h}} = \sqrt{\frac{2 \times 6{,}000 \times 600}{9 \times 0.18}} = 2{,}108\,\text{units}$$

Since $Q_3^* < b_2 = (4{,}000)$, proceed to next step.

Step 2: Compute Q_2^*. Here, the highest discount available is $P_2 = 15$. Now,

$$Q_2^* = \sqrt{\frac{2DO}{P_2 \times h}} = \sqrt{\frac{2 \times 6{,}000 \times 600}{15 \times 0.18}} = 1{,}633\ \text{units}$$

Here, $Q_2^* < b_1 = (2{,}000)$, so go to next step.

Step 3: Compute Q_1^*. Here, $P_1 = 20$. Now,

$$Q_1^* = \sqrt{\frac{2DO}{P_1 \times h}} = \sqrt{\frac{2 \times 6{,}000 \times 600}{20 \times 0.18}} = 1{,}414\ \text{units}$$

Since $Q_1^* < b_1$, compare $TC(Q_1^*)$ with $TC(b_1)$ and $TC(b_2)$ to get the optimum purchase quantity.

$$TC(Q_1^*) = (6{,}000 \times 20) + \left(\frac{60 \times 6{,}000}{1{,}414}\right) + \frac{(0.18) \times (20) \times (1{,}414)}{2} = ₹1{,}22{,}800$$

$$TC(b_1) = (6{,}000 \times 15) + \left(\frac{600 \times 6{,}000}{2{,}000}\right) + \frac{(0.18) \times (15) \times (2000)}{2} = ₹94{,}500$$

$$TC(b_2) = (6{,}000 \times 9) + \left(\frac{600 \times 6{,}000}{4{,}000}\right) + \frac{(0.18) \times (9) \times (4000)}{2} = ₹58{,}140$$

The least cost is ₹58,140, hence optimal order quantity is $b_2 = 4000$ units.

4.3. REPLACEMENT MODELS

4.3.1. Introduction

The replacement problem is related with such items like machines, men, electric-light bulbs, etc., which needs replacement because of the weakening efficiency, failure or breakdown of those items. Such weakening efficiency or breakdown may be regular or random.

The replacement theory of operation research is basically meant for decision-making process for replacing old equipment with a new one, generally for better usage. To overcome from the deteriorating property or failure or breakdown of any particular equipment, replacement is quite necessary. There are few cases where replacement theory is genuinely used, i.e., equipment have out-lived, equipment might be demolished by any wear-tear or depreciate or it may not be economical to continue. The above mentioned all conditions can be solved mathematically and also can be categorised on following basis:

1) Items which are demolished with time. **For example,** machine tools, vehicles, equipment buildings etc.
2) Items which are not according to new developed technology such as ordinary weaving looms by automatic; manual accounting by tally etc.

3) Items which do not damage but unable to work after certain amount of use like electronic parts, street lights etc.
4) The current working staff of an organisation are increasingly diminished due to certain reasons such as death, retrenchment etc.

4.3.2. Failure Mechanism of Assets

The term failure has a far wider meaning in business than what it has in everyday life. There are two types of failure:

1) **Gradual Failure:** The literal meaning of gradual means slow or steady. Gradual failure means the efficiency of the item decreases, as time passes, with increase in the life of the equipment which results in decreased productivity, increased operating cost and decrease in the values of the item. **For example,** machines, tyres, ring, etc.

2) **Sudden Failure:** Those items which cannot work properly after some period of use. Any type of equipment which does not show consistency between installation period and its failure but still it follow some frequency distribution which are as follows:

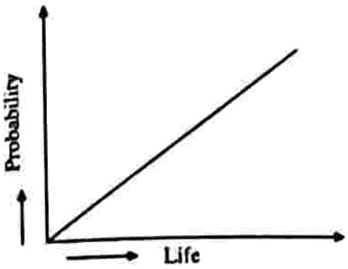


Figure 4.6

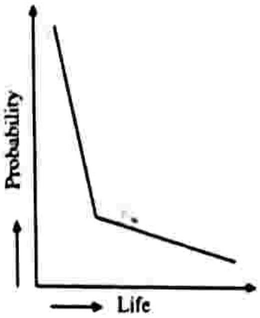


Figure 4.7

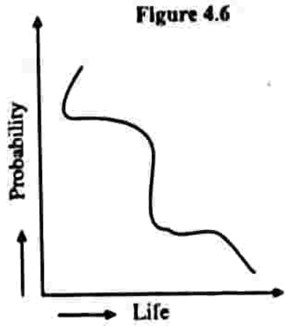


Figure 4.8

i) **Progressive Failure:** In this condition the failure of equipment increases with increase in its life. **For example,** electric bulbs, tubes etc. **Figure 4.6.**

ii) **Retrogressive Failure:** Such type of failure occurs in certain items which have more probability of failure in the beginning of their life and with the passage of time the changes of failure become less or we can say that if the unit services increases in the beginning; its expected life increases, e.g., engines. **Figure 4.7.**

iii) **Random Failure:** The situation where equipment suddenly stop working by any random cause such as any breakage, physical shock etc., which is irrespective of the age of the item. **For example,** vacuum tubes, electronic items etc. **Figure 4.8.**

4.3.3. Need of Replacement

Replacements of items are necessary under certain conditions. Some of them are as follows:

1) When the old item does not work anymore.
2) It is expected that the survival of old item is for few days only.
3) The old item is completely depreciated and needs huge amount for maintenance.
4) With the help of updated technology new and improved design of equipment is developed.
5) To beat the competition.
6) When the current staff of organisation is unable to perform their job due to death, retirement, etc.

The replacement of items are essential just because of following reasons:

1) Due to invention of advanced and improved technology, old equipment does not in use anymore.
2) The current equipment becomes completely obsolete and does not work anymore. **For example,** the electric-light bulb has failed and as such must be replaced. This is a case of sudden failure but the complete failure of an item like a machine may be a gradual one.
3) As the passage of time the continuous use of current equipment make its condition worsened which effects its proper function which further results in expensive maintenance.
4) If it is expected that the current equipment is to fail shortly, in that case it is economically very advantageous to replace the equipment in anticipation of cost failure. We can say that such replacement is known as "preventive replacement".

4.3.4. Assumptions of Replacement Theory

While taking decision about the replacement of the equipment following assumptions are to be made:

1) The quality of the output remains constant.
2) Replacement and maintenance cost remains constant.
3) The operational efficiency of the equipment remains constant.
4) There is no change in technology of the asset under consideration.

4.3.5. Scope of Replacement in Management

1) Identify the problems or faults or failures as soon as it appears and take fast decision to rectify them.
2) By organising the preventive maintenance programmes, it reduces the major break-downs or crisis situations.
3) It increases the reliability of the plant, machinery and equipment in the design stage because it helps in designing and installing equipment in such a way that the equipment failure will be lower in its lifetime. Also Equipments could be designed in such a manner that its maintenance action will take small time, i.e., its 'maintainability' should be high.

4) It helps in making proper replacement policies of equipment and their parts in such a manner that the reliability and availability of system is improved at optimum costs.

5) In order to gain good control over maintenance operations, it offers standard times and standard procedures in maintenance process. It provides motivation to maintenance personnel as well as adequate incentives.

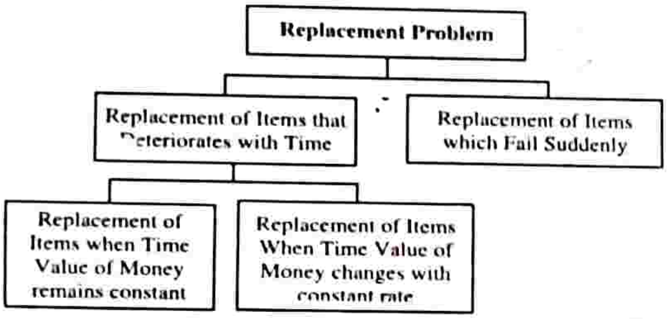
4.4. TYPES OF REPLACEMENT PROBLEMS

Replacement problems can be divided into two categories:

1) **Replacement of Items that Deteriorate with Time:** This is such type of replacement problem where assets are replaced on the basis of anticipation and it is represented by:
   i) Increasing maintenance/operation cost,
   ii) Its waste or scrap value, and
   iii) Damage to item and safety risk.

   The determination of optimum preventive replacement policy is based on the assumption that increased age reduces efficiency.

2) **Replacement of Items which Falls Suddenly:** This type of replacement problem requires expectancy of failure involving probabilities of failure. This replacement policy is adopted for balancing the wasted life of asset before the life of assets gets diminishes it is replaced with new one. To overcome from the problem of obsolescence of technologies and researches, it is necessary to replace the equipment with updated ones.



4.5. REPLACEMENT OF ITEM THAT DETERIORATE WITH TIME

4.5.1. Introduction

It will be economical to replace the old asset with a new one if the operational efficiency of that asset deteriorates with time (gradual failure). There are a number of alternative choices which can be compared with available alternatives on the basis of the running costs (average maintenance and **operating costs**) involved.

The capital assets which become less efficient with time can be replaced. **For example,** machine tools, buses in a transport organisation, planes etc.

4.5.2. Replacement of Item without Time Value of Money

Generally, it is assumed that there is no time value of money. This means that a rupee received today is better than receive after four years.

It is known that the cost of any asset over a given period of time, say n years, have three elements:

Purchase Price – Value Remaining after n Years + Maintenance Cost for n Years

Let, C = Purchase price of the item,
S = Scrap value of the item at the end of n years, and
M_t = Maintenance cost of the item in year t.

The total cost, T(n), of owning and maintaining the item for n years shall be:

$$T(n) = C - S + \sum_{t=1}^{n} M_t$$

Correspondingly, the average cost, A(n), would be defined as, Average investment cost + Average total running cost during period n

$$A(n) = \frac{C-S}{n} + \sum_{t=1}^{n} M_t / n$$

$$\text{Or, } A(n) = \frac{1}{n}\left[C - S + \sum_{t=1}^{n} M_t\right]$$

In above given case, the optimal replacement period would be only that for which average cost A(n) would be minimum. For determination of average cost A(n), first we have to determine the maintenance cost for the n^{th} year and for getting such cost, the cost of item and the net of its salvage value would be added. Such two costs, when get aggregated it comes in the form of total cost i.e., T(n). When such total cost T(n) is divided by the number of years n we get average cost.

Example 29: An organisation is purchasing a machine at a cost of ₹ 13,000 and using it. The installation charge of this machine is ₹3,600 and its scrap value is ₹1,600 as the organisation has monopoly of such type of work. The following table shows the maintenance cost this machine in different years:

Year	1	2	3	4	5	6	7	8	9
Cost	250	750	1000	1500	2100	2900	4000	4900	6500

Suppose that machine can be replaced only at the end of years. Find after how many years later, the organisation will replace the machine on economic considerations?

Solution: We have,

Total machine cost, C = Purchasing Cost + Installation Charge of Machine

= 13,000 + 3,600 = ₹16,600

Scrap value of machine S = ₹1600

One can calculate the optimum replacement period as follows:

Table 4.1: Calculation of Optimal Replacement Period

Year n (1)	Maintenance Cost M_t (2)	Cumulative Maintenance Cost $\sum M_t$ (3)	C – S (4)	T(n) (5) = (3) + (4)	A(n) (6) = (5)/n
1	250	250	15000	15250	15250
2	750	1000	15000	16000	8000
3	1000	2000	15000	17000	5667
4	1500	3500	15000	18500	4625
5	2100	5600	15000	20600	4120
6	2900	8500	15000	23500	3917
7	4000	12500	15000	27500	3929
8	4900	17400	15000	32400	4050
9	6500	23900	15000	38900	4322

From the above table, it is shown that lowest total average cost per year is ₹ 3917 during the sixth year. Hence the organisation should replace the machine after every six year. If organisation is not replacing the machine after six year, then average cost per year would become high.

Example 30: The purchasing price of a machine is ₹ 60,000. **Table below** shows the data related to running a machine:

Year	1	2	3	4	5
Resale Value (₹)	42,000	30,000	20,400	14,400	9,650
Spares Cost (₹)	4,000	4,270	4,880	5,800	6,850
Labor Cost (₹)	14,000	16,000	18,000	21,000	25,000

Find-out the optimal period after which the machine should be replaced?

Solution: The running (operational or maintenance) cost is the sum of spares and labor cost. The following table illustrates the resale prices and running costs of a machine in consecutive years:

Year	1	2	3	4	5
Resale Value (₹)	42,000	30,000	20,400	14,400	9,650
Running Cost (₹)	18,000	20,270	22,880	26,800	31,850

Table 4.2 shows the determination of average running cost per year of a machine during its lifecycle:

Table 4.2: Calculation of Optimal Replacement Period

Year of Service n (1)	Running Cost (₹) M_t (2)	Cumulative Running Cost (₹) ΣM_t (3)	Resale Value (₹) S (4)	Depreciation Cost (₹) C – S (5) =60,000 – (4)	Total Cost (₹) T(n) (6) = (3) + (5)	Average Cost (₹) A(n) (7) = (6)/(1)
1	18,000	18,000	42,000	18,000	36,000	36,000.00
2	20,270	38,270	30,000	30,000	68,270	34,135.00
3	22,880	61,150	20,400	39,600	1,00,750	33,583.30
4	26,800	87,950	14,400	45,600	1,33,550	33,387.50
5	31,850	1,19,800	9,650	50,350	1,70,150	34,030.00

From the above table, it is shown that minimum average cost per year is ₹33,387.50 during the fourth year. Thus, machine should be replaced after every fourth year; if it is not done properly then average cost per year would become higher.

Example 31: Let consider that purchasing cost of equipment X is ₹2, 00,000. The resale value and running costs per year is shown in **table below:**

Year	1	2	3	4	5	6	7
Running Cost (₹)	30,000	38,000	46,000	58,000	72,000	90,000	1,20,000
Resale Value (₹)	1,00,000	50,000	25,000	12,000	8,000	8,000	8,000

i) 'Determine optimal period during which the equipment should be replaced?

ii) When equipment X has become 2 year old, other equipment Y is available for same purpose. The average costs of equipment Y is ₹72,000 and optimal period of replacement of this equipment is 4 years. Should equipment X should be replaced with equipment Y. If yes, then find when?

Solution: **Table 4.3** shows the determination of average running costs per year of equipment X during its life:

Table 4.3: Determination of Optimal Replacement Period

Year of Service n	Running Cost (₹) M_t	Cumulative Running Cost (₹) $\sum M_t$	Resale Price (₹) S	Depreciation Cost (₹) C – S	Total Cost (₹) T(n)	Average Cost (₹) A(n)
(1)	(2)	(3)	(4)	(5) = 2,00,000 – (4)	(6) = (3) + (5)	(7) = (5)/(1)
1	30,000	30,000	1,00,000	1,00,000	1,30,000	1,30,000.00
2	38,000	68,000	50,000	1,50,000	2,18,000	1,09,000.00
3	46,000	1,14,000	25,000	1,75,000	2,89,000	96,333.33
4	58,000	1,72,000	12,000	1,88,000	3,60,000	90,000.00
5	72,000	2,44,000	8,000	1,92,000	4,36,000	**87,200.00**
6	90,000	3,34,000	8,000	1,92,000	5,26,000	87,666.66
7	1,20,000	4,54,000	8,000	1,92,000	6,46,000	92,285.71

From the above table, it is shown that minimum average cost per year is ₹87,200 during the fifth year. Thus, equipment X should be replaced after every fifth year.

Now we have to find the time of replacement of equipment X with equipment Y. The average cost of equipment X in the consecutive years can be calculated as shown in **table 4.4**:

Table 4.4

Year of Service	Running Cost (₹)	Depreciation Cost (₹)	Total Cost (₹)	Cumulative Cost (₹)	Average Cost (₹)
3	46,000	50,000 – 25,000=25,000	71,000	71,000	71,000
4	58,000	25,000 – 12,000=13,000	71,000	1,42,000	**71,000**
5	72,000	4000	76,000	2,18,000	72,666.66
6	90,000	–	90,000	3,08,000	77,000
7	1,20,000	–	1,20,000	4,28,000	85,600

Table above shows that lowest average cost per year is ₹71, 000, thus equipment X should be replaced with equipment Y after every four year. If replacement is not done, then average costs per year have become higher.

Example 32: A firm is considering replacement of a machine whose cost price is ₹12,200 and the scrap value ₹200. The running (maintenance and operating) costs in rupees are found from experience to be as follows:

Year	1	2	3	4	5	6	7	8
Running Cost	200	500	800	1200	1800	2500	3200	4000

When should the machine be replaced?

Solution: It is given that,

Cost of the Machine, C = ₹12,200

Scrap value of Machine S = ₹200

The optimum replacement period can be calculated as follows:

Table 4.5: Determination of Optimal Replacement Period

Year n	Maintenance Cost M_t	Cumulative Maintenance Cost $\sum M_t$	C – S	T(n)	A(n)
(1)	(2)	(3)	(4)	(5) = (3) + (4)	(6) = (5)/n
1	200	200	12000	12200	12200
2	500	700	12000	12500	6350
3	800	1500	12000	13500	4500
4	1200	2700	12000	14700	3675
5	1800	4500	12000	16500	3300
6	2500	7000	12000	19000	**3167**
7	3200	10200	12000	22200	3171
8	4000	14200	12000	26200	3275

It is given that the lowest total average cost per year is ₹ 3167 during the sixth year. So the organisation should replace the machine after every six year. If machine is not replaced after six year then average cost per year would become high.

Example 33: The maintenance cost and resale value per year of a machine whose purchase price is ₹7,000 is given below:

Year	1	2	3	4	5	6	7	8
Operating Cost	900	1200	1600	2100	2800	3700	4700	5900
Resale Value (₹)	4000	2000	1200	600	500	400	400	400

When should the machine be replaced?

Solution: Given that, Capital cost C = ₹7,000.

Let it be profitable to replace the machine after n years. Then n should be determined by the minimum value of T_{ave}.

Values of T_{ave} for various years are computed in **table 4.6**:

Table 4.6

(1) Years of Service (n)	(2) Resale Value (S) (₹)	(3) Purchase Price – Resale Value (C – S) (₹)	(4) Annual Maintenance Cost f(t) (₹)	(5) Summation of Maintenance Cost $\sum_{t=0}^{n} f(t)$ (₹)	(6) Total Cost $\left[C - S + \sum_{t=0}^{n} f(t)\right]$ (₹)	(7) Average Annual Cost $\frac{1}{n}\left[C - S + \sum_{t=0}^{n} f(t)\right]$ (₹)
1	4,000	3,000	900	900	3,900	3,900
2	2,000	5,000	1,200	2,100	7,100	3,550
3	1,200	5,800	1,600	3,700	9,500	3,166.67
4	600	6,400	2,100	5,800	12,200	3,050
5	500	6,500	2,800	8,600	15,100	**3,020**
6	400	6,600	3,700	12,300	18,900	3,150
7	400	6,600	4,700	17,000	23,600	3,371.43
8	400	6,600	5,900	22,900	29,500	3,687.50

From the above table it is given that, average annual cost is minimum (₹3,020) in the 5th year. Hence the machine should be replaced at the end of 5 years of service.

Example 34: The cost of a machine is ₹6,100 and its scrap value is ₹100. The maintenance cost is found from experience to be:

Year	1	2	3	4	5	6	7	8
Maintenance Cost (in ₹)	100	250	400	600	900	1,200	1,600	2,000

At what year is the replacement due?

Solution: Given that,
C = ₹6,100 and
S = ₹100.

Optimum replacement period is determined as shown in **table below:**

Year (n) (1)	Maintenance Cost $C_m(t)$ (2)	Total Maintenance Cost or Cumulative Cost ΣC_m (t) (3)	C – S (4)	Total Cost = C – S + ΣC_m (t) (5) = (3) + (4)	Average Cost A(n) = $\left(\frac{5}{n}\right)$ (6)
1	100	100	6000	6100	6100.00
2	250	350	6000	6350	3175.00
3	400	750	6000	6750	2250.00
4	600	1350	6000	7350	1837.50
5	900	2250	6000	8250	1650.00
6	1200	3450	6000	9450	**1575**
7	1600	5050	6000	11050	1578.57
8	2000	7050	6000	13050	1631.25

This table shows that the value A(n) during the sixth year is minimum. Hence, the machine should be replaced after every 6th year.

4.5.3. Replacement of Item with Time Value of Money

In this case it is assumed that the value of money is constant throughout. But at the same time it is also true that late returns are less profitable because early returns can be again invested and give output in the form of surplus returns. The process of finding alternative discounting of further returns and future cost is compulsory in determination of that age at which the item must be replaced and also for selecting the most profitable return. Now for determining the economic life of the item it is essential to find the present value of future cost and future returns. At the same time it is also accepted that NPV is also compared with investment i.e., investment would be good if NPV is positive and *vice versa*. The rate which is given for determination of NPV is discounting rate.

Here operating cost increases with time and the value of money decreases at a constant rate. Suppose

C = Initial cost;

O_i = Operating cost in i^{th} year;

r = Rate of interest or discounting factor;

V = Present value of ₹1 to be spent in n^{th} year V $= \frac{1}{(1+r)^n}$

Then year wise PV of expenditure on the item in the successive cycle of 'n' years can be calculated as follows:

$PV(n) = (C + O_1) + O_2v + O_3 v^2 + ... O_n v^{n-1} + (C + O_1)v^n + O_2v^{n+1} + O_3v^{n+2} ...... + O_n v^{3n-1} + (C + O_1)v^{2v} + O_2v^{2n+1} + O_3v^{2n+2} .... + O_n v^{2n-1}$ ... so on.

Where, PV (n) is the present value required to pay all the future costs of purchasing and operating the item when it is renewed every 'n' year. If PV(n) < P (n + 1) then replacing each 'n' year is preferable to replacing (n + 1) year. If the best policy is replacement every 'n' years then the inequalities must hold good,

$PV(n + 1) - PV(n) > 0$ (zero)

$PV(n - 1) - PV(n) < 0$ (zero)

When $PV(n) < \frac{(C+O_1)+O_2v+O_3v^2+....O_{n-1}v^{n-2}}{(1+v+v^2+....v^{n-2})}$

And $PV(n+1) > \frac{(C+O_1)+O_2v+O_3v^2+....+O_n v^{n-1}}{(1+v+v^2+....v^{m-2})}$

If the operating cost of the next period is greater than the weighted average of the previous costs then with the help of above given two equations we can replace the item but if it is not so then in that case we cannot replace the items.

Procedure

Step 1: Operating costs of the item for different years are multiplied by the present value factor at the given rate.

Step 2: Cumulative PV for all the years are calculated and total cost is calculated after adding this value in the original cost of the item.

Step 3: The discount factors are then cumulated.

Step 4: Divide the total cost of step 2 by the corresponding value of the cumulated discount factor for each of the year.

Step 5: Replace the item in the latest year that the last column exceeds the total cost.

4.5.3.1. Value of Money Criterion

In case of replacement decision if we consider the time value of money criterion, the replacement decision analysis must be based upon an equivalent annual cost. **For example,** if we invest ₹100 and the rate if interest is 10% per year then the value of ₹100 after 1 year will be ₹110, which is called **value of money**. It is also known that the value of money that decreases with constant rate is called **discount factor.** The amount of money which is required to pay at the time of the policy decision for building up funds at compound interest is called **discounted value.**

For example, if the interest rate on ₹100 is r per cent per year, then present value (or worth) of ₹100 to be spent after n years will be,

$$d = \left(\frac{100}{100+r}\right)^n$$

Where d is called the **discount rate** or d**epreciation value.** After having the idea of **discounted operating cost** the objective should be to determine the critical age at which an item should be replaced so that the sum of all **discounted operating costs** is minimum.

Note: Operating costs are expenses associated with the maintenance and administration of a machine on a day-to-day basis. Operating costs vary directly with the rate of work. These costs include the costs of fuel, lubricants, tires, equipment maintenance and repairs. **For example,** we may have an item that is not subject to failure but whose operating cost increases with use. To reduce this operating cost, a replacement can be performed.

Example 35: Assume that the value of the money is 10% per year and also consider that machine A is replaced at the end of every three years. While the machine B is replaced at the end of every six years. The costs (₹) of machine A and B in successive years are shown in table below:

Year	1	2	3	4	5	6
Machine A	1,000	200	400	1,000	200	400
Machine B	1,700	100	200	300	400	500

Find-out whether machine A should be purchased or machine B and why?

Solution: Table 4.7 shows the discounted cost at 10 % rate per year for machine A:

Table 4.7: Discounted Cost of Machine A

Year	Discounted cost of 10% rate (₹)		
	Cost	Present worth	
1	1,000	$1,000 \times 1.000$	=1,000
2	200	$200\left(\frac{100}{100+10}\right) = 200 \times 0.9091$	=181.82
3	400	$200\left(\frac{100}{100+10}\right)^2 = 400 \times 0.8264$	= 330.56
		Total	₹ 1,512.38

From the above table, it is shown that average cost of machine A is ₹1512.38/3 = ₹ 504.13 per year.

Table 4.8 shows the discounted cost at 10 % rate per year for machine B:

Table 4.8: Discounted Cost of Machine B

Year	Discounted cost of 10% rate (₹)		
	Cost	Present worth	
1	1,700	$1,700 \times 1.00$	= 1,700
2	100	$100 \times (10/11) = 100 \times 0.9091$	= 90.91
3	200	$200 \times (10/11)^2 = 200 \times 0.8264$	= 165.28
4	300	$300 \times (10/11)^3 = 200 \times 0.7513$	= 225.39
5	400	$400 \times (10/11)^4 = 400 \times 0.6830$	= 273.20
6	500	$500 \times (10/11)^5 = 500 \times 0.6209$	= 310.45
		Total	₹2,765.23

Thus, average cost of machine B is ₹2765.23/6 = ₹460.87 per year.

Since the average yearly cost of machine B is low, hence the purchasing of machine B will be beneficial. As the interval for both machines are different, therefore we have to determine the total present worth of machine A for six years as follows:

Total Present Worth = 1,000 + 200 × 0.9091 + 400 × 0.8264 + 1,000 × 0.7513 + 200 × 0.6830 + 400 × 0.6209 = ₹ 2648.64

Thus machine A should be purchased because present worth of machine A is lower than the machine B.

Example 36: A pipeline is due for repairs. The repair would cost ₹10,000 and would last for three years, alternatively, a new pipeline can be laid at a cost of ₹30,000 which would last for 10 years. Assuming the interest rate to be 10% and ignoring salvage value, which is better alternative?

Solution: Let us assume that for new pipeline, there are two types of pipelines for infinite replacement cycles of 10 years and three years for the existing pipeline.

As discount rate is 10 per cent, the present worth of money can be calculated for one year as follows:

$$d = \frac{10}{(100+10)} = 0.901$$

Let us assume that D_n shows the discounted value of all future costs with a policy of replacing the items after n years. Thus we have the following formula:

$$D_n = c + c \times d^n + c \times d^{2n} + \ldots$$
$$= c\,(1 + d^n + d^{2n} + \ldots)$$
$$= \frac{c}{1-d^n} \text{ (sum of infinite GP)} \quad \ldots(1)$$

Here, c is the initial cost.

Now placing the values of c, d's and n in equation (1), we get the following:

$$D_3 = \frac{10{,}000}{1-(0.9091)^3} = ₹4{,}021, \text{ for existing pipeline}$$

and

$$D_{10} = \frac{30{,}000}{1-(0.9091)^{10}} = ₹48{,}820, \text{ for new pipeline}$$

As value of D_3 is less than D_{10}, hence the existing pipeline should be continued. Alternatively, the comparison may be made over 3 × 10 = 30 years.

4.5.3.2. Present Worth Factor Criterion

In this case the optimal value of replacement age of item can be determined under two different situations:

1) The running cost of an item increases monotonically which deteriorates over a period of time and the value of money decreases with a constant rate. If r is the interest rate, then

$$Pwf = (1 + r)^{-n}$$

is called the **Present worth factor** (Pwf) or present value of one rupee spent in n years from time now onwards. But if n = 1 the Pwf is given by

$$d = (1 + r)^{-1}$$

where, d = discount rate or depreciation value.

2) The money to be spent is taken on loan for a certain period at a given rate under the condition of repayment in instalments. The replacement of items on the basis of Present worth factor (Pwf) includes the present worth of all future expenditure and revenues for each replacement alternatives. An item for which present worth factor is less is preferred.

Let, C = Purchase cost of an item
M = Annual running cost/Maintenance cost
n = Life of the item in years
S = Scrap (or salvage) value of the item at the end of its life
r = Annual interest rate

Then the present worth of the total cost during n years is given by:

Total cost = C + M (Pwf for r% interest rate for n years) – S (Pwf for r% interest rate for n years)

For any item if the running cost of the item is different in its different operational life then the present worth of the total cost during *n* years is given by:

Total cost = C + M (Pwf for r% interest rate for i years) – S (Pwf for r% interest rate for i years)

where, i = 1, 2, ..., n.

Example 37: A firm decides to purchase a new machine having purchase price ₹15,000. The life of machine is considered to be 8 years on average and its salvage value is ₹3,000 at the end of life.

The running cost of this machine is ₹7,000 per year on average.

1) Find the present worth of future costs of proposed machine by considering an interest rate of 5%.
2) Differentiate the new machine with the present machine having the operating cost ₹5,000 per year and maintenance cost in the second year is ₹1,500 and this increases in coming years annually ₹500 till the life of machine.

Solution:

1) **New Machine:**
   i) Purchasing Cost, C = ₹15,000
   ii) Annual Present Worth of Operating Cost,
   = 7,000 × Present worth factor (Pwf) at 5% interest for 8 years= 7,000 × 6.4632
   = ₹ 45,242.4
   iii) Present Worth of the Salvage Value
   = 3,000 × Present worth factor (Pwf) at 5% Interest for 8 Years
   = 3,000 × 0.6768 = ₹ 2,030.4

Thus present worth of total future costs for new machine is 15,000 + 45,242.4 + 2,030.4
= ₹ 62,272.8.

Old Machine: The present worth of the old machine can be calculated as follows:

Table 4.9: Present Worth of Old Machine

Year of Service (1)	Operating Cost (₹) (2)	Repair Cost (₹) (3)	Total Operating and Repair Cost (4)= (2) + (3)	Pwf for Single Payment (5)	Present Worth (₹) (6) = (5) × (4)
1	5,000	–	5,000	0.9524	5,000 × 0.9524 =4762.00
2	5,000	1,500	6,500	0.9072	5896.80
3	5,000	2,000	7,000	0.8638	6046.60
4	5,000	2,500	7,500	0.8227	6710.25
5	5,000	3,000	8,000	0.7835	6268.00
6	5,000	3,500	8,500	0.7462	6342.70
7	5,000	4,000	9,000	0.7107	6396.30
8	5,000	4,500	9,500	0.6778	6439.10
				Total	48,321.75

The present worth of old machine is ₹48,321.75 as shown in **table 4.9**. This is lower than the present worth of new machine ₹ 62,272.8, thus purchasing of new machine will not be beneficial.

Example 38: A person has his own factory for which he/she wants to purchase a machine. The data related to machines A, B and C is shown in table below:

	Machine A	Machine B	Machine C
Present Investment (₹)	10,000	12,000	15,000
Total Annual Cost (₹)	2,000	1,500	1,200
Life (Years)	10	10	10
Salvage Value (₹)	500	1,000	1,200

Let consider that you are an advisor to the buyer, then which machine you suggest him to choose, assuming 12% normal rate of return.

We have:

i) Single payment present worth factor (Pwf) is 0.322 at 12% interest for 10 years.

ii) Annual series present worth factor (Pwf) is 5.650 at 12% interest for 10 years.

Solution: The present values of total cost of machines A, B and C for an interval of 10 years are shown in **table 4.10**:

Table 4.10: Present Value of Total Cost for Machines A, B and C

Machine (1)	Present Investment (2)	Preset Value of Total Annual Cost (3)	Present Value of Salvage Value (4)	Net Cost (₹) (5) = (2) + (3) – (4)
A	10,000	2,000 × 5.65 = 11,300	500 × 0.322 = 161.00	21,139.00
B	12,000	1,500 × 5.65 = 8,475	1,000 × 0.322 = 322.00	20,153.00
C	15,000	1,200 × 5.65 = 6,780	1,200 × 0.322 = 386.40	21,393.60

From the **table 4.10**, it is shown that present value of net cost for machine B is ₹ 20,153.00, which is minimum than the machine A and C, hence machine B should be suggested for purchasing.

Example 39: A machine A has purchasing cost ₹5,000. The maintenance cost of this machine is ₹1,000 for the first four years, and then rises by ₹200 in next coming years. The other machine B has purchasing cost ₹8,000 and their maintenance cost is ₹200 for the first year and rises by ₹400 in next coming years. Assume that:

1) There is no salvage value of both machines.
2) Time value of money is 10% per annum.
3) Operating and maintenance costs are incurred in the start of each year.

Determine the optimal replacement time for both machines and also find which machine should be preferred?

Solution: The calculation of average cost for machine A for different years are shown in **table 4.11**:

Table 4.11: Calculation of Average Cost for Machine X

Year	M(t)	P.V. Factor	P.V. of M(t)	ΣM(t)	Total Cost TC_n	Cumulative P.V	Average Cost W_n
1	1,000	1.0000	1,000	1,000	6,000	1.00	6,000
2	1,000	0.9091	909.1	1,909.1	6,909.1	1.9091	3,619
3	1,000	0.8264	826.4	2,735.50	7,735.50	2.7355	2,828
4	1,000	0.7513	751.3	3,486.80	8,486.80	3.4868	2,434
5	1,200	0.6830	819.6	4,306.40	9,306.40	4.1698	2,232
6	1,400	0.6209	969.26	5,175.66	10,175.66	4.7907	2,124
7	1,600	0.5645	903.20	6,078.86	11,078.86	5.3552	2,069
8	1,800	0.5132	923.76	7,002.62	12,002.62	5.8684	2,045
9	2,000	0.4665	933	7,935.62	12,935.62	6.3349	**2,042**
10	2,200	0.4241	933.02	8,868.64	13,868.64	6.7590	2,052
11	2,400	0.3855	925.02	9,793.84	14,793.84	7.1445	2,071
12	2,600	0.3505	911.3	10,705.14	15,705.14	7.4950	2,075

From the above table, it is clear that optimal replacement time for machine A is ninth year as the average cost is minimum i.e., ₹2042.

The calculations of average cost for machine B for different years are shown in **table 4.12**:

Table 4.12: Calculation of Average Cost for Machine B

Year	M(t)	P.V. Factor @10%	P.V. of M(t)	M(t)	TC_n	Cumulative P.V	Average Cost W_n
1	200	1.0000	200	200	8200	1.000	8,200
2	600	0.9091	545.46	745.46	8,745.46	1.9091	4,581
3	1,000	0.8264	826.40	1,571.86	9,571.86	2.7355	3,499
4	1,400	0.7513	1,051.82	2,623.68	10,623.68	3.4868	3,047
5	1,800	0.6830	1,229.40	3,853.08	11,853.08	4.1698	2,843
6	2,200	0.6209	1,365.98	5,219.06	13,219.06	4.7907	2,759
7	2,600	0.5645	1,467.70	6,686.76	14,686.76	5.3553	**2,743**
8	3,000	0.5132	1,539.60	8,226.36	16,226.36	5.8684	2,765
9	3,400	0.4664	1,586.10	9,812.46	17,812.46	6.3349	2,812

As the minimum average cost for machine B is ₹2,743 in seventh year, hence optimal replacement time for machine B is at the end of seventh year. The minimum average cost of machine A (₹2042) is lower than the minimum average cost of machine B (₹2,743), hence machine A should be selected.

Example 40: A machine costs ₹15,000 and its running costs for different years are given below. Find optimum replacement period if the capital is worth 10% and the machine has no salvage value.

Year	1	2	3	4	5	6	7
Running Cost (₹)	2,500	3,000	4,000	5,000	6,500	8,000	10,000

Solution: The present worth factor can be calculated as follows:

$$v = \frac{100}{100+10} = \frac{10}{11} = 0.9091$$

W(n) for one rupee to be spent in n years can be determined as shown in the following table:

Year (n)	Running Cost R_n	v^{n-1} (pwf)	$R_n v^{n-1}$	$\sum R_n v^{n-1}$	$C + \sum R_n v^{n-1}$	$\sum v^{n-1}$	W(n)
1	2,500	1000	2,500	2,500	17,500	1.000	17,500
2	3,000	0.9091	2,727	5,227	20,227	1.9091	10,595
3	4,000	$(0.9091)^2 = 0.8264$	3,306	8,533	23,533	2.7355	8,603
4	5,000	$(0.9091)^3 = 0.7513$	3,756	12,289	27,289	3.4868	7,826
5	6,500	$(0.9091)^4 = 0.6830$	4,440	16,729	31,729	4.1698	**7,609**
6	8,000	$(0.9091)^5 = 0.6209$	4,967	21,696	36,696	4.7907	7,660
7	10,000	$(0.9091)^6 = 0.5645$	5,645	27,341	42,341	5.3552	7,906

From above it is shown that $R_4 < W(5) < R_5$, i.e., ₹6,500 < ₹7,609 < ₹8,000, thus optimum replacement period is the 5th year.

Example 41: A manufacturer is offered two machines A and B. A has cost price of ₹2,500 its running cost is ₹400 for each of the first 5 years and increases by ₹100 every subsequent year. Machine B, having the same capacity as A, cost ₹1,250, has running cost of ₹600 for 6 years, increasing by ₹100 per year thereafter. If money is worth 10% per year, which machine should be purchased?

Solution: Since the money is worth 10% per year, hence the discount rate for machine A and B can be determined as follows:

$$v = \frac{1}{1+r} = \frac{1}{1+0.10} = 0.9091.$$

The calculations for machines A and B are determined in **tables 4.13 and table 4.14** respectively.

Table 4.13: Machine A

(1) Years of service (r)	(2) Running cost (R_r) (₹)	(3) Discount factor (v^{r-1})	(4) Discounted running cost ($R_r v^{r-1}$) (₹)	(5) $C + \sum_{r=1}^{n} R_r v^{r-1}$ (₹)	(6) $\sum_{r=1}^{n} v^{r-1}$	(7) $\frac{(5)}{(6)}$ (₹)
1	400	1.0000	400.00	2,900.00	1.0000	2,900.00
2	400	0.9091	363.64	3,263.64	1.9091	1,709.45
3	400	0.8264	330.56	3,594.20	2.7355	1,313.84
4	400	0.7513	300.52	3,894.72	3.4868	1,116.93
5	400	0.6830	273.20	4,167.92	4.1698	999.50
6	500	0.6209	310.45	4,478.37	4.7907	934.80
7	600	0.5645	338.70	4,817.07	5.3552	899.40
8	700	0.5132	359.24	5,176.31	5.8684	881.92
9	800	0.4665	373.20	5,549.51	6.3349	**875.86 Replace**
10	900	0.4241	381.69	5,931.20	6.7590	877.35

From **table 4.14**, it is shown that for machine A, ₹ 800 < ₹ 875.86 < ₹ 900, where ₹ 800 is the running cost during 9th year and ₹ 900 is that in 10th year. So machine A should be replaced after 9th year.

Table 4.14: Machine B

(1) Years of service (r)	(2) Running cost (R_r) (₹)	(3) Discount factor (v^{r-1})	(4) Discounted running cost ($R_r v^{r-1}$) (₹)	(5) $C+\sum_{r=1}^{n} R_r v^{r-1}$ (₹)	(6) $\sum_{r=1}^{n} v^{r-1}$	(7) $\frac{(5)}{(6)}$ (₹)
1	600	1.0000	600.00	1,850.00	1.0000	1,850.00
2	600	0.9091	545.46	2,395.46	1.9091	1,254.75
3	600	0.8264	495.84	2,891.30	2.7355	1,056.95
4	600	0.7513	450.78	3,342.08	3.4868	958.49
5	600	0.6830	409.80	3,751.88	4.1698	899.77
6	600	0.6209	372.54	4,124.42	4.7907	860.92
7	700	0.5645	395.15	4,519.57	5.3552	843.96
8	800	0.5132	410.56	4,930.13	5.8684	**840.11 Replace**
9	900	0.4665	419.85	5,349.98	6.3349	844.52
10	1,000	0.4241	424.10	5,774.08	6.7590	854.28

From **table 4.14**, it is shown that for machine B, ₹800 < ₹840.11< ₹900, where ₹ 800 is the running cost during 9th year and ₹900 is that in 10th year. So machine B should be replaced after 9th year.

4.6. REPLACEMENT OF ITEM THAT FAILS SUDDENLY

4.6.1. Introduction

There are many items which fail abruptly or fully in place of depreciate with time. A system usually consists of a large number but low cost items which are gradually responsible for failure of items with time, e.g., failure of radio, television and computers. Sometimes the failure of any item may be happens due to the complete collapse of the system. In such cases, the cost of an item is always higher in comparison to the cost of failure.

For example, a tube or a condenser in an aircraft costs little but its failure may result in total collapse of the aircraft. Likewise, failure of industrial item such as a pump in a refinery may close down the entire system and may cause heavy losses due to loss in production, wastage and other costs.

It is actually very important to know in advance that when failure of an item is likely to take place so that item can be replaced before it actually breakdown. To overcome from such situation, cautious examination of the item is necessary for detection of imminent failure and in such type of case, preventive replacement becomes very economical. The main purpose of such replacement is to minimise the cost of breakdown of the item as well as the total cost of the item. There are many conditions where the replacement of entire system works out to be less cost in comparison to the replacement of an individual unit of a system.

Therefore, upto a certain point, when the time interval increases the total cost of replacement starts diminishing and again it starts rising where the individual replacement component of the cost exerts a greater influence on cost.

In this figure, the sum of the cost group replacement per unit of time (G/T) is represented by GR and cost of individual replacement of items which fail during the replacement interval (also per unit of time) and, hence, equal to I(T/T).

In the **figure**, the optimal group replacement interval is determined as T*, corresponding to the lowest point on the GR curve.

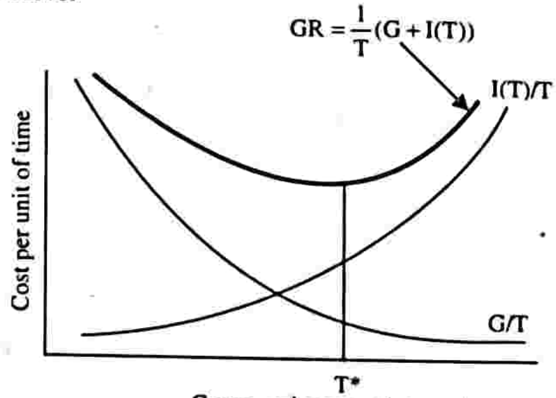


Figure 4.9: Determination of Optimal Replacement Interval

The analysis of replacement cost of both the items, i.e., the items, which fails suddenly and the determination of probability of human deaths and expected premium income and liability of claims due to death of policy holders, are similar.

Such analysis can appear in two ways. First is the number of items surviving upto certain time period and second is the number of items failing upto certain period.

The probabilities of the replacement of the items are based upon either past experience or on tests conducted under controlled conditions. Now, for minimising the total cost which involved in the system, we have to find the optimal value of time 't'.

4.6.2. Mortality Tables

There is no prediction about the failure of any item in the system. For obtaining probability distribution of failure of items in the system, we have to make an assumption that failure of the item occur only at the end of the given time period.

Now the main objective is to find out such optimum time period in which item can be replaced by another one in such a way so that the sum of the costs involved should be minimum.

These are used to derive the probability distribution of the lifespan of item.

Let,
$M(t)$ = Number of survivors at any time t
$M(t-1)$ = Number of survivors at any time $(t-1)$, and
N = Initial number of items

Then the probability of failure during time period t is given by:

$$P(t) = [M(t-1) - M(t)]/N$$

The probability that an item survived till age $(t-1)$ will fail during the interval $(t-1)$ to t, can be defined as the conditional probability of failure. It is given by:

$$P_c(t) = M(t-1) - M(t)/M(t-1)$$

The probability of survival till age t is given by:

$$P_s(t) = M(t)/N.$$

4.6.3. Replacement Policy

Let N be the number of items in a system which are to be replaced whenever any of these fails and even if it has not yet failed when it reaches a service life T. Let us consider the following notations in order to derive an optimum replacement policy:

$S(t)$ = Fraction of items surviving a service life, say t
$F_1(t) = 1 - S(t)$ = Fraction of items failed (first failure)
I = Total operating time
C_f = Cost of replacement after failure
C_p = Cost of preventive replacement

The cost of replacement during the service life T is given by:

$NF_1(T)\,C_f$ (for items replaced after failure)

and

$N\{1 - F_1(T)\}C_p$ (for items replaced before failure)

Thus, the total replacement cost per unit operating time will be:

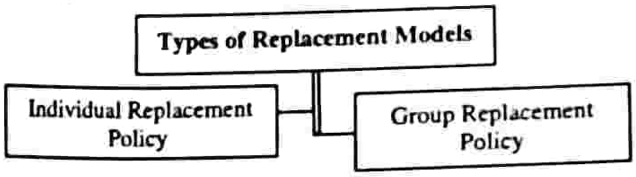
$$\frac{NF_1(T)C_f + N\{1-F_1(T)\}C_p}{NI} = (C_f - C_p)\left\{\frac{F_1(T) + C_p/(C_f - C_p)}{I}\right\}$$

This cost will be minimum only if the factor $C_p/(C_f - C_p)$ is minimum. Hence, we can replace an item when $\{F_1(T) + C_p/(C_f - C_p)\}/I$ is minimum. The value of I is given by

$$I = \int_0^T S(t)dt, \text{or } S(t) = \frac{dI}{dt}$$

4.6.4. Types of Replacement Models

Two types of replacement policies are considered when dealing with different situations (**figure below**):



4.6.4.1. Individual Replacement Models

According to this replacement policy, any item is replaced as soon as it gets fail in it use. Such replacement policy is adopted when the operating efficiency of an item to fail increases with the time when it has been in service. Therefore, to know the service life of the item, a record is maintained which easily clears how long the item has been in service and it is replaced when either it fails or it reaches some service life, say T, but not failed.

When number of items are used simultaneously, in that case such replacement policy is useful where the main aim is to determine the optimum value of service life of the item, i.e., T. It happens sometimes that single preventive replacement policy increases the number of replacements without reducing the cost of each replacement. It happens so as each preventive replacement means that the remaining service life of the item is lost.

For example, let us consider the problem of replacing light bulbs in a factory. The replacements at different locations will not be simultaneous if single replacement policy is adopted. If all bulbs are replaced at the same time, then on allocations where a failure occurs before the preventive replacement is due, the resulting replacement will be out of order with the remaining.

During the next period of preventive replacement some remaining bulbs will be failed. Hence it can be said that the single replacement will not be cheaper because replacement of bulbs on different locations will occur at different times.

Example 42: Let consider the following table that shows the mortality rates for specific type of light bulbs:

Time (Weeks)	0	1	2	3	4	5	6
Number of Bulbs Still Working	500	490	475	460	425	370	310

There are total 500 light bulbs that are operating and ratio of costs of replacement of a bulb after and before it has burned out are given as:

$$C_f/C_p = 5.$$

Find when the bulb should be replaced?

Solution: We have, $C_f/C_p = 5$

Now the total operating cost per bulb operating time will be as follows:

$$\frac{F_1(T) + C_p/(5C_p - C_p)}{I} = \frac{F_1(T) + 0.25}{I}$$

Under single replacement policy, the determination of optimum replacement period is shown in **table 4.15**:

Table 4.15

Time	Number of Bulbs Working	S(T)	$F_1(T) = 1-S(T)$	I	$F_1(T) + 0.25$	$[F_1(T) + 0.25]/I$
0	500	1.00	0.00	0.00	0.25	–
1	490	0.98	0.02	0.99	0.27	0.27
2	475	0.95	0.05	1.95	0.30	0.15
3	460	0.92	0.08	2.89	0.33	0.11
4	425	0.85	0.15	3.78	0.40	**0.10**
5	370	0.74	0.26	4.57	0.51	0.11
6	310	0.62	0.38	5.25	0.63	0.12

It is shown from **table 4.15** that cost is lowest during fourth week, so after every 4 weeks the bulbs should be replaced for economic benefits. If the replacement is not done after 4 weeks then the cost will start increasing.

4.6.4.2. Group Replacement Models

Under the group replacement policy, all the items are replaced ignoring the fact that whether all have failed or not with a provision that if any item fails before the optimal time then it may be replaced individually. In group replacement policy, there are more replacements in comparison of single replacement policy.

For example, if an item fails only just before its preventive replacement is due and single replacement policy is used, then the replacement of the failed item would not itself be replaced until service life period T had elapsed. However, with group replacement policy, it would be replaced when the next replacement is due, although it will then have had only a short period of service.

If the item has been in service for a short period and is therefore unlikely to fail then it will not be used. There will be more preventive replacements when group replacement policy is used than single replacement policy for a given service life T. It will be justified if each of these replacements costs much less.

The probability of items failure increases with time or age because a system having a large number of identical low cost items. In this case, there is a set-up cost which is independent of the number of replacements. In this system, it is profitable to replace all the items after a fixed interval and this policy is called group replacement policy.

Theorem (Group Replacement)

Let us consider that the all the items are replaced in a system after a time interval 't' with a provision that the single replacement can be applied if any item fails before this time interval. If the cost of single replacement for the period is greater than the average cost per unit time period at the end of t periods then the group replacement policy must be selected.

If the cost of single replacements at the end of period t – 1 is less than the average cost per unit period at the end of period t, then group replacement is not suggested at the end of period t.

Proof: Let,

N = Total number of items in the system

C_2 = Cost of replacing an single item

C_1 = Cost of replacing an item in group

C(t) = Total cost of group replacement after time period t

f(t) = Number of failures during time period t

Then, clearly, $C(t) = NC_1 + C_2 \sum_{x=0}^{t-1} f(x)$

The average cost of group replacement per unit period of time during a period *t*, is thus given by:

$$A(t) = \frac{C(t)}{t} = \left[NC_1 + C_2 \sum_{X=0}^{t-1} F(x)\right]/t$$

We shall determine the optimum *t* so as to minimise C(t)/t

Note that whenever $\frac{C(t-1)}{t-1} > \frac{C(t)}{t}$ and $\frac{C(t+1)}{t+1} > \frac{C(t)}{t}$, it is better to replace all the items after time period *t*

Now, $\frac{C(t+1)}{t+1} - \frac{C(t)}{t} > 0 \Rightarrow C_2 f(t) > C(t)/t;$

and, $\frac{C(t-1)}{t-1} - \frac{C(t)}{t} > 0 \Rightarrow C_2 f(t-1) < C(t)/t;$

$\therefore \quad tC_2 f(t-1) < C(t) < t.C_2 f(t).$

Or $\quad tf(t-1) - \sum_{x=0}^{t-1} f(x) < \frac{NC_1}{C_2} < tf(t) - \sum_{x=0}^{t-1} f(x)$

Difference between Individual and Group Replacement Policies

Table below shows the difference between individual and group replacement policies:

Group Replacement Policy	Individual Replacement Policy
In group replacement policy, all items are replaced at the same time after a particular amount of time. In is not considered that how many items in group are failed.	In individual replacement policy, replacement of items is done after the failure.
It is uneconomical in nature.	It is economical in nature.
When the machine is normally non-operative, then group replacement is done.	When machine is operative then individual replacement is done.
There is cost associated in term of lost production.	There is no cost associated in term of lost production.

Example 43: In a digital computer, the failure rates of a specific type of transistors are shown in **table below:**

End of the Week	1	2	3	4	5	6	7	8
Probability of Failure	0.05	0.13	0.25	0.43	0.68	0.88	0.96	1.00

The replacement cost of a failed transistor is ₹1.25. It is observed that all transistors are replaced concurrently at fixed intervals. The replacement should be done when a single transistor fails in service. If the group replacement cost per transistor is 30 paisa, then find the best period between group replacements? Also determine the group replacement price per transistor at which strictly single replacement policy would become superior to the adopted policy?

Solution: Let consider that 1000 transistors are in use. Suppose p_i is the probability of a new transistor used for working, fails during the i[th] week of its life. Therefore, we have:

$p_1 = 0.05$ $\quad p_2 = 0.13 - 0.5 = 0.08$

$p_3 = 0.25 - 0.13 = 0.12$ $\quad p_4 = 0.43 - 0.25 = 0.18$

$p_5 = 0.68 - 0.43 = 0.25$ $\quad p_6 = 0.88 - 0.68 = 0.20$

$p_7 = 0.96 - 0.88 = 0.08$ $\quad p_8 = 1.00 - 0.96 = 0.04$

Assume that total replacements done at the end of i[th] week is N_i. Now, we have:

N_0 = Number of Transistors at the Beginning	= 1,000
$N_1 = N_0p_1 = 1{,}000 \times 0.05$	= 50
$N_2 = N_0p_2 + N_1p_1 = 1{,}000 \times 0.08 + 50 \times 0.05$	= 82
$N_3 = N_0p_3 + N_1p_2 + N_2p_1 = 1{,}000 \times 0.12 + 50 \times 0.08 + 82 \times 0.05$	= 128
$N_4 = N_0p_4 + N_1p_3 + N_2p_2 + N_3p_1$	= 199
$N_5 = N_0p_5 + N_1p_4 + N_2p_3 + N_3p_2 + N_4p_1$	= 289
$N_6 = N_0p_6 + N_1p_5 + N_2p_4 + N_3p_3 + N_4p_2 + N_5p_1$	= 272
$N_7 = N_0p_7 + N_1p_6 + N_2p_5 + N_3p_4 + N_4p_3 + N_5p_2 + N_6p_1$	= 194
$N_8 = N_0p_8 + N_1p_7 + N_2p_6 + N_3p_5 + N_4p_4 + N_5p_3 + N_6p_2 + N_7p_1$	= 195

From above, it is shown that average number of transistors failing increases till the 5[th] week and thereafter decreases and later again increasing from 8[th] week.

Hence, N_i will increase and decrease until the steady state is not obtained. The average life of every transistor is given by:

$1 \times 0.5 + 2 \times 0.08 + 3 \times 0.12 + 4 \times 0.18 + 5 \times 0.25 + 6 \times 0.20 + 7 \times 0.08 + 8 \times 0.04 = 4.62$ weeks.

Average number of failures per week = 1,000/4.62 = 216.45 ≈ 216

Hence, the single replacement cost per week = 216 × 1.25 = ₹ 270.00

As the replacement cost of 1,000 transistors is 30 paisa per transistors when replaced simultaneously and replacement cost of single transistor is ₹1.25 when it fails, thus average cost for different group replacement policies will be as follows:

End of Week	Single Replacement	Total Cost (₹) Single + Group	Average Cost (₹)
1	50	50 × 1.25 + 1000 × 0.30 = 363	363
2	132	132 × 1.25 + 1000 × 0.30 = 465	232.50
3	260	260 × 1.25 + 1000 × 0.30 = 625	208.30
4	459	459 × 1.25 + 1000 × 0.30 = 874	218.50

From above table, it is shown that optimal interval between group replacements is three weeks as the average cost for third week is minimum i.e., ₹208.30. Since it is less than the single replacement cost (₹ 270), hence the group replacement policy would be preferable.

Example 44: There are 1000 resistors in a computer. When any resistor fails, then it is replaced by new one. The replacement cost of a resistor is ₹ 1 when single replacement is done. In case all resistors are replaced simultaneously then the per resistor replacement cost is decreased to 35 paisa. Assume that S(t) is the percentage of operating resistors after t months and P(t) is the probability of failed resistors during the month t. The value of S(t) and P(t) are shown in table below:

T	0	1	2	3	4	5	6
S(t)	100	97	90	70	30	15	0
P(t)	–	0.03	0.07	0.20	0.40	0.15	0.15

Find the optimal replacement plan?

Solution: Let consider that there are N_i resistors replaced at the end of i[th] month. Then values of N_i are determined as follows:

N_0 = Total Resistors in the Beginning = 10,000

N_1 = Resistors replaced at the end of first month
$= N_0p_1 = 10{,}000 \times 0.03 = 300$

$N_2 = N_0p_2 + N_1p_1$
$= 10{,}000 \times 0.07 + 300 \times 0.03 = 709$

$N_3 = N_0p_3 + N_1p_2 + N_2p_1$
$= 10{,}000 \times 0.20 + 300 \times 0.07 + 709 \times 0.03 = 2042$

$N_4 = N_0p_4 + N_1p_3 + N_2p_1$
$= 10{,}000 \times 0.40 + 300 \times 0.20 + 709 \times 0.07 + 2042 \times 0.03 = 4171$

$N_5 = N_0p_5 + N_1p_4 + N_2p_3 + N_3p_2 + N_4p_1$
$= 10{,}000 \times 0.15 + 300 \times 0.40 + 709 \times 0.20 + 2042 \times 4171 \times 0.03 = 2030$

$N_6 = N_0p_6 + N_1p_5 + N_2p_4 + N_3p_3 + N_4p_2 + N_5p_1$
$= 10{,}000 \times 0.15 + 300 \times 0.15 + 709 \times 0.40 + 2042 \times 0.20 + 4171 \times 0.07 + 2030 \times 0.03 = 2590$

From above calculation, it is shown that average resistors deteriorating every month increase till the fourth month and thereafter decrease in the sixth month. Thus, it oscillates until the steady state is not obtained by the system. The formula for calculation of average life of each resistor is given by:

$$\text{Expected life} = \sum_{i=1}^{6} x_i p(x_i)$$
$$= 1 \times 0.03 + 2 \times 0.07 + 3 \times 0.20 + 4 \times 0.40 + 5 \times 0.15 + 6 \times 0.15$$
$$= 4.02 \text{ months}$$

Now Average number of failures per month is as below:

$$\frac{N}{\text{Expected life}} = \frac{10{,}000}{4.02} = 2487.5 = 2488 \text{ Resistors}$$

When single replacement policy is applied at the cost of ₹ 1 per resistor, then net cost will be as follows:

(2488 × 1) = ₹ 2488

Table below shows the average cost per month when all resistors are replaced simultaneously:

End of Month	Total Cost of Group Replacement (₹)	Average Cost per Month (₹)
1	300 × 1 + 10,000 × 0.35 = 3,800	3,800.00
2	(300 + 709) × 1 + 10,000 × 0.35 = 4,509	2254.50
3	(300 + 709 + 2042) × 1 + 10,000 × 0.35 = 6,551	**2183.66**
4	(300 + 709 + 2042 + 4171) × 1 + 10,000 × 0.35 = 10,722	2680.50
5	(300 + 709 + 2042 + 4171 + 2030) × 1 + 10,000 × 0.35 = 12,752	2550.40
6	(300 + 709 + 2042 + 4171 + 2030 + 2590) × 1 + 10,000 × 0.35 = 15,442	2557.00

From above table, it is shown that average cost per month is lowest for the third month i.e., ₹ 2183.66, so optimal group replacement will be done after every third month.

Example 45: A decorative light lamp consists of a series of 10 bulbs. If any bulb fails, then it is replaced by new bulb. When individual replacement policy is used then replacement cost is ₹1 per bulb. If all bulbs are replaced simultaneously, then replacement cost would be 35 paisa per bulb. Assume that percentage of bulbs surviving after t month is S(t) and P(t) is the probability of failed bulbs during the month t. The values of S(t) and P(t) are shown in table below:

t	0	1	2	3	4	5	6
S(t)	100	97	90	70	30	15	0
P(t)	–	0.03	0.07	0.2	0.4	0.15	0.15

Determine the optimal replacement plan?

Solution: First calculate average cost for Individual replacement and group replacement separately and then in order to draw conclusions compare them.

Average life of every bulb can be calculated as follows:

$$\text{Average life of a Bulb} = \sum_{i=0}^{6} X_i P_i$$

where X_i = No. of months

Probability of failure P_i

$= (1 \times 0.03) + (2 \times 0.07) + (3 \times 0.20) + (4 \times 0.40) + (5 \times 0.15) + (6 + 0.15)$

$= 0.03 + 0.14 + 0.60 + 1.60 + 0.75 + 0.90 = 4.02$ months

Average Number of Replacement

$$= \frac{\text{Total Bulbs}}{\text{Average life}} = \frac{10{,}000}{4.02} = 2487.56 \approx 2488 \text{ bulbs}$$

When individual replacement is used then average cost will be as follows:

$= 2488 \times 1 =$ ₹ 2488

The percentage of surviving bulbs S(t) is zero during 6th month, it means that no bulb continues more than 6 months.

Let N_i = Number of bulbs replaced at the end of i^{th} month

N_o = Number of New Bulbs at the Beginning = 10,000

$N_1 = N_0P_1 = 10{,}000 \times 0.03 = 300$

Similarly,

$N_2 = N_0P_2 + N_1P_1$

$= 10{,}000 \times 0.07 \times 300 \times 0.03 = 700 + 9 = 709$

$N_3 = N_0P_3 + N_1P_2 + N_2P_1$

$= 10{,}000 \times 0.20 + 300 \times 0.07 + 709 \times 0.03$

$= 2{,}000 + 21 + 21 = 2042$

$N_4 = N_0P_4 + N_1P_3 + N_2P_2 + N_3P_1$

$= 10{,}000 \times 0.40 + 300 \times 0.20 + 709 \times 0.07 + 2042 \times 0.03$

$= 4{,}000 + 60 + 50 + 61 = 4{,}171$

$N_5 = N_0P_5 + N_1P_4 + N_2P_3 + N_3P_2 + N_4P_1$

$= 10000 \times 0.15 + 300 \times 4.0 + 709 \times 0.20 + 2042 \times 0.07 + 4{,}171 \times 0.03$

$= 1500 + 120 + 142 + 143 + 125 = 2030$

$N_6 = N_0P_6 + N_1P_5 + N_2P_4 + N_3P_3 + N_4P_2 + N_5P_1$

$= 10{,}000 \times 0.15 + 300 \times 0.15 + 709 \times 0.40 + 2042 \times 0.20 + 4{,}171 \times 0.07 + 2030 \times 0.03$

$= 1{,}500 + 45 + 284 + 408 + 292 + 61 = 2{,}590$

The calculation of average cost of group replacement is shown in **table 4.16**:

Table 4.16: Determination of Average Cost of Group Replacement

End of Month	Total Number of Bulbs	Cumulative Number of Failure	Cost of Individual Replacement @ ₹ 1/-	Cost of Group Replacement @ ₹ 0.35	Total Cost TC_n	Av. Cost per Month ATC_n
1	300	300	300	3,500	3,800	3,800
2	709	1,009	1,009	3,500	4,509	2,255.50
3	2,042	3,051	3,051	3,500	6,551	**2,183.67**
4	4,171	7,222	7,222	3,500	10,722	2,680.50
5	2,030	9,252	9,252	3,500	12,752	2,550.40
6	2,590	11,842	11,842	3,500	15,342	2,557.00

The lowest average cost per month is ₹ 2,183.67 at the end of third month as shown in **table 4.16**. It is less than average cost of individual replacement, i.e., ₹ 2,488, therefore group replacement policy would be better over individual replacement policy. Hence, Group Replacement is recommended.

Example 46: An electronic equipment contains 500 resistors. When any resistor fails, it is replaced. The cost of replacing a resistor individually is ₹20. If all the resistors are replaced at the same time, the cost per resistor is ₹5. The percentage of surviving S(i) at the end of month i is given in table.

Table 4.17: Per cent Survival Rate

Month i:	0	1	2	3	4	5
S(i):	100	90	75	55	30	0

What is the optimum replacement plan?

Solution: Let consider that the probability of failure during the month i is represented by p_i. Then we get the following values:

$p_1 = \frac{100-90}{100} = 0.10$; $p_2 = \frac{90-75}{100} = 0.15$

$p_3 = \frac{75-55}{100} = 0.20$; $p_4 = \frac{55-30}{100} = 0.25$

$p_5 = \frac{30-0}{100} = 0.30$

The resistor that has survived for four months will fail in the fifth month because from above it is clear that no resistor can survive after 5 months. Suppose that the resistors failing during a month are considered at the end of the month.

Let N_i be the number of resistors replaced at the end of the i^{th} month. Then we have:

$N_0 = 500$

$N_1 = N_0p_1 = 500 \times 0.1 = 50$

$N_2 = N_0p_2 + N_1p_1 = 500 \times 0.15 + 50 \times 0.1 = 80$

$N_3 = N_0p_3 + N_1p_2 + N_2p_1 = 500 \times 0.20 + 50 \times 0.15 + 80 \times 0.10 = 116$

$N_4 = N_0p_4 + N_1p_3 + N_2p_2 + N_3p_1 = 500 \times 0.25 + 50 \times 0.20 + 80 \times 0.15 + 116 \times 0.10 = 159$

$N_5 = N_0p_5 + N_1p_4 + N_2p_3 + N_3p_2 + N_4p_1 = 500 \times 0.30 + 50 \times 0.25 + 80 \times 0.20 + 116 \times 0.15 + 159 \times 0.10 = 212$

Determination of Individual Replacement Cost

Average life of every resistor $= \sum_{i=1}^{5} x_i p(x_i)$

$= 1 \times 0.1 + 2 \times 0.15 + 3 \times 0.20 + 4 \times 0.25 + 5 \times 0.30$

$= 3.5$ months

Average number of failures/month $= \frac{500}{3.5} = 143$ (approx.)

Therefore, The cost of individual replacement

= No. of Failures/Month × Individual Replacement Cost/ Resistor

= 143 × 20 = ₹2860

Determination of Group Replacement Cost

The cost per resister when replaced simultaneously = ₹5

The cost per resistor when replaced individually = ₹20

The costs of group replacement policy for all replacement periods are given in **table 4.18**:

Table 4.18: Calculations of Costs for Preventive Maintenance

End of Month (N)	Cost of Replacing 500 Resistors at a Time (₹)	Cost of Replacing Resistors Individually during given Replacement Period (₹)	Total Cost [(II) + (III)] (₹)	Average Cost/Month (₹)
(I)	(II)	(III)	(Iv)	(v)
1	2500	50 × 20 = 1000	3500	3500
2	2500	(50 + 80) × 20 = 2600	5100	2550
3	2500	(50 + 80 + 116) × 20 = 4920	7420	2473
4	2500	(50 + 80 + 116 + 159) × 20 = 8100	10600	2850
5	2500	(50 + 80 + 116 + 159 + 212) × 20 = 12340	14840	2968

From **table 4.17**, it is shown that average cost/month is lowest in the third month. Hence the group replacement period is three months.

Example 47: A factory has a large number of bulbs, all of which must be in working condition. Mortality rate is given below:

Week	1	2	3	4	5	6
Proportion which Failed	0.10	0.15	0.25	0.35	0.12	0.03

If a bulb fails in service, it costs ₹3.50 to replace; if all bulbs are replaced at a time, it costs ₹1.20 each. Find optimum replacement policy.

Solution: Let consider that 1000 bulbs are in use. Suppose p_i is the probability of a new bulb used for working, fails during the i^{th} week of its life. Therefore, we have:

$p_1 = 0.10$ $p_2 = 0.15$

$p_3 = 0.25$ $p_4 = 0.35$

$p_5 = 0.12$ $p_6 = 0.03$

Assume that total replacements done at the end of i^{th} week is N_i. Now, we have:

N_0 = Number of Transistors at the Beginning $= 1{,}000$

$N_1 = N_0p_1 = 1{,}000 \times 0.10$ $= 100$

$N_2 = N_0p_2 + N_1p_1 = 1{,}000 \times 0.15 + 100 \times 0.10$ $= 160$

$N_3 = N_0p_3 + N_1p_2 + N_2p_1 = 1{,}000 \times 0.25 + 100 \times 0.15 + 160 \times 0.10$ $= 281$

$N_4 = N_0p_4 + N_1p_3 + N_2p_2 + N_3p_1 = 1000 \times 0.35 + 100 \times 0.25 + 160 \times 0.15 + 281 \times 0.1$ $= 427$

$N_5 = N_0p_5 + N_1p_4 + N_2p_3 + N_3p_2 + N_4p_1 = 1000 \times 0.12 + 100 \times 0.35 + 160 \times 0.25 + 281 \times 0.15 + 427 \times 0.1$ $= 280$

$N_6 = N_0p_6 + N_1p_5 + N_2p_4 + N_3p_3 + N_4p_2 + N_5p_1 = 1000 \times 0.03 + 100 \times 0.12 + 160 \times 0.35 + 281 \times 0.25 + 427 \times 0.15 + 280 \times 0.1$ $= 260$

Hence, N_i will increase and decrease until the steady state is not obtained. The average life of every transistor is given by:

$1 \times 0.10 + 2 \times 0.15 + 3 \times 0.25 + 4 \times 0.35 + 5 \times 0.12 + 6 \times 0.03 = 3.33$ weeks.

Average number of failures per week = 1,000/3.33
= 300.30
= 300

Hence, the single replacement cost per week = 300 × 3.50
= ₹ 1050.00

As the replacement cost of 1,000 transistors is ₹1.20 per bulb when replaced simultaneously and replacement cost of single bulb is ₹3.50 when it fails, thus average cost for different group replacement policies will be as follows:

End of Week	Total Cost (₹) Group Replacement	Average Cost (₹)
1	100 × 3.50 + 1000 × 1.20 = 1550	1550
2	(100 + 160) × 3.50 + 1000 × 1.20 = 2110	1055
3	(100 + 160 + 281) × 3.50 + 1000 × 1.20 = 3093.5	**1031.17**
4	(100 + 160 + 281 + 427) × 3.50 + 1000 × 1.20 = 4588	1147

5	(100 + 160 + 281 + 427 + 280)× 3.50 + 1000 × 1.20 = 5568	1113.6
6	(100 + 160 + 281 + 427 + 280 + 260)× 3.50 + 1000 × 1.20 = 6478	1079.67

From **above table**, it is shown that optimal interval between group replacements is three weeks as the average cost for 3rd week is minimum i.e., ₹1031.17. Since it is less than the single replacement cost (₹ 1050.00), hence the group replacement policy would be preferable.

4.7. EXERCISE

4.7.1. Short Answer Type Questions

1) Define inventory.
2) State the two types of inventory.
3) List the elements of carrying cost.
4) What is purchasing cost?
5) Explain the following terms in inventory management:
   i) Carrying Cost
   ii) Shortage Costs
6) Define the following terms: Lead time, Shortage costs.
7) What do you mean by inventory management?
8) What is re-order level?
9) What is EOQ?
10) Mention the assumptions of EOQ.
11) What is economic batch quantity?
12) Briefly explain the various types of inventory models.
13) Explain the concepts of EOQ of with and without shortages.
14) Discuss the EBQ model of with and without shortages.
15) Explain the quantity discount model in detail.
16) What is meant by a quantity discount?
17) Derive EOQ formula for simple inventory model with no shortages and instantaneous replenishment.
18) Define replacement.
19) What are the types of replacement policies?
20) What is Gradual Failure?
21) What are the assumptions of Replacement Theory?
22) What are the types of Replacement Decisions?
23) What are mortality tables?
24) What is Individual Replacement Policy?
25) What is Group Replacement Policy?
26) Explain with examples the failure mechanism of items.

4.7.2. Long Answer Type Questions

1) Find out the economic ordering quantity from the following particulars:

Annual usage	1,20,000 units
Cost of placing and receiving one order	₹60
Cost of materials per unit	₹8,000
Annual carrying cost	10% of inventory value

[Ans: EOQ = 18,000 units**]**

2) From the following data prepare economic ordering quantity for the item:

Total Annual Consumption	1,20,000 units
Purchase Price (Per unit)	₹1.50 per unit
Ordering Cost	₹80 per order
Cost of Carrying Inventory	20% per annum

[Ans: EOQ = 5,657 units**]**

3) The daily demand for an electronic machine is approximately 25 times every time an order is placed, a fixed cost of ₹25 is increased. The daily holding cost per item inventory is ₹0.40. If the lead time is 16 days determine the economic lot size and the re-order point.

[Ans: EOQ = 56**]**

4) A company manufactures a product from a raw material, which is purchased at ₹60 per kg. The company incurs a handling cost of ₹360 plus freight of ₹390 per order. The incremental carrying cost of inventory of raw material is ₹0.50 per kg per month. In addition, the cost of working capital finance on the investment in inventory of raw material is ₹9 per kg per annum. The annual production of the product is 1,00,000 units and 2.5 units are obtained from one kg of raw material.
Calculate the economic order quantity of raw materials.

[Ans: EOQ = 2,000 kg.**]**

5) A manufacturing company purchases 24,000 pieces of a component from a sub-contractor at ₹500 per piece and uses them in its assembly department, at a steady rate. The cost of placing an order and following it up is ₹2,500. The estimated stock holding cost is approximately 1% of the value of average stock held. The company is at present placing orders which at present vary between an order placed every two months (i.e., six orders p.a.) to one order per annum. Which policy would you recommend?

[Ans: EOQ = 4,900 pieces**]**

6) Footwear Company uses 4,000 nos. per annum of special leather material in manufacturing of its product. The leather is procured from a local manufacturer at the basic price of ₹10 each.
The Inventory Cost Data is:
Procurement Cost per order = ₹40.
Inventory Carrying Cost = 20%.

The supplier offers following discounts on the basic price for ordered quantities of:

Order Quantity	Discount
400 – 799	2%
800 – 1599	4%
1600 and above	5%

What quantity should the company order to derive optimum cost benefit?

[Ans: EOQ = 400 units; Optimum order quantity = 800 units**]**

7) Calculate the optimum order quantity using the following information:

Quantity	Purchasing cost per unit (₹)
$0 \le Q_1 < 200$	40
$200 \le Q_2 < 400$	36
$400 \le Q_3$	32

The average monthly demand is expected to be 800 units. The storage cost is likely to be 40% of the unit cost of the product. The ordering cost is ₹50.00 per month.

[Ans: EOQ of Q_3 = 79 units, EOQ of Q_2 = 75 units, EOQ of Q_1 = 71 units**]**

8) An auto industry purchases spark plugs at the rate of ₹25 per piece. The annual consumption of spark plugs is 18,000 numbers. If the ordering cost is ₹250 per order and carrying cost is 25 per cent per annum. What would be the EOQ? If the supplier of spark plugs offer a discount of 5 per cent for order quantity of 3,000 numbers per order, do you accept the discount offer?

[Ans: EOQ = 1,200 nos.; Yes, the offer is acceptable**]**

9) A trader stocks a particular seasonal product at the beginning of the season and cannot re-order. The item costs him ₹25 each and he sells at ₹50 each. For any item that cannot be met on demand, the trader has estimated a goodwill cost of ₹15. Any item unsold will have a salvage value of ₹10.

Holding cost during the period is estimated to be 10 per cent of the price. The probability distribution of demand is as follows:

Units Stocked	2	3	4	5	6
Probability of Demand, p(D = Q)	0.35	0.25	0.20	0.15	0.05

Determine the optimal number of items to be stocked.
[Ans: Optimal order 4]

10) Zentron uses 500 cases of flour polish per year. The ordering cost is ₹15 per order. The inventory carrying cost is 20% per year. The price schedule, which includes the transformation cost, shows that orders of less than 50 cases will cost ₹49.95 per case; between 50 and 79 cases will cost ₹44.95 per case and 80 cases or more will cost ₹39.95 per case. Prices apply inclusively to all units bought. What is the optimum purchase order size that should be placed and what is the total cost?

[Ans: Optimum purchase order size = 80 units; Total cost = ₹20,388.35]

11) The demand for a certain item is 16 units per period. Unsatisfied demand causes a shortage cost of ₹0.75 per unit per short period. The cost of initiating purchasing action is ₹15.00 per purchase and the holding cost is 15% of average inventory valuation per period. Item cost is ₹8.00 per unit. (Assume that shortages are being back ordered at the above mentioned cost). Find the minimum cost purchase quantity.

[Ans: Minimum Cost of Purchase Quantity = ₹15 approximately]

12) A firm is considering when to replace its machine whose price is ₹12,200. The scrap value of the machine is ₹200 only. From past experience the maintenance costs of the machine are as under:

Year	1	2	3	4	5	6	7	8
Maintenance Cost in (₹)	200	500	800	1,200	1,800	2,500	3,200	4,000

Find when the new machine should be purchased.
[Ans: After 6th year]

13) The cost of a truck is ₹10,000. The salvage value and the running costs are given below. Find the most economical age for replacement.

Year	1	2	3	4	5	6	7
Running Cost (₹)	3,000	3,200	3,600	4,200	5,000	5,800	6,800
Resale Value (₹)	7,000	5,000	3,400	2,400	1,600	1,000	1,000

[Ans: After 4th year]

14) Machine A costs ₹45,000 and the operating costs are estimated at ₹1,000 for the first year increasing by ₹10,000 per year in the second and subsequent years. Machine B costs ₹50,000 and operating cost are ₹2,000 for the first year increasing by ₹4,000 per year in the second and subsequent years. If you now have a machine of type A, should you replace it with B? If so, when? Assume that both the machines have no resale value, and that the future costs are not discounted.

[Ans: Machine A should be replaced by B, when its age is 2 years]

15) Find the cost per period of individual replacement policy of an installation of 300 electric bulbs given the following:
i) Cost of replacing individual bulb is ₹3.
ii) Conditional probability of failure is given below:

Week	0	1	2	3	4
Conditional Probability of Failure	0	1/10	1/3	2/3	1

[Ans: ₹206]

16) A computer contains 10,000 resistors. When any resistor fails it is replaced. The cost of replacing a resistor individually is ₹1. If all the resistors are replaced at the same time, the cost per resistor would be reduced to 35 paise. The percentage of surviving resistors, say, s(t) at the end of month t and p(t), the probability of failure during the month t are:

t	0	1	2	3	4	5	6
s(t)	100	97	90	70	30	15	0
p(t)	–	0.03	0.07	0.20	0.40	0.15	0.15

What is the optimum replacement plan?

[Ans: Group replacement at the end of 3rd month]

17) A factory has a large number of bulbs, all of which must be in working condition. The mortality of bulbs is given in the following table:

Week	1	2	3	4	5	6
Proportion of Bulbs Failing	0.10	0.15	0.25	0.35	0.12	0.03

If a bulb fails in service, it costs ₹3.50 to be replaced; but if all the bulbs are replaced at a time it costs ₹1.20 each. find the optimum group replacement policy.

[Ans: After every third week]

18) It has been suggested by a data processing firm that a company adopt the policy of periodically replacing all the tubes in a certain piece of equipment. A given type of tube is known to have mortality distribution as shown in the following table:

Tube Failures/Week	1	2	3	4	5
Probability of Failure	0.3	0.1	0.1	0.2	0.3

The cost of replacing the tubes on an individual basis is estimated to be ₹1.00 per tube and the cost of a group replacement policy average ₹0.30 per tube. Compare the cost of preventive replacement with that of remedial replacement.

[Ans: It is optimal to have a group replacement after every fourth week alongwith individual replacements]

Unit 5

Queuing Theory and Simulation

5.1. QUEUING THEORY

5.1.1. Introduction

The mathematical study of **waiting lines (or queues)** is called queuing theory. The mathematical analysis of several related processes waiting in the queues, which is a common phenomenon is enabled by this theory. **For example,** people waiting in a queue on a milk booth to get milk, waiting at the cash counter to deposit the electricity and water bill, to pay fees at the fee counter in the college etc. Obviously, waiting line crops up because the service may not be rendered immediately as a 'customer' (the one seeking service) reaches the service facility.

Thus, absence of required service facility causes the waiting line of customers. The service demand can be met with instantly if service capacity is high and hence, it meets the peak demand. It is very costly to create such a capacity level and it will be very uneconomical as well because the system will be free when there are no customers or some customers. Thus, one looks for a proper level of service which means that neither too low (when there are long queues) nor too high (which is a costly affair). The theory applied to this problem is known as the **waiting line or queuing theory.**

5.1.2. Queue/ Waiting Line

The word queue comes, *via.* French, from the Latin cauda, meaning "tail".

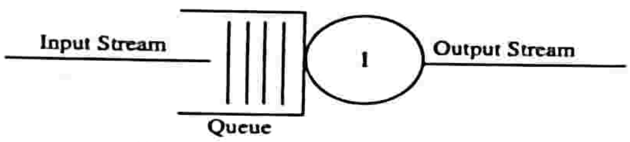


Figure 5.1: Simple Queuing System

A **queue** is a group of items which are waiting to receive service. When the demand for a service is more than the capacity of service facility, a queue is formed. Customers arrive at a higher rate than the service facility available in a queue.

In this situation, a queue starts building up and after some time queue will become very large so that the customers may leave the queue and new customers may join the queue. Due to this, business faces the loss and customer wastes his time staying in a queue which is known as **waiting time cost.** When the queues are not present then the service facility may remain idle (no work).

This waiting phenomenon is the direct result of randomness in the operation of service facility. Otherwise the operation of customer's arrival and the operations of service time are not known in advance. Hence, the main objective of the study of queuing theory is to maintain a balance between the waiting time cost and service cost so that these two could be minimised.

Some **examples** of queue systems are as mentioned below:
1) Waiting to cash a check in the bank.
2) Toll booths for a bridge or toll road.
3) Doctors available for clinic calls.
4) Repair persons servicing machines.
5) Landing strips for aircraft.
6) Docks for ships.
7) Service windows for a post office.

5.1.3. Operating Characteristics of Queuing System

An analysis of a given queuing system involves a study of its different operating characteristics. Some of the more commonly considered characteristics are discussed below:

1) **Queue Length:** Queue length means the average number of customers who are waiting in the queue to get service. If the queue is large, then it means that the performance of server is poor while the small queue indicates that the performance of server is good.
2) **System Length:** System length means the average number of customers existing in the system which includes those who are waiting to be served and those who are being served. Large number of system length indicates congestion and possible customer dissatisfaction and required to improve the service capacity.
3) **Waiting Time in the Queue:** The waiting time in a queue means the average time that a customer has spent in the queue to get service. Long waiting time indicates that the customers are dissatisfied and the loss of future revenue while small waiting time indicates very good service capacity.
4) **Total Time in the System:** Total time in the system means the average time that a customer spends in the system from entry in the queue to completion of service. Large total time indicates the requirement to make adjustment in the capacity.
5) **Server Idle Time:** Server idle time is the relative frequency with which the service system is idle. Idle time is directly related to cost. However, reducing idle time may have adverse effects on the other characteristics mentioned above.

5.1.4. Elements of Queuing System

The general structure of a queuing system is shown in figure 5.2:

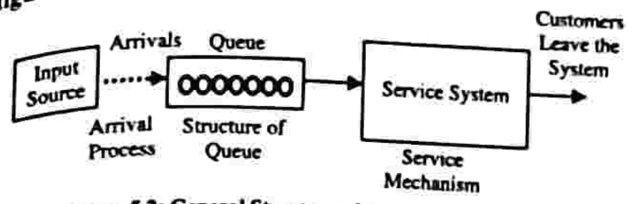


Figure 5.2: General Structure of Queuing System

Elements of Queuing System

Elements of the queuing system are also called the elements of the queuing theory. The following are the important elements:

1) **Calling Population (Input Source)/ Input Process / Arrival Process:** The source of the customers to the waiting line system is referred to as the calling population and it can be either:
   i) **Infinite Calling Population:** It assumes a large number of potential customers as it is always possible for one more customer to arrive to be served. **For example,** the department store, a grocery store, a bank and a service station are assumed to have infinite calling populations, i.e., the whole town or geographic area.
   ii) **Finite Calling Population:** It has a specific, countable number of potential customers. All the customers may be waiting in line at the same time, i.e., it may occur that there is not one more customer to be served. **For example,** a repair person in a shop who is responsible for a fixed number of machines to work on, a trucking terminal that services a fleet of ten trucks or a nurse assigned to attend to only twelve patients.

Calling Population Characteristics

i) **Size of the Calling Population:** The size of the calling population may be finite or infinite:
   a) **Infinite Population:** It can be said that nothing is infinite but practical example of infinite population could be on a national highway the arrival of vehicles at a toll bridge, patients arriving at the OPD of a hospital and people waiting to buy ticket at a cinema hall on the opening day of a popular film.
   b) **Finite Population:** On the other hand, finite population can be four printing machines in a printing press which requires to be serviced by a mechanic periodically, two coffee machines available at a coffee shop and four trucks with a transport company. The probability of an arrival is significantly changed when one member of the calling population is receiving service, the calling population is finite.

ii) **Arrival Characteristics of the Calling Population:** Arrival of the members of a calling population for service is in random fashion. They may also arrive in some organized pattern.

iii) **Customer Behaviour in Queue:** Customers generally behave in the following ways:
   a) **Balking:** Due to the long queue length, lack of time or due to unavailability of sufficient space, a customer may not like to wait in the queue.
   b) **Reneging:** Some customers wait for some time but leave the queue due to impatience without getting the service.
   c) **Collusion:** Some of the customers join together but one of them may stay in the queue. **For example,** at the park ticket window one person may join the queue and purchase ticket for his friends.
   d) **Jockeying:** If there is more than one queue available then one customer may shift from one queue to another queue to improve their position. This occurs generally in the super market.

iv) **Pattern of Arrivals at the System (Arrival Time Distribution):** The time interval between the arriving customers is referred as Arrival Time Distribution, which is usually a random variable. It is represented by λ. Hence, the inter-arrival time (i.e., the time interval between the consecutive arrivals) has an exponential distribution with mean $\frac{1}{\lambda}$ (by property of Poisson process).

2) **Queuing Process:** The number of queues and their respective lengths are represented by the queuing process. The layout of a service system is the basis of number of queues i.e., single, multiple or priority queues. The operational situations such as physical space, legal restrictions and attitude of the customers are the basis of the length (or size) of a queue. In various cases, a service system is not able to accommodate new customers after the accommodation of a required number of customers. No extra customers are allowed to enter in the system until more space is available to the accommodation of new customers. This situation is referred to as **finite queue source.** The situation refers to **infinite queue source** if a system is able to accommodate any number of customers at a time.

3) **Queuing Discipline/Queue Structure/Queuing Rule:** Queue structure is also considered after considering the arrival process and service system. Queue discipline is the important factor which should be known. The order in which the customers are entering the system and are serviced is known as the queue discipline. The queue discipline specifies the manner in which the customers form the queue or equivalently the manner in which they are selected for service, when a queue has been formed. Some of these are:
   i) **Static Queue Disciplines/Rule:** The individual customer's status in the queue is the basis of static queue disciplines. Few of such disciplines are:
      a) **First-Come, First-Served (FCFS):** Under this system, the customers are served in the order in which they have arrived in the queue. In this system the head of the queue is

always served next. **For example,** at any airport the queue of prepaid taxi is an example of this system, where taxi is engaged on a first-come, first-served basis.

b) **Last-Come-First-Served (LCFS):** Under this system, the customers are served in the reverse order of their arrival This means that the customer who joins at the last is served first. **For example,** in government offices, where the file which comes on table last gets cleared first.

ii) **Dynamic Queue Disciplines/Rule:** The individual customer attributes in the queue are the basis of dynamic queue disciplines. Few of such disciplines are:

a) **Service in Random Order (SIRO):** In this rule, the customers are selected for service at random without considering their arrivals in the service system. In this, every customer has the equal chance to be selected. The arrival time of the customers have no significance in such a case. **For example,** when the callers to a mobile operator customer care centre are put on hold then the luck of the draw often determines the next caller serviced by an operator.

b) **Priority Service:** In this rule, customers are classified in various classes on the basis of some attributes. These attributes may be service time, urgency or according to some identifiable characteristics. Each class is served on the basis of FCFS. **For example,** treatment of VIPs in preference to other patients in a hospital is an example of priority service.

c) **Pre-emptive Priority (Or Emergency):** In various situations priority discipline requires to be introduced to allow for realistic queues with high priority arrivals. The server may stop the service of lower priority customer in case of serving higher priority customer and this is called pre-emptive priority. **For example,** a gantry crane working on a container ship may stop the unloading halfway and shift to another load to unload perishable goods of a later arrived ship.

d) **Non- Pre-emptive Priority:** A situation where a service of a low priority customer started before the arrival of a high priority customer is completed and high priority customer only receives a good position in the queue is called non-pre-emptive priority. **For example,** in case of an emergency landing the ground control gives the permission on the condition that it lands next to the airplane being on runway at the moment.

4) **Service Process/Queues in Series and Parallel:** Two aspects are studied such as structure of the service mechanism and speed of the service with respect to the service system:

i) **Arrangement of Service Facilities (Structure of the Service System):** How the service facilities exist is represented by the structure of the service system. There are several possibilities which are as follows:

a) **A Single Service Facility (Series):** In this service facility, there is only one service station to provide the service. This is defined as a single channel service system. A single server and a single line of customers are involved in a single server model. **Figure 5.3** Illustrates such a model.

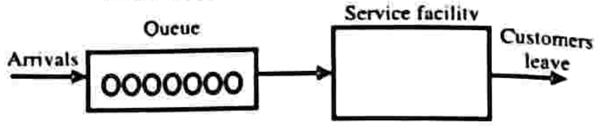


Figure 5.3: Single Server, Single Queue Model

b) **Multiple Parallel Facilities with Single Queue:** In this service facility, there is more than one server. The term parallel means that the each server provides the same type of services. Booking at a service station with the use of several machines and each machine handles one vehicle. It is shown in **figure 5.4.**

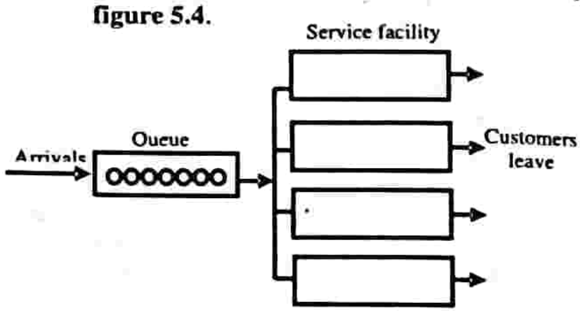


Figure 5.4: Multiple, Parallel Servers, Single Queue Model

c) **Multiple Parallel Facilities with Multiple Queues:** In this service facility, there is more than one server and more than **one** queue, i.e., each server has its own queue. **For example,** different counters for the verification of documents in the admission process of any university. It is shown in **figure 5.5.**

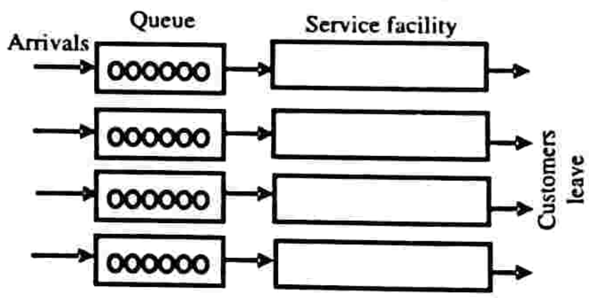


Figure 5.5: Multiple, Parallel Server, Multiple Queue Model

d) **Service Facilities in Series:** In this service facility, a customer enters the first station and gets a portion of service and then moves on the next station.... and so on and in the last after receiving the complete service he leaves the system. **For example,** in the admission process of any university, various functions are to be performed where each function is performed by a single server in a series. **Figure 5.6** shows such a situation.

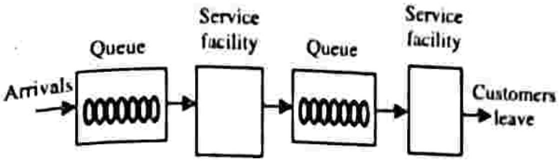


Figure 5.6: Multiple Servers In Series

ii) **Speed of Service:** In a queuing system, the service speed can be expressed with the help of two different ways which are as follows:

a) **Service Rate:** The number of customers served during a particular time period is described by the service rate.

b) **Service Time:** The amount of time needed to service a customer is described by service time.

Service Time Distribution

The number of customers served per unit of time or the inter-service time (i.e., the time required to serve a customer) may be constant or randomly distributed. It is represented by μ. The inter-service time (or simply the service time) follows an exponential distribution with mean $1/\mu$.

5) **Output of Queue/Departure:** The completion of service is referred by the departure which means exit of the unit from the system. Generally, this factor can be ignored but in various situations this may affect the service and/or arrival times. **For example,** if there is only one door to service point, through which people enter and leave after being served, it is possible that people leaving could affect the rate of arrival.

In a single channel facility, the output of the queue does not create any problem for the customer departures after receiving the service but the output of the queue becomes important when the system is of multistage channel facilities because the possibility of a service station breakdown can have repercussions on the queues. The line before the breakdown will be increase in length and the line following the breakdown will be diminished.

5.1.5. Advantages of Queuing Theory

1) Queuing theory is used to study the behaviour of waiting lines through the mathematical techniques with the use of stochastic process.
2) Queuing theory provides the models which are used to determine arrival pattern of customers or most appropriate number of service stations.
3) For maintaining the balance between the two opportunity costs for optimisation of waiting costs and service costs queuing models are used.
4) For developing the adequate service with tolerance, waiting/queuing theory is helpful for understanding of waiting lines.

5.1.6. Disadvantages of Queuing Theory

1) Mathematical distributions, which one assumes while solving queuing theory problems, are only a close approximation of the behaviour of customers, time between their arrival and service time required by each customer.
2) Most of the real life queuing problems are complex situations and it is very difficult to use the queuing theory technique, even then uncertainty will remain.
3) Many situations in industry and service are multi-channel queuing problems. When a customer has been attended to and the service has been provided, he/she may still have to get some other service from another server and may have to fall in queue once again. Here the departure of one channel queue becomes the arrival of the other channel queue. In such situations, the problem becomes more difficult to analyse.
4) Queuing model may not be the ideal method to solve certain very difficult and complex problems and one may have to resort to other techniques like Monte-Carlo simulation method.

5.1.7. Symbols and Notations

The notations used in the analysis of a queuing system are as follows:

n = Number of customers in the system (waiting and in service)

P_n = Probability of n customers in the system

λ = Mean customer arrival rate or average number of arrivals in the queuing system per unit time

μ = Mean service rate or average number of customers completing service per unit time

$\frac{\lambda}{\mu} = \rho$ = Traffic intensity or server utilisation factor (the expected fraction of time in which server is busy).

s = Number of service channels (service facilities)

N = Maximum number of customers allowed in the system

L_s = Mean number of customers in the system (waiting and in service)

L_q = Mean number of customers in the queue (queue length)

L_b = Mean length of non-empty queue

W_s = Mean waiting time in the system (waiting and in service)

W_q = Mean waiting time in the queue

W_b = Mean waiting time of an arrival who has to wait

5.1.8. Terminology of Queuing Model

The various terminologies used in queuing model are as follows:

1) **Traffic Intensity or Utilisation Rate:** This is denoted by following formula:

$$\text{Utilisation rate } (\rho) = \frac{\text{Mean Arrival Rate}}{\text{Mean Service Rate}} = \frac{\lambda}{\mu}$$

This is the probability to have a queue on arriving.

If $\rho > 1$, the traffic intensity is very high and consequently the waiting time will be more.

2) **Idle Rate:** Idle rate is given by following formula:

$$P_0 = 1-\rho = 1-\frac{\lambda}{\mu}$$

Expected Idle Time

$$\text{(for service facility)} = \left(1-\frac{\lambda}{\mu}\right) \times \text{Total Service Time}$$

3) **Expected(Average) Number of Customers in the System, Both Waiting and in Service**

$$L_s = \frac{\lambda}{\mu-\lambda} = \frac{\rho}{1-\rho}$$

4) **Expected(Average) Number of Customers in Queue or Waiting Line**

$$L_q = L_s - \rho = \frac{\lambda}{\mu-\lambda} - \frac{\lambda}{\mu} = \frac{\lambda^2}{\mu(\mu-\lambda)}$$

5) **Expected(Average) Waiting Time in the Queue**

$$W_q = W_s - \frac{1}{\mu} = \frac{1}{\mu-\lambda} - \frac{1}{\mu} = \frac{\lambda}{\mu(\mu-\lambda)}$$

6) **Expected(Average) Time Spent by the Customers in System**

W_s = Expected Time Spent while Waiting for Service + the Service Time

$$= W_q + \frac{1}{\mu} = \frac{1}{\mu-\lambda}$$

7) **Probability of n Customer in the System**

$$P_n = \rho^n (1-\rho)$$

5.2. QUEUING MODELS

5.2.1. Introduction

In queuing theory, a real queuing situation or system is estimated with the use of queuing model to analyse the queuing behaviour mathematically.

Queuing model allows a number of useful steady state performance measures to be determined, including:

1) Average number in the queue or the system,
2) Average time spent in the queue or the system,
3) Statistical distribution of those numbers or time,
4) Probability that the queue is full or empty, and
5) Probability of finding the system in a particular state.

These performance measures are important, as issues or problems caused by queuing situations are often related to customer dissatisfaction with service or may be the root cause of economic losses in a business. Analysis of the relevant queuing models allows the cause of queuing issues to be identified and the impact of proposed changes to be assessed.

5.2.2. Assumptions of Queuing Model

1) The customers arrive according to Poisson distribution with mean arrival rate λ for service at a single service facility at random.
2) The service time has exponential distribution with mean service rate μ.
3) First come, first served (FCFS) is followed by the service discipline.
4) It is assumed that the customer behaviour is normal which means customers join the queue for service, wait for their turn and leave only after getting served; they do not resort to balking, reneging or jockeying.
5) Service facility behaviour is normal. It serves the customers continuously, without break, as long as there is queue. Also, it serves only one customer at a time.
6) The available waiting space is infinite for customers in the queue.
7) The calling source (population) has infinite size.
8) The elapsed time since the start of the queue is sufficiently long so that the system has attained a steady state or stable state.
9) The mean service rate μ is greater than the mean arrival rate λ.

5.2.3. Kendall's Notation for Representing Queuing Models

Figure 5.7 depicts the specialised Poisson queuing situation with c parallel servers.

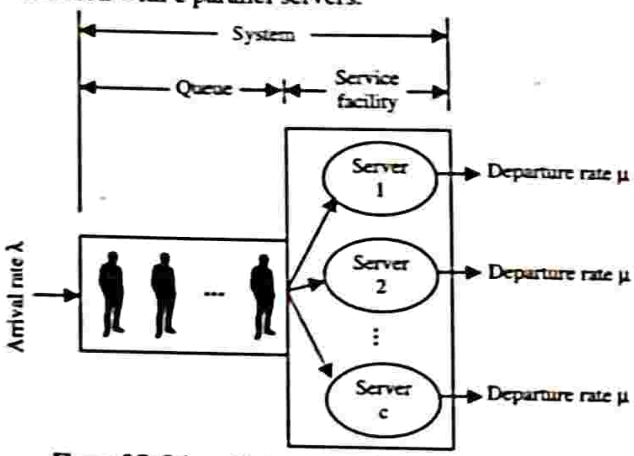


Figure 5.7: Schematic Representation of a Queuing System with c Parallel Servers

A convenient notation for summarising the characteristics of the queuing situation in **figure 5.7** is given by the following format:

(a/b/c): (d/e/f)

Where,

a = Arrivals distribution
b = Departures (service time) distribution
c = Number of parallel servers (= 1, 2, ..., ∞)
d = Queue discipline
e = Maximum number (finite or infinite) allowed in the system (in-queue plus in-service)
f = Size of the calling source (finite or infinite)

The standard notation for representing the arrivals and departures distributions (symbols **a** and **b**) is:

1) **M** = Markovian (or Poisson) arrivals or departures distribution (or equivalently exponential inter-arrival or service time distribution)
2) **D** = Constant (deterministic) time

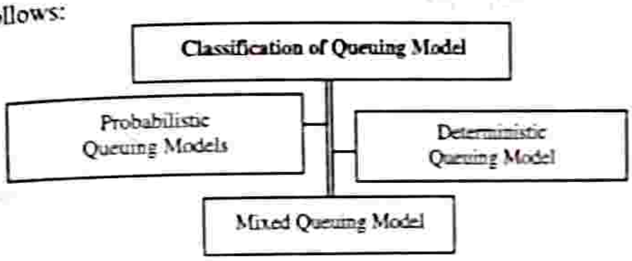
3) E_k = Erlang or gamma distribution of time (or, equivalently, the sum of independent exponential distributions)
4) GI = General (generic) distribution of inter-arrival time
5) G = General (generic) distribution of service time

The queue discipline notation (symbol **d**) includes:
1) FCFS = First Come, First Served
2) LCFS = Last Come, First Served
3) SIRO = Service in Random Order
4) GD = General Discipline (i.e., any type of discipline)

The first three elements of the notation (a/b/c) were devised by **D.G. Kendall** in 1953. In 1966, **A.M. Lee** added the symbols **d** and **e** to the notation. This author added the last element, symbol **f**, in 1968. **For example,** the model (M/D/10): (GD/20/∞) uses Poisson arrivals (or exponential inter-arrival time), constant service time and 10 parallel servers. The queue discipline is GD and there is a limit of 20 customers on the entire system. The size of the source from which customers arrive is infinite.

5.2.4. Classification of Queuing Model

The various types of queuing models can be classified as follows:



1) **Probabilistic Queuing Models:** If the service time is known with uncertainty or determined on the basis of probability and each customer at unknown intervals then the queuing model shall be probabilistic in nature. There are the majority of the queuing models which are based on the assumption that one or more elements of the queuing system can be expressed only in probabilistic terms. Hence, nearly all of the queuing models are of probabilistic type.
   i) **Model I (Erlang Model):** This model is symbolically represented by **(M/M/1):(FCFS/∞/∞).**
      M = Markovian (Poisson) arrival distribution.
      M = Markovian (exponential) service distribution.
      1 = Single channel
      FCFS = Service discipline (First come first served)
      Since, the Poisson and exponential distributions are related to each other, both of them are denoted by the symbol 'M' due to Markovian property of exponential distribution.
   ii) **Model II (General Erlang Model):** This model is also represented by (M/M/1): **(FCFS/∞/∞),** it is a general queuing model in which the arrival and service rates are based on the length of the queue. Some people who desire service but may not join the queue due to the long queue length and hence arrival rate is affected. Thus, length of the queue affects the service rate.
   iii) **Model III:** This model is represented by **(M/M/1): (SIRO/∞/∞).** This model is same as the model I except that the service discipline followed in this model is SIRO instead of FCFS.
   iv) **Model IV:** This model is represented by **(M/M/1): (FCFS/N).** N represents the capacity of the system which is limited or finite. Hence, the number of arrivals cannot exceed N.
   v) **Model V:** This model is represented by **(M/M/1): (FCFS/n/M).** It is a finite-population or limited source model. In this model, the number of potential customers available to enter the system is the basis of the probability of an arrival.
   vi) **Model VI:** This model is represented by **(M/M/c): (FCFS/∞/∞).** This is same as model I but there are c service channels working in parallel.
   vii) **Model VII:** This model is represented by **(M/E_k/1): (FCFS/∞/∞).** In this model the Erlang service time with k phases is used instead of exponential service time.
   viii) **Model VIII:** This model is represented by **(M/M/1): (GD/m/n),** where m ≤ n. Machine repair problem with a single repairman is represented by this model. There are total n machines out of which m are broken down and create a queue. GD represents a general service discipline.
   ix) **Model IX:** This model is represented by **(M/M/c): (GD/m/n),** m ≤ n. It is same as model VIII except that there are c repairmen, c < n.
   x) **Model X:** This is called power supply model.
2) **Deterministic Queuing Model:** The queuing model would be deterministic in nature if the arrival of each customer is performed at known intervals and the service time is known with certainty. This contains the following model:

   Model XI: This model is represented by **(D/D/1): (FCFS/∞/∞).** In this model, inter-arrival time and service time are known with certainty. Therefore, this model is called **deterministic model.**
3) **Mixed Queuing Model:** This contain the following model:
   Model XII: This model is represented by **(M/D/1): FCFS/∞/∞).** Here, arrival rate is Poisson distributed while the service rate is deterministic or constant.

5.2.5. Applications of Queuing Model

The queuing theory is applied for better service to the customer in the following areas as follows:

1) **Fast Food:** Queuing model has a wider scope in case of fast food restaurant. As, in the queuing system, there are a number of customers expected to arrive as well as the amount of time customers usually spend in the restaurant together with the number of parties expected to stay at their table before finish eating. The basic need of using queuing theory for fast food restaurant analysis is to mainly recover that time for

which customers have to wait for their orders. Such need can be fulfilled by increasing more servers at certain times of the day, such as on busier days or on any special events taking place in the area. If it is not possible to speed up service, you can also work to improve the waiting experience.

2) **Retail Stores and Banks:** As like fast food restaurants, retail stores and banks are also trying to improve as well as increase the satisfaction of customers by serving them in a better way by increasing the speed of the checkout and cashier lines. Same condition applies to retail stores where a lot of customers are waiting for their services to be fulfilled.
3) **Medical:** The healthcare system seems to be a complex queuing network in which delays can be reduced through:
   i) Management of work among service stages (e.g., coordination of tests, treatments, discharge processes).
   ii) Proper scheduling of resources (e.g., doctors and nurses) according to the arrival pattern.
   iii) Continuous monitoring of the system (e.g., tracking number of patients waiting by location, diagnostic grouping and acuity) for taking immediate actions.
4) **Airports:** Proper scheduling of landing and take-off of planes from airport with heavy duty of air traffic and limited facilities can be done with the help of queuing system. **For example,** check-in, baggage collection, runway delays etc.
5) **Traffic:** Proper scheduling of a limited transport fleet for a large number of traffic users.

5.3. MODEL I: SINGLE CHANNEL QUEUING MODEL (M/M/1)

5.3.1. Introduction

The simplest queuing system and the one that is encountered most often is a single channel single server system. The arrival rate, that is number of arrivals per unit time, follows a Poisson distribution and the service time follows an exponential distribution. The calling population is large enough to be considered infinite and the length of the queue is also unlimited or infinite.

The customers are served on the basis of FCFS (First-Come- First-Served). Such queues are referred to as M/M/1 queues. The description is an abbreviated form of Kendall-Lee's notations for queuing system and the meaning is as follows:

1) The arrivals follow a Poisson distribution which is described by the first M.
2) The service time is exponentially distributed which is described by the second M.
3) The 1 in the notation implies single channel single server queue system.

The complete notation describing the model is M/M/1: FCFS/∞/∞. Queue discipline is first come first served, the length in service, i.e., callers waiting and those being provided service are infinite and the calling population is infinite and this is described by the last three symbols. The second part of the notation is generally omitted. **For example,** the doctor acts as a single server at his clinic.

5.3.2. Assumptions of Single Channel Queuing Model

1) Customers are served on the basis of First Come First Serve (FCFS).
2) Balking or Reneging does not exist in this system. All the customers wait until they are served, no one jumps out of the queue and no one leaves the queue.
3) Arrival rate is constant and does not change with time.
4) Arrival of new customers is independent of the earlier arrivals.
5) Arrivals follow Poisson's distribution and infinite population is finite.
6) Rate of serving is known.
7) All customers have different service time requirements and are independent of each other.
8) Service time can be described by negative exponential probability distribution.
9) Average service rate is higher than the average arrival rate and over a period of time the queue keeps reducing.

5.3.3. Single Channel and Infinite Population (M/M/1): (∞/FIFO)

In this model a queuing system has a single service channel with Poisson input and exponential service. There is no limit on the system capacity and the customers are served on the basis of "First Come, First Serve".

The solution procedure of this queuing model can be summarised in the following three steps:

Step 1: Construction of differential equation.

Step 2: Deriving the steady state differential equation.

Step 3: Solution of steady state difference equation.

The probability distribution of this model is given by,

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right) = \rho^n (1+\rho)$$

Where, $\lambda/\mu = \rho$ is called the average utilisation or the **traffic intensity.**

5.3.3.1. Characteristics of M/M/1 Queuing System with Infinite Population

1) Probability of queue size being greater than or equal to n, the number of customers is given by,

   $P(\geq n) = \rho^n$

2) Average number of customers in the system(waiting + service) is given by,

   $$L_s = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu - \lambda}$$

3) Average queue length is given by,

$$L_q = L_s - \frac{\lambda}{\mu} = \frac{\lambda}{\mu-\lambda} - \frac{\lambda}{\mu} = \frac{\lambda^2}{\mu(\mu-\lambda)}$$

4) Average length of non-empty queue is given by,

$$L_n = \frac{\text{Average length of queue}}{\text{Probability of non-empty queue}} = \frac{\mu}{\mu-\lambda}$$

5) The fluctuation (variance) of queue length is given by,

$$V(n) = \frac{\rho}{(1-\rho)^2} = \frac{\lambda\mu}{(\mu-\lambda)^2}$$

6) Average waiting time of a customer (in the queue) is given by,

$$W_q = \frac{\rho}{\mu(1-\rho)} = \frac{\lambda}{\mu(\mu-\lambda)}$$

7) Average waiting time of customer in system is given by,

$$W_s = \frac{L_s}{\lambda} = \frac{1}{\mu-\lambda}$$

8) Average waiting time in non-empty queue (average waiting time of an arrival who waits) is given by,

$$W_n = \frac{1}{\mu-\lambda}$$

5.3.3.2. Relations between Average Queue Length & Average Waiting Time – Little's Formula

The following given are the important characteristics of M/M/1 queuing system:

$$L_s = \frac{\lambda}{\mu-\lambda}, \quad L_q = \frac{\lambda^2}{\mu(\mu-\lambda)},$$

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)}, \text{ and } W_s = \frac{1}{\mu-\lambda}$$

Using these expressions, we observe that

$$L_s = \lambda W_s$$

$$L_q = \lambda W_q$$

and $W_s = W_q + \dfrac{1}{\mu}$

i.e.,

[Expected waiting time in system = Expected waiting time in queue + Expected service time]

Example 1: Semi-finished components arrive at a workstation of an assembly line at an average rate of 2 per minute, Poisson distributed. A machine is to be installed at this workstation for the specific operation. Three alternative machine P, Q and R are available. The characteristics of the machines are given below. Whenever a component is idle awaiting the machine to get free, the cost is estimated at ₹18 per minute. Using the concept of single-channel queue system and considering all relevant costs, recommend the machine that would be the best for this work station.

Machine	P	Q	R
Fixed Costs (₹/min.)	36	60	90
Variable Costs (₹/min.)	18	15	8
Processing Rate (Unit/min.)	3	6	12

Solution: Given that, $\lambda = 2$ per minute.

For Machine P

$\mu = 3$ per minute.

Average number of components awaiting the machine to get free,

$$L_q = \frac{\lambda}{\mu} \cdot \frac{\lambda}{\mu-\lambda} = \frac{2}{3} \cdot \frac{2}{3-2} = \frac{4}{3}$$

$\therefore$ Cost per minute = ₹ $\left(\frac{4}{3} \times 18 + 36 + 18\right)$ = ₹ 78.

For Machine Q

$\mu = 6$/minute.

$$L_q = \frac{\lambda}{\mu} \cdot \frac{\lambda}{\mu-\lambda} = \frac{2}{6} \cdot \frac{2}{6-2} = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$$

$\therefore$ Cost per minute = ₹ $\left(\frac{1}{6} \times 18 + 60 + 15\right)$ = ₹ 78.

For Machine R

$\mu = 12$ per minute.

$$L_q = \frac{\lambda}{\mu} \cdot \frac{\lambda}{\mu-\lambda} = \frac{2}{12} \cdot \frac{2}{12-2} = \frac{1}{6} \cdot \frac{1}{5} = \frac{1}{30}$$

$\therefore$ Cost per minute = ₹ $\left(\frac{1}{30} \times 18 + 90 + 8\right)$ = ₹ 98.60

$\therefore$ Either machine P and Q may be installed.

Example 2: A bike mechanic spent time on his jobs of an average of 30 minutes following exponential distribution. He repairs bikes in the order First Come First Serve (FCFS) and the arrival of bikes is approximately Poisson with a mean rate of 10 bikes in 8 hours within a day. Calculate:
1) **Average idle time of mechanic in a day.**
2) **Average number of bikes in the system.**

Solution: From the above problem, we have the following information:

Arrival Rate (λ) = $\frac{10}{8}$ = 5 bikes per 4 hour, and

Service Rate (μ) = $\left(\frac{1}{30}\right) \times 60$ = 2 bikes per hour,

$$\therefore \quad \rho = \frac{\lambda}{\mu} = \frac{5}{4 \times 2} = 0.625$$

1) The probability of the mechanic to be idle is:

   $P_0 = 1 - \rho = 1 - 0.625 = 0.375$

   Hence, average idle time per day = 8 × 0.375 = 3 hours

2) Average number of bikes in the system

   $$L_s = \frac{\rho}{1-\rho} = \frac{0.625}{1-0.625} = 1.666 \approx 2 \text{(approx.) bikes}$$

Example 3: In a bank, customers arrive at the cash counter with an average rate of 20 per hour. They are handled by only a single cashier according to Poisson process. The average time served to customer is 100 seconds according to exponential distribution. Determine the expected (or mean) waiting time of a customer at cash counter.

Solution: From above problem, we have the following information:

Arrival Rate (λ) = 20 customers per hour

Service Rate (μ) = $\frac{60\times60}{100}=36$ customers per hour

The mean waiting time of a customer (in the queue) will be as follows:

$$W_q=\frac{\lambda}{\mu(\mu-\lambda)}=\frac{20}{36(36-20)}$$

$$=\frac{5}{36\times4}\text{ hours or }\frac{5\times3600}{36\times4}\text{, i.e., 125 seconds.}$$

The mean waiting time of a customer (in the system) will be as follows:

$$W_s=\frac{1}{(\mu-\lambda)}=\frac{1}{(36-20)}\text{ or }\frac{1}{16}\text{ hour, i.e., 225 seconds}$$

Example 4: The number of customers arriving in a ladies tailor's shop according to Poisson distribution with an average of 6 customers per hour. Tailor attends customers as they come in the shop and customers have to wait until their number does not come. The tailor provides services to its customers at a mean rate of 10 customers per hour and service time is exponentially distributed. Determine:

1) Probability of the number of arrivals (0 through 5) during:
   i) An interval of 15 minute.
   ii) An interval of 30 minute.
2) The utilization factor.
3) The probability that queuing system is idle.
4) Average time that tailor do not have any customer on a 10 hour working day, i.e., tailor is free.
5) The probability related with customers (0 through 5) in the queuing system.
6) Average number of customers in the shop.

Solution: From the above problem, we have:
Arrival Rate (λ) = 6 customers per hour
Service Rate (μ) = 10 customers per hour

Hence,

1) Probability of the number of arrivals (0 through 5) in two cases (a) when T= 15/60= 1/4 hour and (b) when T= 30/60=1/2 hour are given by the following **table 5.1:**

Table 5.1: Calculation of Probabilities of Arrivals

n	When T= ¼ hour P(n)	When T= ½ hour P(n)
0	0.2231	0.0498
1	0.3347	0.1494
2	0.2510	0.2240
3	0.1255	0.2240
4	0.0471	0.1680
5	0.0141	0.1008

These can be calculated using the following formula:

$$P(n)=e^{-m}\times\frac{m^n}{n!}$$

Where m =λT and e = 2.7183

With λ = 6, we have,
m = 6 × 1/4 =1.5 for first case
m = 6 ×1/2 =3 for second case.

2) Utilization factor $\rho=\frac{\lambda}{\mu}=\frac{6}{10}=0.6$

3) Probability that queuing system is idle:
$$P_0=1-\frac{\lambda}{\mu}=1-0.6=0.4$$

4) Average time in which tailor is free in a 10 hour working day = P_0 × Number of Hours = 0.4 × 10 = 4 hour.

5) Probability of n customers in the system is given by:
$$P_n=\rho^n(1-\rho)$$
Hence using the above formula, we get the following **table 5.2.**

Table 5.2: Calculation of Probabilities of Customers in System

n	Probabilities P(n)
0	0.40
1	0.24
2	0.144
3	0.0864
4	0.05184
5	0.031104

6) Average numbers of customers in the tailor's shop is given by:
$$L_s=\frac{\rho}{1-\rho}=\frac{0.6}{1-0.6}=1.5\approx2\text{ customers}$$

Example 5: A T.V. repairman finds that the time he spent on his job has an exponential distribution with mean 30 minutes. If he repairs the set in the order it arrives, and the arrival rate is approximately Poisson, with an average rate of 10 per 8 hour day, what is the expected idle time of repairman each day? How many jobs are ahead of average before the job just brought in?

Solution: Given,
The average arrival rate, λ = 10 sets/day.
The average service rate, μ = 16 sets/day.
∴ Utilisation factor $\rho=\lambda/\mu$ = 10/16 = 5/8

The probability that repairman is idle can be given by:
$$\rho(0)=1-\rho=1-\frac{5}{8}=\frac{3}{8}$$

∴ Average idle time per day = 8 × 3/8 = 3 hours

We have to find out the expected number of sets in the system in order to determine the expected number of sets ahead of the set just brought in. It is given below:

$$L_s=\frac{\rho}{1-\rho}=\frac{\frac{5}{8}}{1-\frac{5}{8}}=\frac{5}{3}=1\frac{2}{3}\text{ sets}$$

Example 6: Patients arrive at a clinic according to Poisson distribution at the rate of 20 patients per hour. Examination time per patient is exponential with mean rate of 30 per hour.

Find:
i) Find the traffic intensity.
ii) What is the probability that new arrival does not have to wait?
iii) What is the average waiting time of patient before he leaves the clinic?

Solution: Given that,

Arrival rate λ =20 patients/hour and

Service rate μ = 30 patients/hour.

i) Traffic intensity $(\rho) = \lambda/\mu = 20/30 = 0.66$
ii) Probability that new arrival does not have to wait:
P (No Patient Waiting to be Served) = P(0) + P(1)
$P(0) = 1 - \rho = 1 - 0.66 = 0.34$
$P(1) = P(0)\rho = 0.34 \times 0.66 = 0.2244$

Hence, P (No Patient Waiting to be Served) = 0.34 + 0.2244 = 0.5644

iii) Average waiting time of patient before he leaves the clinic,

$$W_s = \frac{L_s}{\lambda} = \frac{1}{\mu - \lambda} = \frac{1}{30-20} = 0.1$$

Example 7: In a bank on an average every 15 minutes a customer arrives for cashing the cheque. The staffs at the payment counter takes 10 minutes for serving a customer on an average.

Calculate:
i) Probability that system is busy.
ii) Average queue length.
iii) Average no. of customers in the bank.
iv) Average waiting time of customer in queue before service.

Solution: Given that,

Arrival Rate $(\lambda) = \frac{1}{15}$ per minute, and

Service Rate $(\mu) = \frac{1}{10}$ = per minute,

i) The probbility that system is busy = P(n > 0)

$$= \frac{\lambda}{\mu} = \frac{1}{15} \times \frac{10}{1} = \frac{2}{3} = 66.66\%$$

ii) Average queue length,

$$L_n = \frac{\mu}{\mu - \lambda} = \frac{1/10}{\left(\frac{1}{10} - \frac{1}{15}\right)} = \frac{1}{2} \times 6 = 3 \text{ customers}$$

iii) Average no. of customers in the bank,

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{1/15}{\left(\frac{1}{10} - \frac{1}{15}\right)} = \frac{1/15}{1/30} = 2 \text{ customers}$$

iv) Average waiting time of a customer in the queue,

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{1/15}{1/10\left(\frac{1}{10} - \frac{1}{15}\right)} = \frac{1/15}{\frac{1}{10} \times \frac{1}{30}}$$

$$= \frac{1}{15} \times \frac{300}{1} = 20 \text{ minutes}$$

Example 8: Arrivals at a telephone booth are considered to be Poisson, with an average time of 10 minutes between one arrival and the next. The length of a phone call is assumed to be distributed exponentially with mean of 3 minutes.
1) What is the probability that person arriving at the booth will have to wait?
2) Average time person spends on the system.

Solution: Given that,

$\lambda = \frac{1}{10}$ minute and $\mu = \frac{1}{3}$ minute

1) Probability that a person has to wait at the Booth

$$\rho = \text{Traffic Intensity} = \frac{\lambda}{\mu} = \left(\frac{1/10}{1/3}\right) = \frac{3}{10} = 0.3$$

2) Average waiting time person spends in the system is given by,

$$W_s = \frac{L_s}{\lambda} = \frac{1}{(\mu - \lambda)} = \frac{1}{\left(\frac{1}{3} - \frac{1}{10}\right)}$$

$$= \frac{30}{(10-3)} = \frac{30}{7} = 4.28 \text{ minutes}$$

Example 9: The tool room (company's quality control department) is operated by a single clerk who takes an average of 5 minutes in checking parts of each of machine coming for inspection. The machine arrives once in every 8 minutes on the average. One hour of the machine is valued at ₹ 15 and a clerk's time are valued at ₹ 4 per hour. What are the average hourly queuing system costs associated with the quality control department?

Solution: Given that,

Arrival Rate $= \lambda = \frac{60}{8} = 7.5$ per hour

Service Rate $= \mu = \frac{60}{5} = 12$ per hour

Thus, the average number of machines in system,

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{7.5}{12 - 7.5} = \frac{7.5}{4.5}$$

Cost per machine hour is ₹15.

Average hourly queuing system cost $= \frac{7.5}{4.5} \times 15 = 25$

Average hourly cost for employee = ₹ 4

Total cost associated with department purpose = 25 + 4 = 29

5.3.4. Single Channel and Finite Population (M/M/1): (FIFO/ N/M)

This model is also based on the same assumption as the previous model but population, from the where customers arrive is assumed to be finite, is not included.

For example, there may be only six mechanics coming to a tool room or the number of machines to be repaired and maintained by the maintenance staff might be fixed. In such cases, the mechanics and the machines are viewed as the customers who require service.

Characteristics of M/M/1 Queuing System with Finite Population

Let the finite number of customers in the source is denoted by M and customers individual arrival rate (not the all customers) is denoted by λ. If service rate is denoted by μ then the operating characteristics of the system can be stated as follows:

1) Probability that the system shall be idle:

$$P_0 = \frac{1}{\sum_{i=0}^{M}\left[\frac{M!}{(M-i)!}\left(\frac{\lambda}{\mu}\right)^i\right]^{-1}}$$

2) Probability that there shall be n customers in the system:

$$P_n = P_0\left(\frac{\lambda}{\mu}\right)^n \frac{M!}{(M-n)!} \qquad 0 < n \le M$$

3) Expected length of the queue:

$$L_q = M - \frac{\lambda+\mu}{\lambda}(1-P_0)$$

4) Expected number of customers in the system:

$$L_s = L_q + [1-P_0]$$

5) Expected waiting time of a customer in the queue:

$$W_q = \frac{L_q}{\mu(1-P_0)}$$

6) Expected time a customer spends in the system:

$$W_s = W_q + \frac{1}{\mu}$$

Example 10: In a bank, 20 customers on the average, are served by a cashier in an hour. If the service time has exponential distribution, what is the probability that it will take more than 10 minutes to serves a customer?

Solution: Here,

μ = 20 customers/ hour,

t = 10 minutes = 1/6 hour

∴ Probability that it will take more than 10 minutes to serve a customer $= e^{-\mu t} = e^{-20\times 1/6} = e^{-10/3} = 0.0357$.

Example 11: A manufacturing plant has five machines. The average of failure of two machines is 5 weeks. Assume that only one machine can be repaired in a week and repairing time of machines is exponentially distributed. Find:

1) The probability in which service will be idle,
2) The probability that there shall be exactly three machines in the system,
3) The average length of the queue,
4) The average number of machines waiting in the system,
5) The average time a machine is waiting in the queue to be repaired, and
6) The average time that a machine spends in the system.

Solution: According to given question, we have the following information:

Total number of Machines (M) = 5 machines,

$\lambda = \frac{2}{5} = 0.4$ machine per week,

$\mu = 1$ machine per week,

Hence $\rho = \frac{\lambda}{\mu} = \frac{0.4}{1} = 0.4$

1) Let P(0) is the probability of service to be idle. The calculation of values used in P(0) is shown in table below:

(i)	$\frac{M!}{(M-i)!}$	$(\lambda/\mu)^i$	$\frac{M!}{(M-i)!}\left(\frac{\lambda}{\mu}\right)^i$
0	1	1	1.0000
1	5	0.4	2.0000
2	20	0.16	3.2000
3	60	0.064	3.8400
4	120	0.02556	3.0720
5	120	0.01024	1.2288
		Total	**14.3408**

We can calculate P(0) as follows:

$$P(0) = \frac{1}{\left[\frac{M!}{(M-i)!}\left(\frac{\lambda}{\mu}\right)^i\right]} = \frac{1}{14.3408} = 0.0697$$

Hence, the probability in which service will be idle is given by:

$P(0) = 0.0697$

2) The probability that there shall be exactly 3 machines in the system is given by:

$$P_3 = P(0)\left(\frac{\lambda}{\mu}\right)^3 \frac{M!}{(M-3)!}$$

$$= 0.0697 \times \left(\frac{0.4}{1}\right)^3 \times \frac{5!}{(5-3)!}$$

$$= 0.0697 \times 0.064 \times 60 = 0.27$$

3) The average length of the queue,

$$L_q = M - \frac{\lambda+\mu}{\lambda}(1-P(0))$$

$$= 5 - \frac{0.4+1}{0.4}(1-0.0697)$$

$$= 5 - \frac{1.4}{0.4}(0.9303) = 1.74 \text{ machines}$$

4) The average number of machines waiting in the system,

$L_s = L_q + (1 - P(0)) = 1.74 + (1 - 0.0697) = 1.74 + 0.93$
$= 2.67 \approx 3$ machines

5) The average time a machine is waiting in the queue to be repaired,

$$W_q = \frac{L_q}{\mu(1-P(0))} = \frac{1.74395}{1(1-0.0697)}$$
$= 1.8746 \approx 1.87$ weeks

7) The average time that a machine spends in the system,

$$W_s = W_q + \frac{1}{\mu} = 1.8746 + \frac{1}{1} \approx 2.87 \text{ weeks.}$$

Example 12: Assume that a bus station handles only one bus at a time. The bus yard is only sufficient for two buses and others buses have to wait until these buses has not left from station. The average arrival rate of buses at the station is 6 per hour while the station handles 12 buses per hour. Suppose system is following Poisson arrivals and exponential service distribution. Determine the steady-state probabilities for several numbers of buses in the system. Also calculate the expected (average) waiting time of a new bus which is coming into the yard.

Solution: Here, we have the following information:

$\lambda = 6$ buses per hour and

$\mu = 12$ buses per hour

Hence $\rho = 6/12 = 1/2 = 0.5$

The maximum queue length is 2, i.e., the maximum number of buses in the system is 3(= n).

1) The probability that there is no bus in the system is given by the following formula:

$$P_0 = \frac{1-\rho}{1-\rho^{n+1}} = \frac{1-0.5}{1-(0.5)^{3+1}} = 0.53.$$

As $P_n = P_0 \rho^n$, hence we have,

$P_1 = (0.53)(0.5) = 0.27$

$P_2 = (0.53)(0.5)^2 = 0.13$ and

$P_3 = (0.53)(0.5)^3 = 0.07.$

2) Average number of buses in the system is given by:

$L_s = 1(0.27) + 2(0.13) + 3(0.07) = 0.74$

Thus, every bus takes an expected $\frac{1}{12}(=0.08)$ hour for getting service. As arrival of new bus assumes to find an average of 0.74 buses in the system before it, hence the expected waiting time of a new bus is given by:

$W_s = (0.74)(0.08) = 0.0592$ hours or 3.5 minutes.

Example 13: According to the previous records of a factory, consisting five machines, it is shown that breakdown occurs between 2 days on average and at random. Assume that a mechanic repairs only one machine in a day and it is distributed exponentially. Find:

1) The probability that the service will be idle.
2) Probability that different machines to be and being repaired.
3) The average queue length.
4) The average number of machines waiting to be, and being repaired.
5) The average time of a machine this is waiting in the queue for repairing.
6) The average time of a machine that is in the system.

Solution: Here, we have following information:

M= 5 machines,

$\lambda = 1/2$ machine per day,

$\mu = 1$ machine per day,

$\therefore$ Utilisation factor $\rho = \lambda/\mu = ½ = 0.5$

1) Let P_0 is the probability of service to be idle. The calculation of values used in P_0 is shown in **table 5.3**:

Table 5.3: Calculation of P_0

I	$M!/(M-I)!$	$(\lambda/\mu)^i$	$\frac{M!}{(M-i)}\left(\frac{\lambda}{\mu}\right)^i$
0	1	1.00000	1.00
1	5	0.50000	2.50
2	20	0.25000	5.00
3	60	0.12500	7.50
4	120	0.06250	7.50
5	120	0.03125	3.75
		Total	**27.25**

Now, we can calculate P_0 using the following formula:

$$P_0 = \left[\sum_{i=0}^{M}\left(\frac{M!}{(M-i)!}\left(\frac{\lambda}{\mu}\right)^i\right)\right]^{-1} = \frac{1}{27.25} = 0.0367$$

Hence, the probability that service will be idle is given by:

$P_0 = 0.0367$

2) Probability of different machines(n) of system can be obtained using the following formula:

$$P_n = P_0\left(\frac{\lambda}{\mu}\right)^n\left(\frac{M!}{(M-n)!}\right)$$

The probabilities of machines are shown in **table 5.4**:

Table 5.4: Probabilities of Various Number of Machines on the System

n	$\left(\frac{\lambda}{\mu}\right)^n\left(\frac{M!}{(M-n)!}\right)$	Probability
0	1.00	0.036700
1	2.50	0.091750
2	5.00	0.183500
3	7.50	0.275250
4	7.50	0.275250
5	3.75	0.137620

Note: The sum of probabilities should be equal to 1. But here it is not equal to 1 because of approximation error.

3) Average length of the queue is given by:

$$L_q = M - \frac{\lambda+\mu}{\lambda}(1-P_0) = 5 - \frac{0.5+1}{0.5}(1-0.0367)$$
$= 5 - 2.89 = 2.11 \approx 2$ machines

4) The average number of machines waiting to be, and being repaired is given by:

$$L_s = M - \frac{\mu}{\lambda}(1-P_0) = 5 - \frac{1}{0.5}(1-0.0367)$$
$$= 5 - 1.93 = 3.07 \approx 3 \text{ machines}$$

5) The average time of a machine waiting in the queue for repairing is given by:

$$W_q = \frac{1}{\mu}\left[\frac{M}{1-P_0} - \frac{\lambda+\mu}{\lambda}\right] = \frac{1}{1}\left[\frac{5}{1-0.0367} - \frac{0.5+1}{0.5}\right]$$
$$= \frac{5}{0.9633} - 3 = 5.1905 - 3 = 2.19 \text{ days}$$

7) The average time of a machine that will be in the system, is given by:

$$W_s = W_q + \frac{1}{\mu} = 2.19 + \frac{1}{1} = 3.19 \text{ days}$$

Example 14: A maintenance service facility has Poisson arrival rates, negative exponential service times and operates on first time served queue discipline. Break down occurs on an average of three per day with a range of zero to eight. The maintenance crew can service, on an average, six machines per day, with a range from zero to seven find the:

i) Utilisation factor of the service facility
ii) Mean waiting time in the system
iii) Mean number machine in the system
iv) Mean waiting time of machine in the queue

Solution: Arrival rate (λ) = 3 per day

Service Rate (μ) = 6 per day

i) Utilization factor of the service facility

$$(\rho) = \frac{\lambda}{\mu} = \frac{3}{6} = 0.5$$

ii) Mean waiting time in the system

$$(W_s) = \frac{1}{(\mu-\lambda)} = \frac{1}{(6-3)} = \frac{1}{3}\text{ day}$$

iii) Mean number of machine in the system

$$(L_s) = \frac{\lambda}{\mu-\lambda} = \frac{3}{(6-3)} = \frac{3}{3} = 1 \text{ machine}$$

iv) Mean waiting time of machine in the queue

$$(W_q) = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{3}{6(6-3)} = \frac{1}{6}\text{ day}$$

Example 15: Consider a single server queuing system with Poisson input and exponential service times. Suppose the mean arrival rate is 3 calling units per hour, the expected service time is 0.25 hours and the maximum permissible calling units in the system is two. Derive the steady state probability distribution of the number of calling units in the systems, and then calculate the expected number in the system.

Solution: Given that:

$\lambda = 3$ units per hour

$\mu = 4$ units per hour, and

$N = 2$

Then traffic intensity, $\rho = \lambda/\mu = 3/4 = 0.75$

The steady-state probability distribution of the number of n customers (calling units) in the system is:

$$P_n = \frac{(1-\rho)\rho^n}{1-\rho^{N+1}} = \frac{(1-0.75)(0.75)^n}{1-(0.75)^{2+1}} = (0.43)(0.75)^n;\ \rho \neq 1$$

and $P_0 = \frac{(1-\rho)}{1-\rho^{N+1}} = \frac{1-0.75}{1-(0.75)^{2+1}} = \frac{0.25}{1-(0.75)^3} = 0.431$

The expected number of calling units in the system is given by:

$$L_s = \sum_{n=1}^{N} nP_n = \sum_{n=1}^{2} n(0.43)(0.75)^n$$
$$= 0.43\sum_{n=1}^{2} n(0.75)^n = 0.43\left[(0.75) + 2(0.75)^2\right] = 0.81$$

Example 16: The machine in production shop breaks-down at an average of 2 per hour. The non-production time of any machine cost ₹30 per hour. If the cost of repairman is ₹50 per hour and the repair rate is 3 per hour. Calculate:

i) Number of machines not working at any point of time.
ii) Average time that a machine is waiting for the repairman.
iii) Cost of non-production time of the machine operator.
iv) Expected cost of system per hour.

Solution: Given that,

$\lambda = 2$ and $\mu = 3$

Non-production machine cost = ₹30/hour

Cost of repairman = ₹50/hour.

i) No. of machines not working at any point of time

$$= \frac{\lambda}{\mu-\lambda} = \frac{2}{3-2} = 2 \text{ machines}$$

ii) Average time that a machine is waiting for the repairman

$$= \frac{\lambda}{\mu(\mu-\lambda)} = \frac{2}{3(3-2)} = \frac{2}{3}\text{ hour}$$

iii) Cost of non-production time of the machine = 2 × 30 = 60

iv) Expected cost of system per hour = 60 + 50 = ₹110

Example 17: A company's quality control department is manned by a single inspector who takes an average of 5 minutes in checking parts of each machine coming for inspection. The machines arrive once in every 8 minutes on the average. One hour of machine is valued at ₹15 and the inspector's time is valued at ₹4 per hour. What are the hourly average queuing system costs associated with the quality control department?

Solution: Given that,

Mean Arrival Rate = $\lambda = \frac{60}{8}$ per hour = 7.5 per hour

Mean Service Rate = $\mu = \frac{60}{5}$ = 12 per hour

$$W_s = \frac{1}{\mu-\lambda} = \frac{1}{12-7.5} = \frac{2}{9} \text{ hour}$$

Average Queuing Cost per machine = ₹$\frac{2}{9} \times 15$ = ₹$\frac{10}{3}$

Average Queuing Cost/hour (D) = ₹$\frac{10}{3} \times 7.5$ = ₹ 25

Average Hourly Cost for the inspector (E) = ₹4

Total Cost (D + E) = ₹29 per hour

5.4. MODEL II: MULTI-CHANNEL QUEUING MODEL(M/M/C)

5.4.1. Introduction

Multi-channel queuing model is the common facilities system where there are more than one service facilities and the customers arriving for service are attended by these facilities. **For example,** Hospitals, banks, etc. This model works on the first come first serve basis.

This model provides parallel service points in front of which there are customers waiting for their number in a queue. Such parallel service point reduces the length of queue especially where there was only one service station. This creates the biggest advantage for customers by shifting them from a longer queue to shorter queue where they spend lesser time in the queue and can be served very fast.

5.4.2. Assumptions of Multi-Channel Queuing Model

1) It is assumed that the input population is infinite, i.e., the customers arrive out of a large number and follow Poisson's distribution.
2) Arriving customers form one queue.
3) Customers are served on the basis of 'First Come First Served'.
4) Service time follows an exponential distribution.
5) There are a number of service stations (C) and each one provides identical service.
6) The service rate of all the service stations put together is more than arrival rate.

5.4.3. Multi-Channel with Infinite Number of Customers and Infinite Calling Source (M/M/C): (∞/FIFO)

In this model, there are C parallel service channels and hence it is called a special case of **Birth Death Process.** The arrival rate is λ and the service rate per channel is μ.

Assume a queuing system having more than one service station which provides identical service. **For example,** there are more than one counter for submission of water tax, more than one counter of a hospital OPD, more than one cash counter at the bank, etc. When there is more than one queue then the problem can be handled as C different single-server queuing systems (if there are C service facilities).

5.4.3.1. Assumptions of Multi-Channel with Infinite Population (M/M/C)

1) Arrival of customers follows the Poisson law.
2) Service times are distributed exponentially.
3) Only a single waiting line exists.
4) There are multiple (C), identical service facilities.
5) Customers are from infinite population.
6) The arrival rate λ is smaller than the combined service rate (Cμ) of all C service facilities.

5.4.3.2. Derivation of (M/M/C): (GD/N/∞) Queuing Model

This types of queuing system deals with parallel service channels (such as reservation counters) where every server is working independently and follow distributed exponential service distribution.

This queuing system follows Poisson arrival. This model is different from the first model in the manner that it has several service channels as compared to single service channel.

Let consider that there are C servers working together, and then two cases arises:

1) In case, the customers arriving in the system is less than C, i.e. n < C, then only n of the C servers will be busy and so, the expected service rate will be nμ.
2) In case, the customers arriving in the system is greater than C, i.e., n > C, then every C servers will be busy and so, the expected service rate will be Cμ.

This means that,

$$\mu_n = n\mu, \quad \text{if } 0 \le n < C$$
$$= c\mu, \quad \text{if } n \ge C$$

and also $\lambda_n = \lambda$

The steady-state difference equations are determined as follows:

$$P_0(t+\Delta t) = P_0(t)(1-\lambda\Delta t) + P_1(t)\mu\Delta t, \text{ for } n = 0$$

$$P_n(t+\Delta t) = P_n(t)[1-(\lambda+n\mu)\Delta t] + P_{n-1}(t)\lambda\Delta t + P_{n+1}(t)(n+1)\mu\Delta t, \text{ for } n = 1,2,3,\ldots,c-1$$

$$= P_n(t)[1-(\lambda+C\mu)\Delta t] + P_{n-1}(t)\lambda\Delta t + p_{n+1}(t)C\mu\Delta t,$$

for n = C, C + 1, C + 2, C + 3, ...

Dividing the equation by Δt and taking the limit as Δt →0, the difference equation are simplified to

$$P'_0(t) = -\lambda P_0(t) + \mu P_1(t), \text{ for } n = 0$$

$$P'_n(t) = -(\lambda+n\mu)P_n(t) + \lambda P_{n-1}(t) + (n+1)\mu p_{n+1}(t), \text{ for } n = 1,2,3,\ldots,C-1$$

$$P'_n(t) = -(\lambda+c\mu)P_n(t) + \lambda P_{n+1}(t) + c\mu P_{n-1} \text{ for } n = C, C+1, C+2, C+3,\ldots$$

Considering the case of steady-state independent of t as $P'_n(t) \to 0$ for all n, we get from the above equations,

$$0 = -\lambda P_0 + \mu P_1, \text{ for } n = 0 \quad \ldots(1)$$

$$0 = -(\lambda+n\mu)P_n + \lambda P_{n-1} + (n+1)\mu P_{n+1}, \text{ for } 1 \le n < C \quad \ldots(2)$$

$$0 = -(\lambda+C\mu)P_n + \lambda P_{n-1} + C\mu P_{n+1}, \text{ for } n \ge C \quad \ldots(3)$$

From equation (1), we get

$$P_1 = \frac{\lambda}{\mu} P_0$$

Substituting n = 1, 2, 3,....., c in equation (2) and simplifying, we get

$$P_2 = \frac{\lambda}{2\mu} P_1 = \frac{1}{2!}\left(\frac{\lambda}{\mu}\right)^2 P_0$$

$$P_3 = \frac{\lambda}{3\mu} P_2 = \frac{1}{3!}\left(\frac{\lambda}{\mu}\right)^3 P_0$$

$$P_4 = \frac{\lambda}{4\mu} P_3 = \frac{1}{4!}\left(\frac{\lambda}{\mu}\right)^4 P_0$$

In general,

$$P_n = \frac{\lambda}{n\mu} P_{n-1} = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0 \quad 1 \le n < C$$

Again from equation (3), we get

$$P_C = \frac{\lambda}{C\mu} P_{C-1} = \frac{1}{C!}\left(\frac{\lambda}{\mu}\right)^C P_0$$

$$P_{C+1} = \frac{\lambda}{C\mu} P_C = \frac{1}{C}\frac{1}{C!}\left(\frac{\lambda}{\mu}\right)^{C+1} P_0$$

$$P_{C+2} = \frac{\lambda}{C\mu} P_{C+1} = \frac{1}{C^2}\frac{1}{C!}\left(\frac{\lambda}{\mu}\right)^{C+2} P_0$$

The general formula will be as follows:

$$P_n = P_{C+(n-C)} = \frac{1}{C^{n-C}}\frac{1}{C!}\left(\frac{\lambda}{\mu}\right)^n P_0, \; n \ge C$$

We know that the total probability is always equal to 1, hence:

$$\sum_{n=0}^{\infty} P_n = 1 \text{ Or } \sum_{n=0}^{C-1} P_n + \sum_{n=c}^{\infty} P_n = 1$$

$$\text{Or} \quad \sum_{n=0}^{C-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0 + \sum_{n=c}^{\infty}\frac{1}{C!C^{n-C}}\left(\frac{\lambda}{\mu}\right)^n P_0 = 1$$

$$\text{Or} \quad \left[\sum_{n=0}^{C-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{C^C}{C!}\sum_{n=C}^{\infty}\left(\frac{\lambda}{C\mu}\right)^n\right] P_0 = 1$$

$$\text{Or} \quad \left[\sum_{n=0}^{C-1}\frac{C^n}{n!}\left(\frac{\lambda}{C\mu}\right)^n + \frac{1}{C!}\sum_{n=C}^{\infty}\frac{C^n}{C^{n-C}}\left(\frac{\lambda}{C\mu}\right)^n\right] P_0 = 1$$

$$\text{Or} \quad \left[\sum_{n=0}^{C-1}\frac{(C\rho)^n}{n!} + \frac{C^C}{C!}\sum_{n=C}^{\infty}\rho^n\right] P_0 = 1, \; \rho = \frac{\lambda}{C\mu}$$

$$P_0 = \left[\sum_{n=0}^{C-1}\frac{(C\rho)^n}{n!} + \frac{C^C}{C!}\sum_{n=C}^{\infty}\rho^n\right]^{-1}, \quad \rho = \frac{\lambda}{C\mu} < 1$$

$$\therefore \; P_0 = \left[\sum_{n=0}^{C-1}\frac{(C\rho)^n}{n!} + \frac{(C\rho)^C}{C!(1-\rho)}\right]^{-1}, \quad \rho = \frac{\lambda}{C\mu} < 1 = \frac{\lambda}{C\mu} < 1$$

Therefore, the **steady-state probability distributions** of n arrivals in the system are:

$$P_n = \frac{\lambda}{n\mu} P_{n-1} = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0, \quad 0 \le n < C$$

$$= P_{C+(n-C)} = \frac{1}{C^{n-C}}\frac{1}{C!}\left(\frac{\lambda}{\mu}\right)^n P_0, \quad n \ge C$$

Note: System busy = server busy = $\rho = \frac{\lambda}{C\mu}$ is also called **traffic intensity** (M/M/C-Model). The condition for the existence of steady-state solution for M/M/C infinite capacity model is: $\rho = \frac{\lambda}{C\mu} < 1$

5.4.3.3. Operating Characteristics of M/M/C Queuing Model with Infinite Population

The characteristics of M/M/C queuing model are given below:

1) P (n ≥ C) = Probability that an arrival has to wait,

$$= \sum_{n=C}^{\infty} P_n = \sum_{n=C}^{\infty}\frac{1}{C!\,C^{n-C}}(\lambda/\mu)^n P_0 = \frac{(\lambda/\mu)^C C\mu}{C!\,(C\mu-\lambda)} P_0$$

2) Probability that an arrival enters the service without wait,

$$= 1 - P(n \ge C) \text{ or } 1 - \frac{C(\lambda/\mu)^C}{C!\,(C-\lambda/\mu)} P_0$$

3) Average queue length is given by,

$$= \frac{\lambda\mu(\lambda/\mu)^C P_0}{(C-1)!(C\mu-\lambda)^2}$$

4) Average number of customers in the system is given by,

$$L_s = L_q + \frac{\lambda}{\mu} = \frac{\lambda\mu(\lambda/\mu)^C P_0}{(C-1)!(C\mu-\lambda)^2} + \frac{\lambda}{\mu}$$

5) Average waiting time of an arrival is given by,

$$W_q = \frac{\mu(\lambda/\mu)^C . P_0}{(C-1)!(C\mu-\lambda)^2}$$

6) Average waiting time an arrival spends in the system is given by,

$$W_s = W_q + \frac{1}{\mu} = \frac{\mu(\lambda/\mu)^C P_0}{(C-1)!(C\mu-\lambda)^2} + \frac{1}{\mu}$$

Average number of idle servers = C – Average number of customers served.

Example 18: Luthra & Luthra is tax consulting firm consists of three counters and solve the person's problem related to income, wealth and sales tax respectively. The expected 48 persons are arriving in the firm within 8 hour working per day. On average 15 minutes service is provided by each tax adviser to persons. Assume that service is exponentially distributed and arrival is poissonly distributed. Calculate the following:

1) Expected number of persons in the system,
2) Expected number of persons waiting for service,
3) The expected time a person spends in the system.

Solution: Here, we have the following:
Number of counters C =3,

Arrival rate $\lambda = \frac{48}{8} = 6$ persons per hour ,

Service rate $\mu = \frac{1}{15} \times 60 = 4$ persons per hour

Probability P_o can be calculated using the following formula:

$$P_o = \left[\sum_{n=0}^{C-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{C!}\left(\frac{\lambda}{\mu}\right)^C \cdot \frac{C\mu}{(C\mu-\lambda)}\right]^{-1}$$

$$= \left[\sum_{n=0}^{2} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{3!}\left(\frac{\lambda}{\mu}\right)^3 \cdot \frac{3\mu}{(3\mu-\lambda)}\right]^{-1}$$

$$= \frac{1}{\left[1+\frac{\lambda}{\mu}+\frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2\right]+\frac{(\lambda/\mu)^3}{6}\cdot\frac{3\mu}{(3\mu-\lambda)}}$$

$$= \frac{1}{\left[1+\frac{3}{2}+\frac{1}{2}\left(\frac{3}{2}\right)^2\right]+\frac{(3/2)^3}{6}\cdot\frac{3\times4}{(3\times4-6)}}$$

$$= \frac{1}{\left[1+\frac{3}{2}+\frac{9}{8}\right]+\frac{27}{48}\cdot\frac{12}{(12-6)}} = \frac{1}{\left(\frac{29}{8}+\frac{9}{8}\right)} = \frac{8}{38} = 0.21$$

Hence,

1) Expected number of persons in the system is given by:

$$L_s = \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^C}{(C-1)!(C\mu-\lambda)^2} \cdot P_0 + \frac{\lambda}{\mu}$$

$$= \frac{6\times4\times(3/2)^2}{2!(12-6)^2}\times(0.21)+\frac{3}{2} = 1.74$$

2) Expected number of persons waiting for service is given by:

$$L_q = L_s - \frac{\lambda}{\mu} = 1.74 - \frac{3}{2} = 0.24$$

3) The expected time a person spends in the system is given by:

$$W_s = \frac{L_s}{\lambda} = \frac{1.74}{6} = 0.29 \text{ hour} = 17.4 \text{ minutes}$$

Example 19: A titan shop has two counters from where girls are providing services to their customers with an average of 6 customers per hour in exponential fashion. The customers are arriving in the shop on an average rate of 12 customers per hour in a Poisson manner. Determine:

1) The probability that a customer waits for service,
2) The expected number of customers in the whole system, and
3) The expected time that a customer spent in the shop.

Solution: This follows M/M/C: (∞/FIFO) model. From this problem, we have the following:

C = 2

λ = 12 customers per hour, and

μ = 10 customers per hour.

The probability P_0 can be calculated using the following formula:

$$\therefore \quad P_0 = \left[\sum_{n=0}^{2-1} \frac{1}{2!}\left(\frac{12}{10}\right)^n + \frac{1}{2!}\left(\frac{12}{10}\right)^2 \frac{2\times10}{20-12}\right]^{-1}$$

$$= \frac{1}{4} = 0.25$$

Hence,

1) The probability that a customer wait for service is given by:

$$P(n \geq 2) = \frac{1}{C!}(\lambda/\mu)^C \frac{C\mu}{(C\mu-\lambda)} P_0$$

$$= \frac{1}{2}\left(\frac{12}{10}\right)^2 \frac{2\times10}{(20-12)} \times \frac{1}{4} = 0.45$$

2) The expected length of queue is given by:

$$L_q = \frac{\lambda\mu(\lambda/\mu)^C P_0}{(C-1)!(C\mu-\lambda)^2}$$

$$= \frac{12\times10\times(1.2)^2\times0.25}{(2-1)!(20-12)^2} = \frac{27}{40}$$

Thus, the expected number of customers in the whole system is given by:

$$L_s = L_q + \frac{\lambda}{\mu} = \frac{27}{40} + \frac{12}{10} = 1.87 = 2 \text{ customers}$$

3) The expected time that a customer spent in the shop is given by:

$$W_s = \frac{L_s}{\lambda} = \frac{1.87}{12} = 0.156 \text{ hours} = 9.3 \text{ hours.}$$

Example 20: The patients are arriving in a hospital to get the token from the registration center in First-Come First-Served (FCFS) basis. After registration, patients can visit the any of the five doctors in the general ward. The patients are arriving with an average rate of 36 per hour. The doctor solves a patient's problem in 6 minutes on average.

Assuming that patients' arrivals are Poisson distributed and doctors' service times are distributed exponentially, determine,

1) Utilisation factor of the system,
2) The probability that all doctors shall be free,
3) The probability there shall be exactly 4 patients in the hospital,
4) The probability that there shall be 10 patients in the hospital,
5) The expected number of patients waiting in the queue,
6) The expected number of patients in the hospital,
7) The average waiting time in the queue, and
8) The average time a patient spends in the hospital in waiting and getting doctor's service.

Solution: We have, $\lambda = 36$ patients per hour

$\mu = 60/6 = 10$ patients per hour

$n = 5$

1) Utilisation factor of the system,

$$= \frac{\lambda}{n \times \mu} = \frac{36}{10 \times 5} = 0.72$$

2) Probability that all doctors are free,

$$P_0 = \left[\sum_{n=0}^{C-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{C!}\left(\frac{\lambda}{\mu}\right)^C \cdot \frac{C\mu}{(C\mu - \lambda)}\right]^{-1}$$

The denominator is composed of two parts. We first evaluate that second part

$$\frac{1}{5!}\left(\frac{36}{10}\right)^5 \cdot \frac{5 \times 10}{(5 \times 10 - 36)} = 18$$

And, now, the first part of it

For n = 0,	$\frac{(3.6)^0}{0!} = 3.6$
For n = 1,	$\frac{(3.6)^0}{1!} = 3.6$
For n = 2,	$\frac{(3.6)^2}{2!} = 6.48$
For n = 3,	$\frac{(3.6)^3}{3!} = 7.776$
For n = 4,	$\frac{(3.6)^4}{4!} = 6.9984$
	25.8544

$$\text{Thus } P_0 = \frac{1}{18 + 25.8544} = \frac{1}{43.8544} = 0.0228$$

3) Probability that there shall be 4 patients in the hospital:

$$P(4) = P_0 \frac{\rho^4}{4!} = 0.0228 \times \frac{3.6^4}{4!} = 0.16 \quad (\text{since } 4 < n)$$

4) Probability that there shall be 10 patients in the hospital,

$$\frac{P_0\left(\frac{\lambda}{\mu C}\right)^{-n} C^C}{n!} \quad (\text{Since } 10 > n)$$

$$= \frac{0.0228 \times (0.72)^{10} \times 5^5}{5!} = 0.0222$$

5) Expected number or patients waiting in the queue

$$L_q = \frac{P_0 \rho^C \frac{\lambda}{\mu C}}{n!\left(1 - \frac{\lambda}{\mu C}\right)^2}$$

$$= \frac{0.0228(3.6)^5 \times 0.72}{5!(1 - 0.72)^2} = 1.055 \text{ Patients}$$

6) Average number of patients in the hospital is given by:

$L_s = L_q + \rho = 1.055 + 3.6 = 4.655$ patients

7) Average waiting time in the queue is given by:

$$W_q = \frac{L_q}{\lambda} = \frac{1.055}{36} = 0.029 \text{ hours} = 1.76 \text{ minutes}$$

8) Average time a patient spends in the hospital is given by:

$$W_s = W_q + \frac{1}{\mu} = 0.029 + \frac{1}{10} = 0.129 \text{ hours or } 7.76 \text{ minutes.}$$

Example 21: A car washing unit has two cleaning bays manned by a three-man crew.

Cars arrive at an average rate of 10 cars per hour and the arrival rate is Poisson distributed. The under chassis cleaning of a car takes 4 minutes on an average and can be assumed to be exponentially distributed. Determine the:

1) Probability that a customer has to wait before being served,
2) Expected percentage of idle time for each bay, and
3) What is the expected waiting time for a car?

Solution: It is given that, $\lambda = 1/6$ per minute,
$\mu = 1/4$ per minute, $C = 2$

$$\text{Hence utilization factor } \rho = \frac{\lambda}{C\mu} = \frac{1}{6} \times \frac{4}{2 \times 1} = \frac{1}{3}$$

Therefore, probability

$$P_0 = \left[\sum_{n=0}^{2-1} \frac{1}{n!}(4/6)^n + \frac{1}{2!}(4/6)^2 \frac{2.(1/4)}{\{2.(1/4) - (1/6)\}}\right]^{-1}$$

$$= \left(1 + \frac{2}{3} + \frac{1}{3}\right)^{-1} = \frac{1}{2}$$

$P_1 = (\lambda/\mu)P_0 = (4/6)(1/2) = (1/3)$

1) Probability of having to wait for services

$$P(n \geq 2) = \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \cdot \frac{s\mu}{s\mu - \lambda} . P_0$$

$$= \frac{1}{2!}\left(\frac{4}{6}\right)^2 \cdot \frac{2(1/4)}{2(1/4) - (1/6)}\left(\frac{1}{2}\right) = \frac{1}{6}$$

2) The fraction of time the servers are busy, $\rho = \lambda/C\mu = 1/3$. Hence, the expected idle time for every bay is $(1 - 1/3) = 2/3 = 67$ per cent.

3) Expected waiting time for a customer in the system:

$$W_s = W_q + \frac{1}{\mu} = \frac{L_q}{\lambda} + \frac{1}{\mu} = \frac{1}{(S-1)!}\left(\frac{\lambda}{\mu}\right)^s \cdot \frac{\mu}{(s\mu - \lambda)^2} . P_0 + \frac{1}{\mu}$$

$$= \left(\frac{4}{6}\right)^2 \frac{1/4}{[(1/2) - (1/6)]^2} \times \frac{1}{2} + 4 = 4.5 \text{ minutes}$$

Example 22: A telephone exchange has two long distance operators. The telephone company finds that during the peak load long distance calls arrive in a Poisson fashion at an average rate of 15 per hour. The length of service on these calls is approximately distributed with mean 5 minutes.

1) What is the probability that subscriber will have to wait for his long distance call during the peak hours of the day?
2) If the subscriber will wait and be serviced in turn, what is the expected waiting time in queue?

Solution: The above problem comes under (M/M/C): (∞/FCFS) queuing model.

Here, $\lambda = \frac{15}{60} = \frac{1}{4}$ calls/minute;

$\mu = \frac{1}{5}$ calls/minute, $C = 2$

Thus, $\rho = \frac{\lambda}{\mu C} = \frac{1/4}{(1/5)\times 2} = \frac{5}{8}$

Now to find P_0, we use the following formula:

$$P_0 = \frac{1}{\sum_{n=0}^{C-1}\frac{(\lambda/\mu)^n}{n!} + \frac{\left(\frac{\lambda}{\mu}\right)^C}{C!}\cdot\frac{\mu C}{C\mu-\lambda}}$$

$$= \frac{1}{\sum_{n=0}^{1}\frac{\left(\frac{5}{4}\right)^n}{n!} + \frac{\left(\frac{5}{4}\right)^2}{2!}\cdot\frac{\frac{1}{5}\times 2}{\left(2\times\frac{1}{5}-\frac{1}{4}\right)}} = \frac{1}{1+\frac{\frac{5}{4}}{1!}+\frac{25\times 8}{32\times 3}} = \frac{3}{13}$$

1) Probability that a subscriber will have to wait for his long distance call

$$= \text{Probability}(n \geq 2) = \sum_{n=2}^{\infty} P_n$$

$$= \sum_{n=0}^{\infty} P_n - \sum_{n=0}^{1} P_n = 1 - P_0 - P_1$$

$$= 1 - \frac{3}{13} - \left[\frac{1}{1!}\left(\frac{5}{4}\right)^1 P_0\right] \cdot \left[P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0, n \leq c\right]$$

$$= 1 - \frac{3}{13} - \frac{5}{4}\times\frac{3}{13} = \frac{25}{52} = 0.48$$

2) Average waiting time is given by the formula:

$$W_q = \frac{L_q}{\lambda} = \frac{\mu\left(\frac{\lambda}{\mu}\right)^C}{(C-1)!(C\mu-\lambda)^2}P_0 = \frac{\frac{1}{5}\left(\frac{5}{4}\right)^2}{1!\left(2\times\frac{1}{5}-\frac{1}{4}\right)^2}$$

$$= \frac{125}{39} = 3.2 \text{ minutes}$$

Example 23: Find P_0 for (M/M/C): (GD/∞/∞) model if
λ = Arrival rate = 18/hour
μ = Service rate = 6/hour and C = 4

Solution: The above problem comes under the (M/M/C): (GD/∞/∞) queuing model. It is given that:
λ = Arrival rate = 18/hour
μ = Service rate = 6/hour and C = 4
The probability P_0 of having no customer in the system is given by:

$$P_0 = \frac{1}{\sum_{n=0}^{C-1}\frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^C}{C!}\cdot\frac{C\mu}{(C\mu-\lambda)}}$$

$$= \frac{1}{\sum_{n=0}^{3}\frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^4}{4!}\cdot\frac{4\mu}{(4\mu-\lambda)}}$$

$$= \frac{1}{\left[1+\frac{\lambda}{\mu}+\frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2+\frac{1}{6}\left(\frac{\lambda}{\mu}\right)^3\right]+\frac{(\lambda/\mu)^4}{24}\cdot\frac{4\mu}{(4\mu-\lambda)}}$$

$$= \frac{1}{\left[1+\frac{18}{6}+\frac{1}{2}\left(\frac{18}{6}\right)^2+\frac{1}{6}\left(\frac{18}{6}\right)^3\right]+\frac{(18/6)^4}{24}\cdot\frac{4\times 6}{(4\times 6-18)}}$$

$$= \frac{1}{\left[1+3+\frac{9}{2}+\frac{27}{6}\right]+\frac{3^4}{24}\cdot\frac{24}{(24-18)}}$$

$$= \frac{1}{\left[1+3+\frac{9}{2}+\frac{9}{2}\right]+\left(\frac{81}{6}\right)} = \frac{1}{[4+9]+\left(\frac{27}{2}\right)}$$

$$= \left(\frac{2}{26+27}\right) = \left(\frac{2}{53}\right) = 0.0377$$

Example 24: An insurance company has three claim adjusters in its branch office. People with claims against the company are found to arrive in a Poisson fashion, at an average rate of 20 per 8-hour day. The amount of time that an adjuster spends with a claimant is found to have exponential distribution with a mean service time of 40 minutes. Claimants are processed in the order of their appearance.

1) How many hours a week can an adjuster expect to spend with claimants?
2) How much time, on an average, does a claimant spend in the branch office?

Solution: Given that,
C = 5,
λ = 20/8 = 5/2 arrivals per hour,
μ = 1/40 service per minute or 3/2 per hour,

Hence, the utilisation factor $\rho = \left(\frac{\lambda}{\mu}\right) = \frac{5}{3}$

Thus, we have probability,

$$P_0 = \left[\sum_{n=0}^{C-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + +\frac{1}{C!}\left(\frac{\lambda}{\mu}\right)^C\cdot\frac{C\mu}{(C\mu-\lambda)}\right]^{-1}$$

$$= \left[1+\frac{5}{3}+\frac{1}{2}\left(\frac{5}{3}\right)^2+\frac{1}{6}\left(\frac{5}{3}\right)^3\cdot\frac{9}{4}\right]^{-1} = \frac{24}{139}$$

$$P_1 = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)P_0 = \frac{5}{3}\times\frac{24}{139} = \frac{40}{139}$$

$$P_2 = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^2 P_0 = \frac{1}{2}\left(\frac{5}{3}\right)\times\frac{24}{139} = \frac{100}{417}$$

Expected numbers of idle adjusters at any given time are given by:
$3P_0 + 2P_1 + 1P_2 =$

$$P_2 = 3\left(\frac{24}{139}\right)+2\left(\frac{40}{139}\right)+\left(\frac{100}{417}\right) = \frac{4}{3} \text{ adjusters.}$$

$$P_2 = 3\left(\frac{24}{139}\right)+2\left(\frac{40}{139}\right)+\left(\frac{100}{417}\right) = \frac{4}{3}\ \frac{4}{3\times 3} = \frac{4}{9}$$

Thus the probability that no adjuster is idle = 1 – (4/9).

This means that average weekly time an adjuster spends with claimants = (5/9) × 40 (consider 5 working days within week) = 22.2 hours.

W_s = Average Time an Adjuster Spends in the System

$$= W_q + \frac{1}{\mu}$$

$$= \frac{\mu(\lambda/\mu)^C}{(C-1)!(C\mu-\lambda)^2} P_0 + \frac{1}{\mu}$$

$$= \frac{(3/2)(5/3)^3 P_0}{(3-1)!\left(\frac{9}{2}-\frac{5}{2}\right)^2} + \frac{2}{3} = \frac{681}{834} \text{ hours or 49 minutes}$$

Example 25: There are four booking counters in a railway station, the arrival rate of customers follows Poisson distribution and it is 30 per hour. The service rate also follows Poisson distribution and it is 10 customers per hour. Find the following:

i) Average number of customers waiting in the system.

ii) Average waiting time of a customer in the system.

Solution: The above problem follows (M/M/C):(GD/∞/∞) model. It is given that:

Arrival rate λ = 30 customers per hour.

Service rate, μ = 10 customers per hour

Number of booking counters, C = 4

$$\text{Therefore, } \rho = \frac{\lambda}{\mu} = \frac{30}{10} = 3$$

P_0 is computed as:

$$P_0 = \left(\sum_{n=0}^{C-1} \frac{\rho^n}{n!} + \frac{\rho^C}{C![1-(\rho/C)]}\right)^{-1}$$

$$P_0 = \left(\sum_{n=0}^{3} \frac{3^n}{n!} + \frac{3^4}{4![1-(3/4)]}\right)^{-1} = 0.03773585$$

i) Expected waiting number of customers in the queue is given by:

$$L_q = \frac{\rho^{C+1}}{(C-1)!\times(C-\rho)^2} P_0$$

$$= \frac{3^5}{(4-1)!\times(4-3)^2} \times 0.03773585 = 1.5283 \text{ Customers}$$

Therefore, the expected waiting number of customers in the system,

$L_S = L_q + \rho$ = 1.5283 + 3 = 4.5283 customers

ii) Expected waiting time per customer in the queue is given by:

$$W_q = \frac{L_q}{\rho} = \frac{1.5283}{30} = 0.0509434 \text{ hour}$$

$$= 3.0566 \text{ minutes}$$

Expected waiting time per customer in the system,

$$W_S = W_q + \frac{1}{\mu} = 0.0509434 + \frac{1}{10} = 0.1509434 \text{ hour}$$

= 9.0566 minutes.

Example 26: Ships arrive at a port at the rate of one in every 4 hours with exponential distribution of interarrival times. The time a ship occupies a berth for unloading has exponential distribution with an average of 10 hours. If the average delay of ships waiting for berths is to kept below 14 hours, how many berths should be provided at the port?

Solution: Here, $\lambda = \frac{1}{4}$ per hour $\mu = \frac{1}{10}$ per hour

For multiple server queuing system $\frac{\lambda}{\mu C} < 1$, to ensure that the queue does not explode.

$$\text{i.e., } \frac{\lambda}{\mu C} < 1 \Rightarrow \frac{10}{4C} < 1 \Rightarrow C > \frac{5}{2}$$

First we calculate the waiting time for

$$C = 3 \left(\text{since } C > \frac{5}{2}\right)$$

$$\text{i.e., } W_q = \frac{1}{\mu}\frac{1}{C.C!}\frac{\left(\frac{\lambda}{\mu}\right)^C}{\left(1-\frac{\lambda}{\mu C}\right)^2} P_0$$

$$\text{where, } P_0 = \left[\sum_{n=0}^{C-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{C!}\left(\frac{\lambda}{\mu}\right)^C \frac{\mu C}{\mu C-\lambda}\right]^{-1}$$

$$= \left[1+\left(\frac{5}{2}\right)^1 + \frac{1}{2!}\left(\frac{5}{2}\right)^2 + \frac{1}{3!}\left(\frac{5}{2}\right)^3\left(\frac{\frac{3}{10}}{\frac{3}{10}-\frac{1}{4}}\right)\right]^{-1} = 0.0449$$

$$\therefore W_q = 10.\frac{1}{3.3!}\frac{\left(\frac{5}{2}\right)^3}{\left(1-\frac{5}{6}\right)^2}(0.0449) = 14.0313 \text{ hours}$$

Which is slightly greater than 14 hours. So three berths (C = 3) are not enough. Hence, 4 berths must be provided.

5.5. SIMULATION

5.5.1. Introduction

Simulation is a numerical solution method that seeks optimal alternatives (strategies) through a trial and error process. The simulation approach can be used to study almost any problem that involves uncertainty, i.e., one or more decision variables can be represented by a probability distribution, like decision making under risk.

According to Shannon, "Simulation is the process of designing a model of a real system and conducting experiments with this model for the purpose of understanding the behaviour (with the limits imposed by a criterion or set of criteria) for the operation of the system".

According to Donald G. Malcolm, "A simulated model may be defined as one which depicts the working of a large scale system of men, machines, materials and information operating over a period of time in a simulated environment of the real world conditions".

According to Levin & Kirk Patrick, "Simulation is an appropriate substitute for mathematical evaluation of a model in many situations. Although it too involves assumptions, they are manageable. The use of simulation enables us to provide insights into certain management problem where mathematical evaluation of a model is not possible."

5.5.2. Scope of Simulation Techniques

Scope of Simulation in different area is as follows:

1) **Healthcare (Clinical) Simulators:** Medical simulators are developed and deployed to teach healing and diagnostic procedures. It is also used to teach medical concepts and decision making to personnel who are involved in the health professions.

2) **Computer Simulators:** Simulators have been proposed as an ideal tool for assessment of students for clinical skills. Programmed patients and simulated clinical situations, including mock disaster drills, have been used extensively for education and evaluation. These "lifelike" simulations are expensive, and lack reproducibility. A fully functional "3Pi" simulator would be the most specific tool available for teaching and measurement of clinical skills. Such a simulator meets the goals of an objective and standardised examination for clinical competence. This system is superior to examinations that use "standard patients" because it permits the quantitative measurement of competence, as well as reproducing the same objective findings.

3) **Military Simulations:** Military simulations are the models in which theories of warfare can be tested and advanced without the need for actual hostilities. It is also known as war games. They exist in different forms with various degree of realism.

4) **Finance Simulation:** In finance computer simulations are often used for scenario planning.

5) **Flight Simulators:** A flight simulator is used for the training of the pilots on the ground. A pilot gets permission by this technique to crash his simulated "aircraft" without being hurt. Pilots are trained with the use of flight simulators to operate aircraft in extremely dangerous situations, such as landings with no engines, or complete electrical or hydraulic failures. High-fidelity visual systems and hydraulic motion systems are included in most advanced simulators. The simulator is normally cheaper to operate than a real trainer aircraft.

6) **Engineering, Technology or Process Simulation:** Simulation is an important feature in engineering systems or any system that involves many processes. **For example,** in electrical engineering, delay lines may be used to simulate propagation delay and phase shift caused by an actual transmission line. Similarly, dummy loads may be used to simulate impedance without simulating propagation, and is used in situations where propagation is unwanted. A simulator may imitate only a few of the operations and functions of the unit it simulates.

5.5.3. Phases of Simulation

Figure 5.8 shows the different phases of simulation:

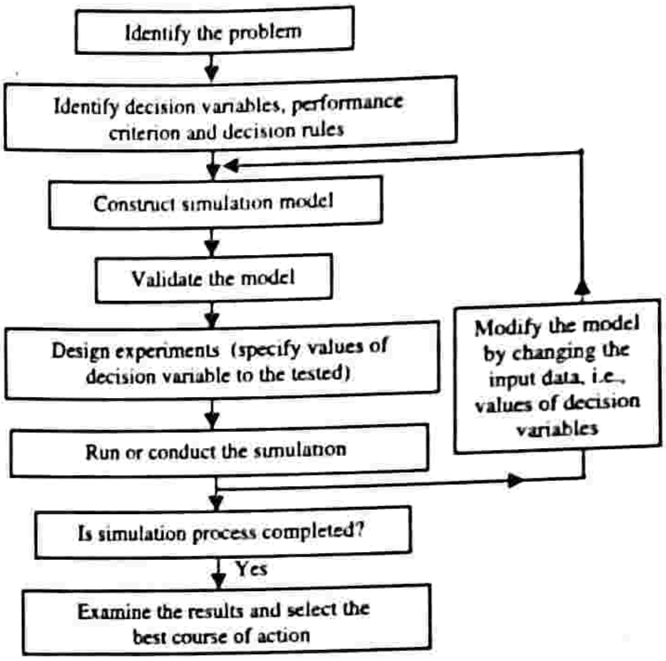


Figure 5.8

Step 1: Identify the Problem: If an inventory system is being simulated, then the problem may concern the determination of the size of order (number of units to be ordered) when inventory level falls up to recorder level (point).

Step 2: i) Identify the Decision Variables

ii) Decide the Performance Criterion (Objective) & Decision Rules

For example, demand (consumption rate), lead time and safety stock are identified as decision variables in inventory problem. These variables shall be liable to measure the performance of the system in terms of total inventory cost under the decision rule – when to order.

Step 3: Construct a Numerical Model: Construct a model which is possible to be analysed on the computer. Sometimes the model is written in a specific simulation language which is suitable for the given problem under analysis.

Step 4: Validate the Model: It is ensuring that the model should be representing the system truly which is analysed and the result will be reliable.

Step 5: Design the Experiments: Design the experiments with the help of simulation model by listing particular values of variables to be tested (i.e., list of courses of action for testing) at each trial (run).

Step 6: Run Simulation Model: For obtaining the results in the form of operating characteristics run the model on the computer.

Step 7: Examine the Results in Terms of Problem Solution: Results are examined as well as their reliability and correctness. Best course of action is selected after completing the simulation process otherwise desired changes are done in model decision variables parameters or design, and return to step 3.

5.5.4. Advantages of Simulation Technique

1) In various cases, mathematical programming and experimentation with the actual system are unable to solve various complex important managerial decision problems and if it is possible then it will be costly. In simulation, solution is obtained by experimentation with a model of the system with affecting the real system.
2) With the help of simulation, management can predict the occurrence of difficulties and bottlenecks due to the introduction of new machines, equipment or process. Thus, simulation eliminates the requirement of costly trial and error method of trying out the new concept on real methods and equipment.
3) Operating personnel and non-technical managers can easily understand the simulation technique because it is relatively free from mathematics. This helps in getting the propose plans accepted and implemented.
4) Simulation models are comparatively flexible and can be changed according to the changing environments of the real situation.
5) Computer simulation can increase the performance of a system over several years and large calculations are done in few minutes of computer running time.
6) In comparison to mathematical model simulation technique is easier to use and it is superior technique to the mathematical analysis.
7) In the operations of complex plans, simulation is used for training the operating and managerial staff. It is very important technique to train the people before putting into their hands in the real system. Simulated exercises have been developed to teach the trainee for gaining sufficient exercise and experience.

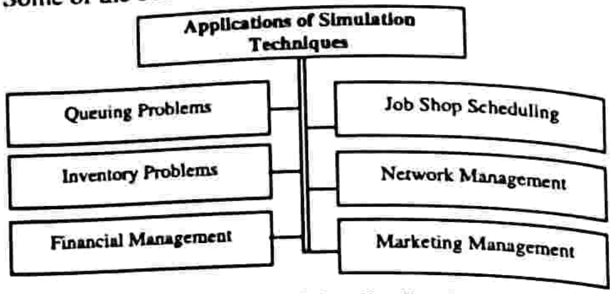
5.5.5. Disadvantages of Simulation Technique

1) Simulation does not produce optimum results. When the model deals with uncertainties, the results of simulation are only reliable approximations subject to statistical errors.
2) Quantification of the variables is another difficulty. In a number of situations it is not possible to quantify all the variables that affect the behavior of the system.
3) In very large and complex problems, the large number of variables and the inter-relationships between them make the problem very unwieldy and hard to program. The number of variables may be too large and may exceed the capacity of the available computer.
4) Simulation is, by no means a cheap method of analysis. In a number of situations, simulation is comparatively costlier and time consuming.

5.5.6. Applications of Simulation Techniques

The range of applications of computer simulation method is very wide, since simulation represents an approach rather than an application of a special technique like linear programming.

Some of the business applications are as follows:



1) **Queuing Problems:** If the distribution of inter-arrival times and service times are known then the queuing theory provides the techniques which are used to determine the measures of effectiveness such as queue length, average waiting time, etc. The problem of establishing a balance between the costs can be determined when these costs are assigned to waiting time of customers and idle time of the service facility.

   Sometimes it is not possible to solve various queuing problems with the use of analytical methods. In this case simulation is the only possible method to solve these types of problems. For solving complex simulation problem Monte Carlo simulation is the important technique. GPSS program is basically structured for queuing simulation model.
2) **Job Shop Scheduling:** Deterministic times for different operations of a given order are involved in the development of a number of job shop simulation programmes. In spite of the absence of probabilistic elements in such a model, the high degree of interaction between orders due to their different processing times for similar operations and to different order operation sequences, makes it difficult to predict the waiting time for a particular order at any given work center. Orders must be scheduled at different work centers with an allowance for waiting time. Simulation is the best suitable technique to make reasonable and accurate estimate of such allowances which are required for efficient scheduling of orders. With the use of simulation technique it is possible to forecast the manpower and machine workloads.
3) **Inventory Problems:** In various inventory problem especially storage problem cannot solved analytically because the distribution followed by demand or supply is very complex.

   The solution can be obtained by using simulation technique. With the use of past data it is possible to determine the probability distribution of the input and output functions and the inventory system run artificially by generating the future observations on the assumption of the same distributions. Trial and error method is used to find the decision for the optimisation problems. With the help of random numbers the artificial sample for the future can be generated. The demand during lead time to provide adequate service to customers is the basis of the selection of reorder point in inventory control.

Simulation technique can be used to investigate the effect of different inventory policies if the lead time and the demand of inventory per unit of time are random variables. For wider applicability as well as those for specialised use a lot of work has been done to develop inventory simulation models. In various inventory problem queuing characteristics can be seen. **For example,** a problem concerned with the optimal inventory of rental cars can be viewed as a queuing problem where the servers are cars.

4) **Network Management:** A number of network simulation models have also been developed. **For example,** simulation of probabilistic activity times in PERT networks. The critical path and the project duration can be evaluated with a randomly selected activity times for each activity. The probability distribution of project completion time and the probability that each of given activity on the critical path can be determined by repeating the process.

5) **Financial Management:** Financial studies involving risky investment and profit planning.

6) **Marketing Management:** Marketing decisions faces uncertainties in various stages such as in new product introduction. Simulation methods are very important for solving marketing problems which have probabilistic outcomes. Generation of random numbers is used to predict the outcomes and results are analysed in terms of the specific decisions the management must make. Simulation is applicable in marketing in various ways such as new product planning.

5.5.7. Simulation Models

Simulation models are mainly of two types:

1) **Continuous Models:** Continuous models are used for the system whose behaviour changes continuously with time. Difference-differential equations are used by this model to describe the interactions among the different elements of the system. A typical example deals with the study of world population dynamics.

2) **Discrete Models:** Discrete models are used for the study of waiting lines, with the objective of determining average waiting time and the length of the queue. When a customer enter or leaves the system then these measures would be changed. At all other instants, nothing from the standpoint of collecting statistics occurs in the system. The instants at which changes take place occur at discrete points in time, giving rise to the name discrete event simulation.

Broadly simulation models can be classified into following four categories:

1) **Deterministic Models:** In these models, input and output variables should not be random variables and exact functional relationship is used to describe the models.

2) **Stochastic Models:** In these models, probability functions are used to describe at least one of the variables or functional relationship.

3) **Static Models:** Representation of a system at a particular time or representation of a system in which time does not play any role is known as static simulation model. **Monte Carlo Models** is an example of static simulation.

4) **Dynamic Models:** A system which changes over the time is represented by a dynamic simulation model such as conveyor system in a factory.

5.5.8. Monte Carlo Simulation

Monte Carlo simulation technique is based on the technique that the given system under analysis is replaced by a system described by some known probability distribution and then random samples are drawn from probability distribution by means of random numbers.

An empirical probability distribution can be constructed if it is not possible to explain a system in terms of standard probability distribution such as Normal, Poisson, Exponential, Gamma, etc.

5.5.8.1. Monte Carlo Simulation Procedure

The Monte Carlo simulation technique consists of the following steps:

Step 1: Clearly Define the Problem

1) Identify the objectives of the problem.
2) Identify the main factors which have the greatest effect on the objectives of the problem.

Step 2: Construct an Appropriate Model

1) Specify the variables and parameters of the model.
2) Formulate the appropriate decision rules, i.e., state the conditions under which the experiment is to be performed.
3) Identify the type of distribution that will be used – Models used either theoretical distributions or empirical distributions to state the patterns the occurrence associated with the variables.
4) Specify the manner in which time will change.
5) Define the relationship between the variables and parameters.

Step 3: Prepare the Model for Experimentation

1) Define the starting conditions for the simulation.
2) Specify the number of runs of simulation to be made.

Step 4: Using Steps 1 to 3, Experiment with the Model

1) Define a coding system that will correlate the factors defined in Step 1 with the random numbers to be generated for the simulation.
2) Select a random number generator and create the random numbers to be used in the simulation.
3) Associate the generated random numbers with the factors identified in Step 1 and coded in Step 4 (a).

Step 5: Summarise and Examine the Results Obtained in Step 4.

Step 6: Evaluate the Results of the Simulation.
Select the best course of action.

5.5.8.2. Random Digits and Methods of Generating Probability Distribution

In Monte Carlo Simulation, a sequence of random numbers is required to generate which is an integral part of the simulation model. The selection of random observations (samples) from the probability distribution is facilitated by these sequences of random numbers. In a sequence of integer number, a random number is a number between 0 to 9, whose probability of occurrence is same as that of any other number in the sequence.

In a simulation model, any decision variable can be represented as a random variable and it is assumed that it must follow some theoretical probability distribution such as normal, Poisson, exponential etc., or an empirical distribution. Simple arithmetic computation and computer generator is used to generate random numbers which is based on some known probability distributions.

Example 27: A sample of 100 arrivals of a customer at a retail sales depot is according to the following distribution:

Time between Arrival (Min)	Frequency
0.5	2
1.0	6
1.5	10
2.0	25
2.5	20
3.0	14
3.5	10
4.0	7
4.5	4
5.0	2

A study of the time required to service customers by adding up the bills, receiving payments and placing packages, yields the following distribution:

Time between Service (min)	Frequency
0.5	12
1.0	21
1.5	36
2.0	19
2.5	7
3.0	5

Estimate the average percentage of customer waiting time and average percentage of idle time of the server by simulation for the next 10 arrivals.

Solution:

Step 1: Convert the frequency distributions of time between arrivals and service time to cumulative probability distributions.

Step 2: Allocate random numbers 00 to 99 for each of the values of time between arrivals and service time, the range allocated to each value corresponding to the value of cumulative probability as shown in **table 5.5** and **5.6.**

Step 3: Using random numbers from table, sample at random time of arrival and service time for 10 sets of random numbers.

Step 4: Tabulate waiting time of arrivals and idle time of servers (**table 5.7**).

Step 5: Estimate the per cent waiting time of arrivals and per cent idle time of servers corresponding to the 10 samples.

Random numbers are allocated to the events in the same proportions as indicated by the probabilities:

Table 5.5: Allocation of random number –Time between Arrivals

Arrivals	Frequency	Cumulative Frequency	Probability	Cumulative Probability	Random Number Allocated
0.5	2	2	0.02	0.02	00-01
1.0	6	8	0.06	0.08	02-07
1.5	10	18	0.10	0.18	08-17
2.0	25	43	0.25	0.43	18-42
2.5	20	63	0.20	0.63	43-62
3.0	14	77	0.14	0.77	63-76
3.5	10	87	0.10	0.87	77-86
4.0	7	94	0.07	0.94	87-93
4.5	4	98	0.04	0.98	94-97
5.0	2	100	0.02	1.00	98-99

Table 5.6: Allocation of Random Numbers – Service Time

Service Time (Min)	Frequency	Probability	Cumulative Probability	Random Number Allocated
0.5	12	0.12	0.12	00-11
1.0	21	0.21	0.33	12-32
1.5	36	0.36	0.69	33-68
2.0	19	0.19	0.88	69-87
2.5	7	0.07	0.95	88-94
3.0	5	0.05	1.00	95-99

Note: The upper bound of random numbers allocated for each value of the parameter is one less than the corresponding cumulative frequency since one has chosen a range of random numbers from 00 to 99.

Table 5.7: Waiting Time of Arrival and Idle Time of Server

Arrival No.	Random No.	Time between Arrivals	Time of Arrival	Random No.	Service Time	Time of Start	Time of Finish	Waiting Time of Arrival	Idle Time of Server
1	78	3.5	3.5	54	1.5	3.5	5.0	–	3.5
2	78	3.5	7.0	26	1.0	7.0	8.0	–	2.0
3	06	1.0	8.0	51	1.5	8.0	9.5	–	–
4	04	1.0	9.0	45	1.5	9.5	11.0	0.5	–
5	97	4.5	13.5	46	1.5	13.5	15.0	–	2.5
6	71	3.0	16.5	84	2.0	16.5	18.5	–	1.5
7	78	3.5	20.0	58	1.5	20.0	21.5	–	1.5
8	61	2.5	22.5	58	1.5	22.5	24.0	–	1.0
9	05	1.0	23.5	60	1.5	24.0	25.5	0.5	–
10	95	4.5	28.0	24	1.0	28.0	29.0	–	2.5
							Total	1.0	14.5

The service facility is made available at clock time zero and the server has to be idle for 3.5 minutes when the service for first arrival starts. The service is completed at 5 minutes and again the server is idle for 2 minutes till the second arrival joins the system. The first three arrivals get immediate service and they do not have to wait, as the server is idle when they arrive. The fourth arrival that joins at 9 minutes has to wait for 0.5 minute when the service for the third is completed. Similarly, the waiting time and idle time can be completed for further arrivals.

Total elasped time = 29 minutes

Waiting time of arrival = 1 minute

Percentage of idle time = $\frac{1\times100}{29}=3.4$

Idle time for server = 14.5 minutes

Percentage of idle time = $\frac{14.5\times100}{29}=50$

Example 28: Bright Bakery keeps stock of a popular brand of cake. Previous experience indicates the daily demand as given here.

Daily Demand	0	10	20	30	40	50
Probability	0.01	0.20	0.15	0.50	0.12	0.02

Consider the following sequence of random numbers:

R. No. 48, 78, 19, 51, 56, 77, 15, 14, 68, 09

Using this sequence, simulate the demand for the next 10 days. Find out the stock situation if the owner of the bakery decides to make 30 cakes every day. Also estimate the daily average demand for the cakes on the basis of simulated data.

Solution: According to the given distribution of demand, the random number coding for various demand levels is shown in **table 5.8**:

Table 5.8: Random Number Coding

Demand	Probability	Cumulative Probability	Random Number Interval
0	0.01	0.01	00
10	0.20	0.21	01-20
20	0.15	0.36	21-35
30	0.50	0.86	36-85
40	0.12	0.98	86-97
50	0.02	1.00	98-99

The simulated demand for the cakes for the next 10 days is given in **table 5.9**. Also given is the stock situation for various days in accordance with the bakery decision of making 30 cakes per day.

Table 5.9: Determination of Demand and Stock Levels

Day Number	Random Demand	Generated Demand	Stock Leave
1	48	30	–
2	78	30	–
3	19	10	20(30 – 10)
4	51	30	20
5	56	30	20
6	77	30	20
7	15	10	40[20 + (30 – 10)]
8	14	10	60[40 + (30 – 10)]
9	68	30	60
10	09	10	80[60 + (30 – 10)]
	Total	**220**	

Hence the Expected demand = 220/10 = 22 units per day

Example 29: Dr. Manoj is a dentist who schedules all his patients for 30 minutes appointments. Some of the patients take more or less 30 minutes depending on the type of dental work to be done. The following summary shows the various categories of work, their probabilities and time actually needed to complete the work:

Table 5.10

Category Of Service	Time Required	Probability of Category
Filling	45	0.40
Crown	60	0.15
Cleaning	15	0.15
Extraction	45	0.10
Checkup	15	0.20

Simulate the dentist's clinic for four hours and determine the average waiting time for the patients as well as the idleness of the doctor. Assume that all the patients show up at the clinic at exactly their schedule arrival time starting at 8.00 am. Use the following random numbers of handling the above problem:

40 82 11 34 25 66 17 79

Solution: The cumulative probability distribution and random number interval for service time are shown in table 5.11:

Table 5.11

Category of Service	Service Time Required (Minutes)	Probability	Cumulative Probability	Random Number Interval
Filling	45	0.40	0.40	00 – 39
Crown	60	0.15	0.55	40 – 54
Cleaning	15	0.15	0.70	55 – 69
Extraction	45	0.10	0.80	70 – 79
Checkup	15	0.20	1.00	80 – 99

The various parameters of a queuing system such as, arrival pattern of customers, service time, waiting time in the context of the given problem is shown in **tables 5.12**:

Table 5.12: Arrival Pattern and Nature of Service

Patient Number	Scheduled Arrival	Random Number	Category of Service	Service Time (Minutes)
1	8.00	40	Crown	60
2	8.30	82	Check up	15
3	9.00	11	Filling	45
4	9.30	34	Filling	45
5	10.00	25	Filling	45
6	10.30	66	Cleaning	15
7	11.00	17	Filling	45
8	11.30	79	Extraction	45

Table 5.13: Computation of Arrivals, Departure and Waiting of Patients

Time	Event (Patient Number)	Patient Number (Time to Exist)	Waiting (Patient Number)
8.00	1 arrives	1(60)	–
8.30	2 arrives	1(30)	2
9.00	1 departs; 3 arrive	2(15)	3
9.15	2 depart	3(45)	–
9.30	4 arrive	3(30)	4
10.00	3 depart; 5 arrive	4(45)	5
10.30	6 arrive	4(15)	5,6
10.45	4 depart	5(45)	6
11.00	7 arrive	5(30)	6,7
11.30	5 depart; 8 arrive	6(15)	7,8
11.45	6 depart	7(45)	8
12.00	End	7(30)	8

The dentist was not idle during the entire simulated period. The waiting times for the patients were as follows:

Table 5.14: Computation of Average Waiting Time

Patient	Arrival Time	Service Starts at	Waiting Time (Minutes)
1	8.00	8.00	0
2	8.30	9.00	30
3	9.00	9.15	15
4	9.30	10.00	30
5	10.00	10.45	45
6	10.30	11.30	60
7	11.00	11.45	45
8	11.30	12.30	60
			280

The average waiting time $= \frac{280}{8} = 35$ minutes.

Example 30: The demand for a bakery product is tabulated below based on previous data:

Daily Demand	0	10	20	30	40	50
Probability	0.01	0.20	0.15	0.50	0.12	0.02

Use the following sequence random numbers simulate the demand for 10 days:
Random Numbers: 25, 39, 65, 76, 12, 05, 73, 89, 19, 49

Solution: Using the daily demand distribution, we obtain a probability distribution as shown in **table below:**

Daily Demand	Probability	Cumulative Probability	Random Number Interval
0	0.01	0.01	00
10	0.20	0.21	01 – 20
20	0.15	0.36	21 – 35
30	0.50	0.86	36 – 85
40	0.12	0.98	86 – 97
50	0.02	1.00	98 – 99

We conduct the simulation experiment for demand by taking a sample of 10 random numbers from a table of random numbers, which represent the sequence of 10 samples. Each random sample number here is a sample of demand. The simulation calculations for a period of 10 days are given in **table below:**

Days	Random Number	Demand
1	25	20
2	39	30
3	65	30
4	76	30
5	12	10
6	05	10
7	73	30
8	89	40
9	19	10
10	49	30
		Total = 240

Expected demand = 240/10 = 24 units per day

Example 31: The rainfall distribution in monsoon seasons is as follows:

Rain in cm.	0	1	2	3	4	5
Frequency	50	25	15	5	3	2

Simulate the rainfall for 10 days using the following random variables:
67, 63, 39, 55, 29, 78, 70, 06, 78, 76 and then find average rainfall.

Solution: The allocation of random number is shown in **table below 5.15**:

Table 5.15: Allocation of Random Number

Rainfall	Frequency	Cumulative Frequency	Probability	Cumulative Probability	Random Number Allocated
0	50	50	0.50	0.50	00-49
1	25	75	0.25	0.75	50-74
2	15	90	0.15	0.90	75-89
3	5	95	0.05	0.95	90-94
4	3	98	0.03	0.98	95-97
5	2	100	0.02	1.00	98-99

The simulated rainfall for the next 10 days is given below:

Day Number	Random Demand	Generated Rainfall
1	67	1
2	63	1
3	39	0
4	55	1
5	29	0
6	78	2
7	70	1
8	06	0
9	78	2
10	76	2
Total		10

Hence, the average rainfall = 10/10 = 1 cm/day

Example 32: At a bus depot every bus should leave with driver. At the terminus they should keep two drivers as reserved if anyone on scheduled duty is sick and could not come, following is the Probability Distribution that the driver becomes sick:

No. of sick drivers	0	1	2	3	4	5
Probability	0.30	0.20	0.15	0.10	0.13	0.12

Simulate for 10 days and find utilisation of reserved drivers. Also find how many days and how many buses cannot run because of non-availability of the drivers.
Use the following random numbers 30, 54, 34, 72, 20, 02, 76, 74, 48, 22.

Solution: According to the given distribution of no. of sick drivers, the random number coding for various levels is shown in **table 5.16**:

Table 5.16: Random Number Coding

No. of Sick Drivers	Probability	Cumulative Probability	RandomNumber Interval
0	0.30	0.30	00 - 29
1	0.20	0.50	30-49
2	0.15	0.65	50-64
3	0.10	0.75	65-74
4	0.13	0.88	75-87
5	0.12	1.00	88-99

The simulated number of sick drivers for the next 10 days is given in **table 5.17**:

Table 5.17: Determination of Number of Buses and days when Buses Not Run

Days	Random Number	Number of Sick Drivers	Number of Buses Not Run
1	30	1	-
2	54	2	-
3	34	1	-
4	72	3	1
5	20	0	-
6	02	0	-
7	76	4	2
8	74	3	1
9	48	1	-
10	22 .	0	-

Hence, the Number of Buses that does Not Run = 4
The Number of days when Buses Not Run = 3

5.6. EXERCISE

5.6.1. Short Answer Type Questions

1) What is queuing theory?
2) What is queuing process?
3) What are advantages of queuing theory?
4) Explain different behavior of customer?
5) What are the elements of queuing theory? Briefly explain.
6) What are the models of queuing models? Explain with example.
7) Explain the characteristics of queuing theory?
8) What are the assumptions Single Channel Queuing Model?
9) What is simulation? What is the scope of simulation?
10) What are the applications of simulation?
11) Give the process of simulation.
12) Give the various types of simulation.
13) What is Monte Carlo Simulation?

5.6.2. Long Answer Type Questions

1) At a certain petrol pump, customers arrive in a Poisson process with an average time of five minutes between successive arrivals. The time taken at the petrol pump to serve customers follows exponential distribution with an average of two minutes. You are required to obtain the following:
   i) Arrival and service rates.
   ii) The utilisation parameter.
   iii) Probability that there shall be four customers in the system.
   iv) Probability that there are more than four customers in the system.
   v) Expected queue length.
   vi) Expected number of customers in the system.
   vii) Expected time that a customer has to wait in the queue.
   viii) Expected time that a customer has to spend in the system.
   ix) It is decided to open a new service point when the customers' expected waiting time rises to three times the present level. What increased flow of customers will justify this?

   [Ans:
   i) $\lambda = 12$ customers/hour, $\mu = 30$ customers/hour
   ii) 0.40
   iii) 0.01536
   iv) 0.01024
   v) 4/15 customer
   vi) 2/3 customer
   vii) 4/3 minutes
   viii) 10/3 minutes
   ix) 20 customers/hour]

2) The Secretary of the Excellent Corporation receives letters for typing, the arrival of which follows the Poisson law with an average arrival rate $\lambda = 3$ letters/hour.
   i) Find the probability that she will receive no more than 2 letters in an hour, atleast one letter in an hour.
   ii) Determine the number of letters expected to be received in a two-hour interval.

   [Ans: i) 0.4232; 0.9502, ii) 6]

3) A foreign bank is considering opening a drive-in window for customer service. Management estimates that customers will arrive for service at the rate of 12 per hour. The teller, whom it is considering to staff the window, can serve customers at the rate of one every three minutes. Assuming Poisson arrivals and exponential service, find:
   i) Utilisation of teller,
   ii) Average number in the system,
   iii) Average waiting time in the line, and
   iv) Average waiting time in the system.

   [Ans: i) 0.6 ii) 1.5 iii) 0.075 hour iv) 0.125 hour**]**

4) A belt snapping for conveyors in an open cast mine at the rate of 2 per 8 hour shift. There is only one hot plate for vulcanizing and it can vulcanize on an average 5 belts snap per shift.
   i) What is the probability that when a belt snaps, the hot plate is readily available?
   ii) What is the average number of belts in the system?
   iii) What is the waiting time of an arrival?
   iv) What is the average waiting time plus vulcanizing time?

   [Ans:
   i) $p_0 = \frac{3}{5}$
   ii) $L_s = \frac{2}{3}$
   iii) $w_q = \frac{2}{15}$ shift
   iv) $w_s = \frac{1}{3}$ shift]

5) The management of ABC company is considering the question of marketing a new product. The fixed cost required in the project is ₹4,000. Three factors are uncertain, *viz.*, the selling price, variable cost and the annual sales volume. The product has a life of only one year. The management has the data on these three factors as under:

Selling Price (₹)	Probability	Variable Cost	Probability (₹)	Sales Volume	Probability (Units)
3	0.2	1	0.3	2,000	0.3
4	0.5	2	0.6	3,000	0.3
5	0.3	3	0.1	5,000	0.4

Consider the following sequence of thirty random numbers:

81, 32, 60, 04, 46, 31, 67, 25, 24, 10, 40, 02, 39, 68, 08, 59, 66, 90, 12, 64, 79, 31, 86, 68, 82, 89, 25, 11, 98, 16

Using the sequence (First 3 random numbers for the first trial, etc.), simulate the average profit for the above project on the basis of 10 trials.

[Ans: Average Profit = ₹2,100]

6) A company manufactures 30 items per day. The sale of these items depends upon demand which has the following distribution:

Sales (Units)	Probability
27	0.10
28	0.15
29	0.20
30	0.35
31	0.15
32	0.05

The production cost and selling price of each unit are ₹40 and ₹50 respectively. Any unsold product is to be disposed off at a loss of ₹15 per unit. There is a penalty of ₹5 per unit if the demand is not met.

Using the following random numbers estimate total profit/loss for the company for the next 10 days: 10, 99, 65, 99, 95, 01, 79, 11, 16, 20.

If the company decides to produce 29 items per day, what is the advantage or disadvantage to the company?

[Ans: No Additional Profit/Loss]

7) A bakery keeps stock of a popular brand of cakes. Previous experience shows the daily demand pattern for the item with associated probabilities, as given:

Daily Demand (Nos.)	0	10	20	30	40	50
Probability	0.01	0.20	0.15	0.50	0.12	0.02

Use the following sequence of random numbers to simulate the demand for next 10 days. Also find out the average demand per day.

Random numbers: 25, 39, 65, 76, 12, 65, 73, 89, 19, 49

[Ans: Average Demand = 26 units/day]

8) A confectioner sells confectionery items. Past data of demand per week in hundred kilograms with frequency is given below:

Demand/Week	0	5	10	15	20	25
Frequency	2	11	8	21	5	3

Using the following sequence of random numbers, generate the demand for next 15 weeks. Also find out the average demand per week.

Random numbers:	35	52	90	13	23	73	34	57
	35	83	94	56	67	66	60	

[Ans: Average Demand = 12.6 units/week]

9) A small retailer has studied the weekly receipts and payments over the past 200 weeks and has developed the following set of information:

Weekly Receipts (₹)	Probability	Weekly Payment (₹)	Probability
3,000	0.20	4,000	0.30
5,000	0.30	6,000	0.40
7,000	0.40	8,000	0.20
12,000	0.10	10,000	0.10

Using the following sequence of random numbers, simulate the weekly pattern of receipts and payments for the 12 weeks of the next quarter, assuming further that the beginning bank balance is ₹8,000. What is the estimated balance at the end of the 12 weekly period? What is the highest weekly balance during the quarter? What is the average weekly balance for the quarter?

Random Numbers

For Receipts	03	91	38	55	17	46	32	43	69	72	24	22
For Payments	61	96	30	32	03	88	48	28	88	18	71	99

[Ans: End Balance = (₹3,000), Highest Balance = ₹7,000, Average Weekly Balance = ₹3,750]

ANNA UNIVERSITY, CHENNAI

MBA - SECOND SEMESTER EXAMINATION, APRIL/MAY - 2015

APPLIED OPERATIONS RESEARCH

Time: 3 Hours **Max. Marks: 100**

Note: Answer All questions.

PART - A (10×2=20)

Ques 1) State the limitation of a graphical method.

Ans: Limitations of a Graphical Method
Refer Unit-1, Page No. 28

Ques 2) How does dual simplex method differ from simplex method?

Ans: Difference between Simplex and Dual Simplex Method
Out of Syllabus

Ques 3) What is an unbalanced transportation problem?

Ans: Unbalanced Transportation Problem
Refer Unit-2, Page No. 87

Ques 4) Which cell will be the first basic variable in case of North-West Corner Method and Least Cost Method?

Ans: North-West Corner Method
Refer Unit-2, Page No. 60

Least Cost Method
Refer Unit 2, Page No. 63

Ques 5) While using IPP technique, what is the fractional part of $-\frac{98}{19}$?

Ans: Fractional value = $-\frac{3}{19}$.

Ques 6) Differentiate between pure and mixed strategies.

Ans: Difference between Pure and Mixed Strategies
Refer Unit-3, Page No. 155

Ques 7) List the elements of carrying cost.

Ans: Elements of Carrying Cost
Refer Unit-4, Page No. 177

Ques 8) What do the terms 'uncertainty' and 'risk' refer?

Ans: Decision Making Under Uncertainty
Refer Unit-3, Page No. 134

Decision Making Under Risk
Refer Unit-3, Page No. 139

Ques 9) In a store with one cashier, nine customers arrive on the average of every five minutes and the cashier can serve them ten in five minutes. Find utilisation factor.

Ans: Here, $\lambda = 5$ and $\mu = 5$.

$$\text{Utilization factor } (\rho) = \frac{\text{Mean Arrival Rate}}{\text{Mean Service Rate}} = \frac{\lambda}{\mu} = \frac{5}{5} = 1.$$

Ques 10) If the money carries an interest rate of 10% per year, what will be the value of one rupee after two years?

Ans: Here, r = 10% and n = 2 years.

$$\text{Since, } A = \left[1+\frac{r}{100}\right]^n = \left[1+\frac{10}{100}\right]^2 = \left[\frac{11}{10}\right]^2 = \frac{121}{100} = 1.21$$

PART - B (5×16=80)

Ques 11 a) i) Solve the following LPP graphically; (08)
Minimise Z = 3x + 2y;
Subject to $x - y \le 1$,
$x + y \ge 3$
and $x, y \ge 0$.

Ans: Refer Unit-1, Page No. 27

Ques 11 a) ii) A person wants to decide the constituents of a diet which will fulfil his daily requirements of proteins, fat and carbohydrates at the minimum cost. The choice is to be made from four different types foods. The yields per unit of these foods are given below:

Food Type	Yield per Unit			Cost per Unit
	Proteins	Fats	Carbohydrates	
1	3	2	6	45
2	4	2	4	40
3	8	7	7	85
4	6	5	4	65
Minimum requirement	800	200	700	

Formulate the LPP for the problem. (08)

Ans: Refer Unit-1, Page No. 19

Or

Ques 11 b) Solve the following LPP Two Phase Simplex Method; (16)

Maximise $Z = -4a - 3b - 9c$;

Subject to $2a + 4b + 6c \geq 15$

$6a + b + 6c \geq 12$

$a, b, c \geq 0$

Ans: Refer Unit-1, Page No. 43

Ques 12 a) Solve the following transportation problem to minimise the total transportation cost for shifting the goods from factories (A, B and C) to warehouses (P, Q and R) where unit transportation cost, availability and demand, at factories and warehouses respectively are given in the following matrix: (16)

		Warehouse			
		P	Q	R	Availability
Factory	A	1	2	0	30
	B	2	3	4	35
	C	1	5	6	35
	Demand	30	40	30	

Find the allocation so that the total transportation cost is minimum.

Ans: Refer Unit-2, Page No. 86

Or

Ques 12 b) A company has 4 territories and four salesmen for assignment. The territories are not equally rich in their sales potential. It is estimated that a typical salesman operating in each territory would bring the following annual sales: (16)

Territory	I	II	III	IV
Annual Sales in (₹)	60,000	50,000	40,000	30,000

The four salesmen are also considered to differ in their ability; it is estimated that working under same condition, their yearly sales would be proportionately as follows:

Salesman	A	B	C	D
Proportion	0.1	0.2	0.3	0.4

If the criteria is to maximise expected sales, what is your intuitive answer and verify your answer with Hungarian method.

Ans: Refer Unit-2, Page No. 118

Ques 13 a) Find the optimum integer solution to the following LPP: (16)

Maximise $Z = 3x_1 + 7x_2$

Subject to $3x_1 + 4x_2 \leq 19$

$3x_1 + 6x_2 \leq 21$

x_1, x_2 non-negative integers.

Ans: Refer Unit-1, Page No. 33

Or

Ques 13 b) i) State the rules of dominance. (04)

Ans: Refer Unit-3, Page No. 161

Ques 13 b) ii) Solve the following game: (12)

Player B

$$\text{Player A} \begin{pmatrix} 1 & 7 & 2 \\ 6 & 2 & 7 \\ 5 & 1 & 6 \end{pmatrix}$$

Ans: Refer Unit-3, Page No. 162

Ques 14 a) i) Derive EOQ formula for simple inventory model with no shortages and instantaneous replenishment. (06)

Ans: Refer Unit-4, Page No. 180

Ques 14 a) ii) Find the optimum order quantity for a product for which the price break is given below:

Quantity	Unit Cost (in ₹)
$0 \leq q1 < 100$	20 per unit
$100 \leq q2 < 200$	18 per unit
$200 \leq q3$	16 per unit

The monthly demand for the product is 400 units. The storage cost is 20% of the unit cost of the product and the cost of ordering is ₹25. (10)

Ans: Refer Unit-4, Page No. 190

Or

Ques 14 b) Concisely explain the criterions used to assist decision-making under uncertainty. (16)

Ans: Decision Criteria: Refer Unit-3, Page No. 135

Ques 15 a) i) A T.V. repairman finds that the time he spent on his job has an exponential distribution with mean 30 minutes. If he repairs the set in the order it arrives, and the arrival rate is approximately Poisson, with an average rate of 10 per 8 hour day, what is the expected idle time of repairman each day? How many jobs are ahead of average before the job just brought in? (04)

Ans: Refer Unit-5, Page No.218

Ques 15 a) ii) A telephone exchange has two long distance operators. The telephone company finds that during the peak load long distance calls arrive in a Poisson fashion at an average rate of 15 per hour. The length of service on these calls is approximately distributed with mean 5 minutes.

1) What is the probability that subscriber will have to wait for his long distance call during the peak hours of the day?
2) If the subscriber will wait and be serviced in turn, what is the expected waiting time in queue? (12)

Ans: Refer Unit-5, Page No. 226

Or

Ques 15 b) A machine costs ₹15,000 and its running costs for different years are given below. Find optimum replacement period if the capital is worth 10% and the machine has no salvage value. (16)

Year	1	2	3	4	5	6	7
Running Cost (₹)	2,500	3,000	4,000	5,000	6,500	8,000	10,000

Ans: Refer Unit-4, Page No.201

ANNA UNIVERSITY, CHENNAI

MBA - SECOND SEMESTER EXAMINATION, MAY/JUNE-2016

APPLIED OPERATIONS RESEARCH

Time: 3 Hours **Max. Marks: 100**

Note: Answer All questions.

PART - A (10×2=20)

Ques 1) How to identify from simplex table an LPP has alternate optimum solution?

Ans: Alternative Optimum Solutions
Refer Unit-1, Page No. 28

Ques 2) What is the use of artificial variable in LP solution?

Ans: Use of Artificial Variables
Refer Unit-1, Page No. 30

Ques 3) What is unbalanced transportation problem?

Ans: Unbalanced Transportation Problem
Refer Unit-2, Page No. 87

Ques 4) What is the Travelling Salesman problem?

Ans: Travelling Salesman Problem
Refer Unit-2, Page No. 123

Ques 5) What is the need for integer programming?

Ans: Need for Integer Programming
Out of Syllabus

Ques 6) Define two person zero sum game.

Ans: Two Person Zero-Sum Game
Refer Unit-3, Page No. 157

Ques 7) What is a decision tree?

Ans: Decision Trees
Refer Unit-3, Page No. 149

Ques 8) Find EOQ if annual demand 15000 units, ordering cost ₹125/ order and carrying cost ₹15/ unit/ year.

Ans: Refer Unit-4, Page No. 179

Ques 9) Find P_0 for (M/M/C): (GD/∞/∞) model if
λ = Arrival rate = 18/hour
μ = Service rate = 6/hour and C = 4

Ans: Refer Unit-5, Page No. 227

Ques 10) Distinguish between breakdown maintenance and preventive maintenance.

Ans: Difference between Breakdown Maintenance and Preventive Maintenance

Breakdown Maintenance	Preventive Maintenance
Breakdown maintenance is done only after the equipment or item breaks down completely.	Preventive maintenance is the periodical inspection to detect and prevent failures before they occur.
Breakdown maintenance cost is low.	Preventive maintenance cost is more as compared to the breakdown maintenance cost.

PART - B (5×13=65)

Ques 11 a) Solve the following LPP:

Maximise $z = 10x_1 + 15x_2 + 20x_3$

S.t. $2x_1 + 4x_2 + 6x_3 \leq 24$

$3x_1 + 9x_2 + 6x_3 \leq 30$

$x_1, x_2, x_3 \geq 0$

Ans: Refer Unit-1, Page No. 34

Or

Ques 11 b) Solve the following LPP using the result of its dual problem

Maximise $z = x_1 - x_2$

S.t. $2x_1 - x_2 \geq 2$

$-x_1 + x_2 \geq 1$

$x_1, x_2, \geq 0$

Ans: Out of Syllabus

Ques 12 a) Obtain an optimal solution to the following transportation problem by U-V method. Use VAM to get the starting BFS.

From \ To	I	II	III	IV	Supply
A	19	30	50	10	7
B	70	30	40	60	9
C	40	8	70	20	18
Demand	5	8	7	14	

Ans: Refer Unit-2, Page No. 76

Or

Ques 12 b) Solve the following assignment problem:

Cost Matrix Table

Jobs \ Machines	A	B	C	D	E
1	11	17	8	16	20
2	9	7	12	6	15
3	13	16	15	12	16
4	21	24	17	28	26
5	14	10	12	11	13

Ans: Refer Unit-2, Page No. 107

Ques 13 a) The non-integer optimal solution of an LPP is given below. Find the all integer optimal solution by Gomory's algorithm.

	C_j	5	8	0	0	
C_B	Basic Variable	x_1	x_2	s_1	s_2	Solution
8	x_2	0	1	4/7	–1/7	22/7
5	x_1	1	0	–1/7	2/7	12/7
	$C_j - z_j$	1	0	–27/7	–2/7	236/7

Ans: Out of Syllabus

Or

Ques 13 b) Consider the payoff matrix with respect to the player A as shown in table. Solve this game optimally using graphical method.

Table

	1	2	3	4	5
1	4	2	1	7	3
2	2	7	8	1	5

Ans: Refer Unit-3, Page No. 164

Ques 14 a) Annual demand for an item is 6000 units. Ordering cost is ₹600 per order. Inventory carrying cost is 18% of the purchase price/unit/year. The price break-ups are as shown below:

Quantity	Price (in ₹) per Unit
$0 \le Q_1 \le 2000$	20
$2000 \le Q_2 < 4000$	15
$4000 \le Q_3$	9

Find the optional order size.

Ans: Refer Unit-4, Page No. 188

Or

Ques 14 b) A newspaper boy has the probability of selling a magazine as shown in table:

No. of Copies Sold	9	10	11	12	13	14
Probability	0.05	0.1	0.15	0.3	0.25	0.15

The cost of a copy sold is ₹30 and the sale price of the magazine is ₹40. The unsold copies fetch a salvage value of ₹5 in the second sale market. How many copies should be ordered to maximise the gain? Use EOL criterion to solve the problem.

Ans: Refer Unit-3, Page No.143

Ques 15 a) There are four booking counters in a railway station, the arrival rate of customers follows Poisson distribution and it is 30 per hour. The service rate also follows Poisson distribution and it is 10 customers per hour. Find the following:
i) Average number of customers waiting in the system.
ii) Average waiting time of a customer in the system.

Ans: Refer Unit-5, Page No. 228

Or

Ques 15 b) An electronic equipment contains 500 resistors. When any resistor fails, it is replaced. The cost of replacing a resistor individually is ₹20. If all the resistors are replaced at the same time, the cost per resistor is ₹5. The percentage of surviving S(i) at the end of month i is given in table.

Table: Per cent Survival Rate

Month i:	0	1	2	3	4	5
S(i):	100	90	75	55	30	0

What is the optimum replacement plan?

Ans: Refer Unit-4, Page No. 206

PART - C (1x15=15)

Ques 16 a) An advertising agency wishes to reach two types of audiences – Customers with annual income greater than ₹15,000 (target audience A) and customers with annual income less than ₹15,000 (target audience B). The total advertising budget is ₹2,00,000. One programme of T.V. advertising cost ₹50,000, one programme on radio advertising ₹20,000. For contract reasons, atleast three programmes ought to be on T.V. and the number of radio programmes must be limited to five. Surveys indicate that a single T.V. programme reaches 4,50,000 customers in target audience A and 50,000 in target audience B. One radio programme reaches 20,000 in target audience A and 80,000 in target audience B. Determine the media mix to minimise the total reach. Use simplex method to solve the problem.

Ans: Refer Unit-1, Page No. 40

Or

Ques 16 b) A person has two independent investment – A and B available to him but he can undertake only one at a time due to certain constraints. He can choose A first and then stop or if A is successful, then take B or *vice versa*. The probability of success of A is 0.6 while for B it is 0.4. Both the investments require an initial capital outlay of ₹10,000 and both return nothing if the venture is unsuccessful. Successful completion A will return ₹20,000 (over cost) and successful completion of B will return ₹24,000 (over cost). Draw decision tree and determine the best strategy.

Ans: Refer Unit-3, Page No. 152

ANNA UNIVERSITY, CHENNAI

MBA - SECOND SEMESTER EXAMINATION, APRIL/MAY - 2017

APPLIED OPERATIONS RESEARCH

Time: 3 Hours **Max. Marks: 100**

Note: Answer All questions.

PART - A (10×2=20)

Ques 1) Define the following terms – Basic variables, artificial variables.

Ans: Basic Variables – Refer Unit-1, Page No. 15

Artificial Variables – Refer Unit-1, Page No. 31

Ques 2) Write the dual of the following LP problem:
Maximise $z = 5x_1 + 6x_2$
Subject to $4x_1 + 7x_2 = 20$
$5x_1 + 2x_2 = 10$
$6x_1 + 8x_2 = 25$
$x_1, x_2 \geq 0$

Ans: Refer Unit-1, Page No. 49, Example 52

Ques 3) Write a linear programming model of the transportation problem.

Ans: Linear Programming Model/Mathematical Model of the Transportation Problem
Refer Unit-2, Page No. 57

Ques 4) What is/are the optimality criterion of the assignment problem?

Ans: Optimality Criterion of the Assignment Problem
Refer Unit-2, Page No. 106

Ques 5) Define pure and mixed integer programming problem.

Ans: Pure Integer Programming Problem
Out of Syllabus

Mixed Integer Programming Problem
Out of Syllabus

Ques 6) Find the value of the following game:

		B 1	B 2
A	1	6	9
	2	8	4

Ans: Refer Unit-3, Page No. 160, Example 32

Ques 7) List the various approaches for decision under uncertainty.

Ans: Approaches for Decision under Uncertainty
Refer Unit-4, Page No. 135

Ques 8) Define the following terms – Lead time, shortage costs.

Ans: Lead Time – Refer Unit-4, Page No. 180

Shortage Costs – Refer Unit-4, Page No. 177

Ques 9) State the Little's formula in queuing theory.

Ans: Little's Formula in Queuing Theory
Refer Unit-5, Page No. 217

Ques 10) Distinguish between breakdown maintenance and preventive maintenance.

Ans: Difference between Breakdown Maintenance and Preventive Maintenance
Refer Solved Paper 2016, Page No. 276

PART - B (5×13=65)

Ques 11 a) i) Solve the following LPP using the Two-phase method. (10)
Minimise $Z = 10x_1 + 6x_2 + 2x_3$
Subject to $-x_1 + x_2 + x_3 \geq 1$
$3x_1 + x_2 - x_3 \geq 2$
and $x_1, x_2, x_3 \geq 0$

Ans: Refer Unit-1, Page No. 44

Ques 11 a ii) Solve the following LPP using graphical method (03)
Maximise $z = 100x_1 + 80x_2$
Subject to $5x_1 + 10x_2 \leq 50$
$8x_1 + 2x_2 \geq 16$
$3x_1 - 2x_2 \geq 6$
and $x_1, x_2 \geq 0$

Ans: Refer Unit-1, Page No. 27

Or

Ques 11 b) Use duality to solve the following LPP
Minimise $Z = 24x_1 + 30x_2$
Subject to $2x_1 + 3x_2 \geq 10$
$4x_1 + 9x_2 \geq 15$
$6x_1 + 6x_2 \geq 20$
and $x_1, x_2 \geq 0$

Ans: Out of Syllabus

Ques 12 a) Consider the following transhipment problem with two sources and three destinations. The unit cost of transportation between different possible nodes is given in the following table. Find the optimal shipping plan such that the total cost is minimised.

		Destination					
		S_1	S_2	D_1	D_2	D_3	Supply
	S_1	0	3	12	4	12	800
	S_2	5	0	3	6	10	700
	D_1	8	10	0	4	20	–
Source	D_2	20	12	5	0	15	–
	D_3	8	10	30	8	0	–
	Demand	–	–	500	400	600	

Ans: Refer Unit-2, Page No. 100

Or

Ques 12 b) The following table represents the factory capacities, store requirements an unit cost (in rupee) of shipping from each factory to each store. Find the optional transportation plan so as to minimise the transportation cost.

		Stores							
		S_1	S_2	S_3	S_4	S_5	S_6	S_7	Factory Capacity
	F_1	5	6	4	3	7	5	4	7000
Factory	F_2	9	4	3	4	3	2	1	4000
	F_3	8	4	2	5	4	8	3	10000
	Store Demand	1500	2000	4500	4000	2500	3500	3000	

Ans: Refer Unit-2, Page No. 82

Ques 13 a) Solve the following integer linear programming problem optimally using Branch-and-Bound technique:

Maximise $z = 6x_1 + 8x_2$

Subject to $4x_1 + 5x_2 \le 22$

$5x_1 + 8x_2 \le 30$

$x_1, x_2 \ge 0$ and integers.

Ans: Out of Syllabus

Or

Ques 13 b) Consider the payoff matrix of player A as shown in table below and solve it optimally using the graphical method.

		Player B				
		1	2	3	4	5
Player A	1	3	6	8	4	4
	2	–7	4	2	10	2

Ans: Refer Unit-3, Page No. 165

Ques 14 a) The demand for an item is 6,000 units per year. Its production rate is 1,000 units per months. The carrying cost is ₹50/unit/year and the set-up cost is ₹2,000 per set-up. The shortage cost is ₹1,000 per unit per year. Find economic batch quantity, maximum inventory, maximum stock out and the period of shortage.

Ans: Refer Unit-4, Page No. 186

Or

Ques 14 b) Consider the following data in daily not profit, find the best order size based on the:

i) Maximum criterion

ii) Savage minimax regret criterion and

iii) Hurwicz criterion Take ($\alpha = 0.4$)

		Demand				
		50	100	150	200	250
Order Size	75	950	1200	575	–675	–1425
	150	50	1700	2000	2250	1600
	225	–850	850	2550	3550	4525
	300	–1800	600	1800	2000	5000

Ans: Refer Unit-3, Page No. 137

Ques 15 a) Ships arrive at a port at the rate of one in every 4 hours with exponential distribution of interarrival times. The time a ship occupies a berth for unloading has exponential distribution with an average of 10 hours. If the average delay of ships waiting for berths is to kept below 14 hours, how many berths should be provided at the port?

Ans: Refer Unit-5, Page No. 228

Or

Ques 15 b) A factory has a large number of bulbs, all of which must be in working condition. Mortality rate is given below:

Week	1	2	3	4	5	6
Proportion which Failed	0.10	0.15	0.25	0.35	0.12	0.03

If a bulb fails in service, it costs ₹3.50 to replace; if all bulbs are replaced at a time, it costs ₹1.20 each. Find optimum replacement policy.

Ans: Refer Unit-4, Page No. 207

PART - C (1×15=15)

Ques 16) For a company engaged in the manufacture of three products X, Y and Z, the available data are given below. Determine the product mix to maximise the profit.

Operations	Time in Hours Required per Unit of Manufacturing			Total Available Hours per Month
	X	Y	Z	
1	1	2	2	200
2	2	1	1	220
3	3	1	2	180
Profit/Units (₹)	10	15	8	
Minimum sales requirements/month in units	10	20	30	

Ans: Refer Unit-1, Page No. 34

ANNA UNIVERSITY, CHENNAI

MBA - SECOND SEMESTER EXAMINATION, APRIL/MAY - 2018

APPLIED OPERATIONS RESEARCH

Time: 3 Hours

Max. Marks: 100

Note: Answer All questions

PART-A (10×2=20)

Ques 1) How will you solve an LPP graphically?

Ans: Steps of Graphical Solution
Refer Unit 1, Page No. 19

Ques 2) What is the role of Slack variables and Surplus variables in the simplex method?

Ans: Slack and Surplus Variables
Refer Unit 1, Page No. 31

Ques 3) Find the basic feasible solution of the following transportation problem.

	1	2	3	4	Supply
1	2	3	11	7	6
2	1	0	6	1	1
3	5	8	15	9	10
Demand	7	5	3	2	

Ans: Refer Unit 2, Page No.67

Ques 4) Write the mathematical formulation of travelling salesman problem.

Ans: Mathematical Formulation of Travelling Salesman Problem
Refer Unit 2, Page No. 123

Ques 5) Write the some of the applications of Integer Programming.

Ans: Out of Syllabus

Ques 6) Find the optimal strategies and game value of the following game.

		Player B H	Player B T
Player A	H	2	−1
	T	−1	0

Ans: Refer Unit 3, Page No. 167, Example 18

Ques 7) Explain the following terms in inventory management.
i) Carrying Cost
ii) Shortage Costs.

Ans:
i) **Carrying Cost:** Refer Unit 4, Page No. 183
ii) **Shortage Cost:** Refer Unit 4, Page No. 183

Ques 8) Differentiate between Decision Making under conditions of uncertainty and risk.

Ans: Difference between Decision Making under Uncertainty and Risk
Refer Unit 4, Page No. 220

Ques 9) In a bank, 20 customers on the average, are served by a cashier in an hour. If the service time has exponential distribution, what is the probability that it will take more than 10 minutes to serves a customer?

Ans: Refer Unit 5, Page No. 246, Example 10

Ques 10) Distinguish between Individual Replacement and Group Replacement policies.

Ans: Difference between Individual and Group Replacement Policies
Refer Unit 5, Page No. 269

PART-B (5×13=65)

Ques 11 a) Use duality to solve the following LPP.
Minimise $Z = x_1 - x_2$
Subject to $2x_1 - x_2 \geq 2$
$-x_1 + x_2 \geq 1$
and $x_1, x_2 \geq 0$.

Ans: Out of Syllabus

Or

Ques 11 b) Solve the following LPP:
Maximise $Z = 1000x_1 + 4000x_2 + 5000x_3$
Subject to $3x_1 + 3x_3 \leq 22$
$x_1 + 2x_2 + 3x_3 \leq 14$
$3x_1 + 2x_2 \leq 14$
and $x_1, x_2 \geq 0$.

Ans: Refer Unit 1, Page No. 35

Ques 12 a) A company manufacturing air-coolers has two plants located at X and Y places with a capacity of 200 units and 100 units per week respectively. The company supplies the air-coolers to its four showrooms situated at A B, C and D which have a maximum demand of 75, 100, 100 and 30 units respectively. Due to the differences in raw material cost and

transportation costs, the profit per unit in rupees differs which is shown in the table below. Plan the production programme so as to maximise the profit. The company may have its production capacity at both plants partly or wholly unused.

	A	B	C	D
X	90	90	100	110
Y	50	70	130	85

Ans: Refer Unit 2, Page No. 92

Or

Ques 12 b) i) A company has a team of four salesman and there are four districts where the company wants to start its business. After taking into account the capabilities of salesman and the nature of districts, the company estimates that the profit per day in rupees for each salesman in each district is as below:

		District 1	2	3	4
Salesman	A	16	10	14	11
	B	14	11	15	15
	C	15	15	13	12
	D	13	12	14	15

Find the assignment of salesman to various districts which will yield maximum profit.

Ans: Refer Unit 2, Page No. 119

Ques 12 b) ii) Explain stepping stone methods for checking the solution for optimality in transportation problems.

Ans: Stepping Stone Method
Refer Unit 2, Page No. 69

Ques 13 a) Solve the following integer linear programming problem by using Gomory fractional cutting plane algorithm.
Maximise $Z = -4x_1 + 5x_2$
Subject to $-3x_1 + x_2 \le 6$
$2x_1 + 4x_2 \le 12$
and $x_1, x_2 \ge 0$ and integers.

Ans: Out of Syllabus

Or

Ques 13 b) i) Solve the following game by graphical method. (08)

		Player B 1	2	3
Player A	1	6	4	3
	2	2	4	8

Ans: Refer Unit 3, Page No. 165

Ques 13 b) ii) Reduce the following game by dominance rule and hence find the game value. (05)

		Player B 1	2	3	4
Player A	1	3	2	4	0
	2	3	4	2	4
	3	4	2	4	0
	4	0	4	0	8

Ans: Refer Unit 3, Page No. 161

Ques 14 a) i) A steel manufacturing company is concerned with the possibility of a strike. It will cost an extra ₹20,000 to acquire an adequate stockpile. If there is a strike and the company has not stockpiled, management estimates an additional expense of ₹60,000 on account of lost sales should the company stockpile or not if it is to use (08)
a) Optimistic criterion
b) Savage criterion
c) Hurwicz criterion for a = 0.4
d) Laplace criterion.

Ans: Refer Unit 3, Page No. 138

Ques 14 a) ii) Compute the EOQ and the total variable cost for the following: (05)
Annual Demand: 25 units
Unit Price: ₹2.50
Order Cost: ₹4.00
Storage Rate: 1% per year
Interest Rate: 12% per year
Obsolescence Rate: 7% per year.

Ans: Refer Unit 4, Page No. 182

Or

Ques 14 b) Find the optimal order quantity for a product for which the price breaks are as follows:

Quantity (q)	Unit Cost (₹)
$0 < q < 500$	10
$500 \le q < 750$	9.25
$750 \le q$	8.75

The monthly demand for the product is 200 units. Storage costs are 2% of the unit cost and cost of ordering is ₹100.

Ans: Refer Unit 4, Page No. 191

Ques 15 a) A manufacturer is offered two machines A and B. A has cost price of ₹2,500 its running cost is ₹400 for each of the first 5 years and increases by ₹100 every subsequent year. Machine B, having the same capacity as A, cost ₹1,250, has running cost of ₹600 for 6 years, increasing by ₹100 per year thereafter. If money is worth 10% per year, which machine should be purchased?

Ans: Refer Unit 4, Page No. 201

Or

Ques 15 b) i) Semi-finished components arrive at a workstation of an assembly line at an average rate of 2 per minute, poisson distributed. A machine is to be installed at this workstation for the specific operation. Three alternative machine P, Q and R are available. The characteristics of the machines are given below. Whenever a component is idle awaiting the machine to get free, the cost is estimated at ₹18 per minute. Using the concept of single-channel queue system and considering all relevant costs, recommend the machine that would be the best for this work station.

Machine	P	Q	R
Fixed Costs (₹/min.)	36	60	90
Variable Costs (₹/min.)	18	15	8
Processing Rate (Unit/min.)	3	6	12

Ans: Refer Unit 5, Page No. 217

Ques 15 b) ii) Derive the steady state probabilities for (M|M|C): (G_D/N/∞), where C > 1.

Ans: Derivation of (M|M|C): (GD/N/∞) Queuing Model
Refer Unit 5, Page No. 223

PART-C **(1×15=15)**

Ques 16 a) ABC printing company is facing a tight financial squeeze and is attempting to cut costs wherever possible. At present it has only one printing contract and luckily, the book is selling well in both the handcover and the paperback editions. It has just received a request to print more copies of this book in either the handcover or the paper back form. The printing cost for the handcover books is ₹600 per 100 books while that for paperback is only ₹500 per 100. Although the company is attempting to economise, it does not wish to lay-off any employee. Therefore, it feels obliged to run its two printing presses-I and II, at least 80 and 60 hours per week respectively. Press I can produce 100 handcover books in 2 hours or 100 paperback books in 1hour. Press II can produce 100 handcover books in 1 Hour or 100 paperbacks books in 2 hours. Determine how many books of each type should be printed in order to minimise cost. (Use Big M method).

Ans: Refer Unit 1, Page No. 39

Or

Ques 16 b) An insurance company has three claim adjusters in its branch office. People with claims against the company are found to arrive in a Poisson fashion, at an average rate of 20 per 8-hour day. The amount of time that an adjuster spends with a claimant is found to have exponential distribution with a mean service time of 40 minutes. Claimants are processed in the order of their appearance.

i) How many hours a week can an adjuster expect to spend with claimants?

ii) How much time, on an average, does a claimant spend in the branch office?

Ans: Refer Unit 5, Page No. 227

ANNA UNIVERSITY, CHENNAI

MBA - SECOND SEMESTER EXAMINATION, APRIL/MAY - 2019

APPLIED OPERATIONS RESEARCH

Time: 3 Hours **Max. Marks: 100**

Note: Answer All Questions.

PART-A **(10×2=20)**

Ques 1) What is a feasible solution in LPP?

Ans: Feasible Solution in LPP
Refer Unit 1, Page No. 15

Ques 2) The dual of the dual problem is the primal problem. Why?

Ans: Refer Unit 1, Page No. 50

Ques 3) What is an unbalanced assignment problem?

Ans: Unbalanced (Non Square Matrix) Assignment Problem
Refer Unit 2, Page No. 114

Ques 4) What is a pure transient node in a transshipment problem?

Ans: Pure Transient Node
Refer Unit 2, Page No. 98

Ques 5) Why is it called a zero sum game in game theory?

Ans: Refer Unit 3, Page No. 157

Ques 6) When can the graphical solution be applied in an IPP?

Ans: Out of Syllabus

Ques 7) What is meant by a quantity discount?

Ans: Refer Unit 4, Page No. 178

Ques 8) What is risk?

Ans: Risk
Refer Unit 4, Page No. 139

Ques 9) Mention any two queuing rules used.

Ans: Refer Unit 5, Page No. 211

Ques 10) What is discounted operating cost?

Ans: Refer Unit 5, Page No. 198

PART-B **(5×13=65)**

Ques 11 a) Solve the following LPP
Maximise $Z = 2x_1 + x_2$

Subject to: $x_1 + 2x_2 \le 10$,
$x_1 + x_2 \le 6$,
$x_1 - x_2 \le 2$,
$x_1 - 2x_2 \le 1$,
$x_1, x_2 \le 0$.

Ans: Refer Unit 1, Page No. 20, Example 8

Or

Ques 11 b) Solve the following LPP
Minimise: $Z = 3x_1 + 2x_2$
Subject to: $5x_1 + x_2 \ge 10$,
$x_1 + x_2 \ge 6$,
$x_1 + 4x_2 \ge 12$,
$x_1, x_2 \ge 0$.

Ans: Refer Unit 1, Page No. 24

Ques 12 a) Minimise the transportation cost,

		Destination				Supply
		A	B	C	D	
Source	I	19	30	50	10	7
	II	70	30	40	60	9
	III	40	8	70	20	18
Demand		5	8	7	14	34

Ans: Refer Unit 2, Page No. 76

Or

Ques 12 b) Assign Jobs to Men

		Men				
		I	II	III	IV	V
Jobs	A	2	9	2	7	1
	B	6	8	7	6	1
	C	4	6	5	3	1
	D	4	2	7	3	1
	E	5	3	9	5	1

Ans: Refer Unit 2, Page No. 120

Ques 13 a) Solve the following IPP.
Minimise $Z = 3x_1 + 2.5x_2$

Subject to: $x_1 + 2x_2 \ge 20$,
$3x_1 + 2x_2 \ge 50$,
$x_1, x_2 \ge 0$ and integers.

Ans: Out of Syllabus

Or

Ques 13 b) Solve the following game for the payoff matrix shown below:

		Player B		
		B1	B2	B3
Player A	A1	20000	30000	60000
	A2	45000	45000	30000

Ans: Refer Unit 3, Example 22, Page No. 163

Ques 14 a) Find the optimum order quantity of an item for which the price breaks are as follows. The monthly demand for the item is 400 units, the cost of storage is 20% of the unit cost and ordering cost is Rupees 50 per order.

Quantity	Purchasing Cost
0 – 100	200
101 – 200	180
Above 200	160

Ans: Refer Unit 4, Page No. 190

Ques 14 b) The demand for a bakery product is tabulated below based on previous data.

Daily Demand	0	10	20	30	40	50
Probability	0.01	0.20	0.15	0.50	0.12	0.02

Using the following random numbers simulate the demand for 10 days. Random numbers: 25, 39, 65, 76, 12, 05, 73, 89, 19, 49.

Ans: Refer Unit 4, Page No. 238

Ques 15 a) A car washing unit has two cleaning bays manned by a three-man crew.

Cars arrive at an average rate of 10 cars per hour and the arrival rate is Poisson distributed. The under chassis cleaning of a car takes 4 minutes on an average and can be assumed to be exponentially distributed. Determine the:

i) Probability that a customer has to wait before being served,

ii) Expected percentage of idle time for each bay, and

iii) What is the expected waiting time for a car?

Ans: Refer Unit 5, Page No. 226

Or

Ques 15 b) A pipeline is due for repairs. The repair would cost ₹10,000 and would last for three years, alternatively, a new pipeline can be laid at a cost of ₹30,000 which would last for 10 years. Assuming the interest rate to be 10% and ignoring salvage value, which is better alternative?

Ans: Refer Unit 4, Page No. 199

PART-C (1×15=15)

Ques 16 a) The Owner of a bus company is planning to provide accommodation for his crew. He has five buses which ply between Chennai and Coimbatore with three crew members in each return trip. The seating capacity in each bus is 50. The crew can either stay in Chennai or in Coimbatore. Suggest an appropriate decision model for this case where the crew can have a home to reside or a temporary place to stay during a trip. Show an illustration with hypothetical data. Make and state the assumptions regarding the time schedules of the trips.

Ans: Refer Unit 2, Page No. 127

Or

Ques 16 b) Mr. Senthil has ₹10,000 invest in one of the three options. A, B or C. The return on his investment depends on whether the economy experiences inflation, recession or no change at all. His possible returns under each economic condition are given below:

	State of Nature		
Strategy	Inflation	Recession	No change
A	2000	1200	1500
B	3000	800	1000
C	2500	1000	1800

What should he decide using the:

i) Maximax Criterion,

ii) Maximin Criterion,

iii) Regret Criterion,

iv) Hurwicz Criterion ($\alpha = 0.5$), and

v) Laplace Criterion?

Ans: Refer Unit 3, Page No. 138

ANNA UNIVERSITY, CHENNAI

MBA - SECOND SEMESTER EXAMINATION, NOVEMBER/DECEMBER-2020

APPLIED OPERATIONS RESEARCH

Time: 3 Hours **Max. Marks: 100**

Note: Answer All Questions

PART-A (10×2=20)

1) Define the following terms – feasible region, optimal feasible solution of Linear Programming Problem.

2) Find the dual problem of the following problem.
Minimise $z = 3x_1 - 2x_2 + 4x_3$

Subject to $3x_1 + 5x_2 + 4x_3 \geq 7$
$6x_1 + x_2 + 3x_3 \geq 4$
$7x_1 - 2x_2 - x_3 \leq 10$
$x_1 + 2x_2 + 5x_3 \geq 3$
$4x_1 + 7x_2 - 2x_3 \geq 2$
$x_1, x_2, x_3 \geq 0$

3) Explain the following in the context of transportation problem. Degenerate transportation problem.

4) Give the Mathematical formulation of an assignment problem.

5) Write some of the practical applications of integer programming.

6) The payoff matrix of a game is given below. Find the solution of the game.

	B				
A	−4	−2	−2	3	1
	1	0	−1	0	0
	−6	−5	−2	−4	4
	3	1	−6	0	−8

7) Write short notes on Laplace criterion.

8) A particular item has a demand of 9000 units/year. The cost of one procurement is ₹100 and the holding cost per unit is ₹2.40 per year. The replacement is instantaneous and no shortages are allowed. Determine the economic lot size.

9) Distinguish between Individual Replacement and Group Replacement policies.

10) Workers come to tool store room to receive special tools for accomplishing a particular project assignment to them. The average time between two arrivals is 60 seconds and the arrivals are assumed to be in Poisson distribution. The average service time is 40 seconds. Determine the mean waiting time of an arrival.

PART-B (5×13=65)

11) a) Use the Simplex Method to solve the following LPP: Maximise $Z = 2x_1 + x_2$. Subject to the constraints $2x_1 - x_2 \leq 8$; $2x_1 + x_2 \leq 12$; $-x_1 + x_2 \leq 3$ and $x_1, x_2 \geq 0$. Find also an alternative optimal solution, if it exists.

Or

b) Solve the following LP problem by the Dual Simplex Method.
Minimise $z = 3x_1 + 5x_2 + 6x_3 + 7x_4$
Subject to the constraints:
$x_1 + x_2 + 2x_3 + 5x_4 \geq 8$
$5x_1 - 6x_2 + x_4 \geq 10$
$-3x_1 + 4x_2 + 5x_3 + 6x_4 \geq 12$
and $x_1, x_2, x_3, x_4 \geq 0$

12) a) Solve the following transportation problem. Use VAM to find the initial basic feasible solution.

	1	2	Supply
A	1	0	20
B	2	4	10
C	5	2	15
D	6	0	10
Demand	10	15	

or

i) A car hire company has one car at each of 6 depots A, B, C, D, E, F. A customer in each of 6 towns 1, 2, 3, 4, 5, 6 requires a car. The distance (in kms.) between the depots and the towns is given in the following distance matrix. How should the cars be assigned to the towns so as to minimise the distance travelled? (05)

	1	2	3	4	5	6
A	20	41	50	30	30	72
B	23	20	16	29	30	19
C	33	27	42	38	40	35
D	22	22	41	50	29	17
E	40	71	55	39	19	12
F	41	15	42	29	62	31

ii) ABC Ice-Cream Company has a distribution depot in greater Chennai for distributing ice cream in Madurai. There are four vendors located in different parts of Madurai who have to be supplied ice-cream everyday. The following matrix displays the distances (in kilometers) between the depot and the four vendors.

	To				
	Debot Vendor	A	B	C	D
Debot From Vendor	–	3.5	3	4	2
A	3.5	–	4	2.5	3
B	3	4	–	4.5	3.5
C	4	2.5	4.5	–	4
D	2	3	3.5	4	–

What route should the company van follow so that the total distance travelled is minimised? (08)

13) a) Solve the following problem by using the Branch and Bound method.

Minimise $z = 2x_1 + 3x_2$
Subject to the constraints.
$6x_1 + 5x_2 \leq 25$
$x_1 + 3x_2 \leq 10$
$x_1, x_2 \geq 0$ and integers

Or

b) i) Solve the following game after reducing it to a 2 × 2 game. (05)

	B_1	B_2	B_3
A_1	1	7	2
A_2	6	2	7
A_3	5	1	6

b) ii) Solve the following game graphically. (08)

	A_1	A_2	A_3	A_4	A_5
B_1	1	4	9	–3	2
B_2	2	5	–7	–4	1

14) a) A businessman has two independent investment portfolios A and B, available to him but he lacks the capital to undertake both of them simultaneously. He can either choose A first and then stop or if A is not successful then take B or *vice versa*. The probability of success of A is 0.6 while for B it is 0.4. Both investment schemes require an initial capital outlay of ₹10,000 and both return nothing if the venture proves to be unsuccessful. Successful completion of A will return ₹20,000 (over cost) and successful completion of B will return ₹24,000 (over cost). Draw a decision tree in order to determine the best strategy.

Or

b) A dealer supplies you the following information with regard to a product that he deals in:
Annual demand = 10000 units
Ordering cost = ₹10 per order
Price = ₹20 per unit
Inventory carrying cost = 20 per cent of the value of inventory per year.

The dealer is considering the possibility of allowing some backorder to occur. He has estimated that the annual cost of backordering will be 25 per cent of the value of inventory.

i) What should be the optimum number of units of the product he should buy in one lot? (03)
ii) What quantity of the product should be allowed to be backordered, if any? (03)
iii) What would be the maximum quality of inventory at any time of the year? (03)
iv) Would you recommend to allow backordering? If so, what would be the annual cost saving by adopting the policy of backordering. (04)

15) a) A Petroleum Company is considering expansion of its one unloading facility at its refinery. Due to random variation in weather, loading delays and other factors – Ships arriving at the refinery to unload crude oil arrive at a rate of 5 ships per week. The service rate is 10 ships per week. Assume arrivals follow a Poisson process and the service time is exponential.

i) Find the average time a ship must wait before beginning to deliver its cargo to the refinery. (03)
ii) If a second berth is rested what will be the average number of ships waiting before being unloaded? (07)
iii) What would be the average time a ship would wait before being unloaded with two berths? (03)

Or

b) A computer contains 10000 resistors. When any resistor fails, it is replaced. The cost of replacing a resistor individually is ₹1 only. If all the resistors are replaced at the same time, the cost per resistor would be reduced to 35 paise. The percentage of servicing resistors say S(t) at the end of month t and the probability of failure P(t) during the month t are as follows:

t:	0	1	2	3	4	5	6
S(t):	100	97	90	70	30	15	0
P(t):	–	0.03	0.07	0.20	0.40	0.15	0.15

What is the optimal replacement plan?

Part-C (1×15=15)

16) a) Solve the following Game by Linear Programming Method.

	B_1	B_2	B_3
A_1	1	7	2
A_2	6	2	7
A_3	5	1	6

Or

b) A company has 3 factories manufacturing the same product and 5 sale agencies in different parts of the country. Production costs differ from factory to factory and the sales prices from agency to agency. The shipping cost per unit product from each factory to each agency is known. Given the following data, find the production and distribution schedules most profitable to the company.

Factory i	Production Cost/Unit ₹	Man. Capacity (No. of Units)
1	18	140
2	20	190
3	16	115

Factory i/j	Agency 1	2	3	4	5	
1	2	2	6	10	5	
2	10	8	9	4	7	Shipping Cost (₹)
3	5	6	4	3	8	
Demand	74	94	69	39	119	
Sales Price (₹)	35	37	36	39	34	

UNIT-WISE CLASSIFICATION OF PREVIOUS YEAR QUESTION PAPERS

UNIT – I

Second Semester Examination – 2015

1) State the limitation of a graphical method.

11 a) i) Solve the following LPP graphically;

Minimise $Z = 3x + 2y;$

Subject to $x - y \le 1,$

$x + y \ge 3$

and $x, y \ge 0.$

11 a) ii) A person wants to decide the constituents of a diet which will fulfil his daily requirements of proteins, fat and c arbohydrates at the minimum cost. The choice is to be made from four different types foods. The yields per unit of these foods are given below:

Food Type	Yield per Unit			
	Proteins	Fats	Carbohydrates	Cost per Unit
1	3	2	6	45
2	4	2	4	40
3	8	7	7	85
4	6	5	4	65
Minimum requirement	800	200	700	

Formulate the LPP for the problem.

11 b) Solve the following LPP Two Phase Simplex Method;

Maximise $Z = -4a - 3b - 9c;$

Subject to $2a + 4b + 6c \ge 15$

$6a + b + 6c \ge 12$

$a, b, c \ge 0$

Second Semester Examination – 2016

1) How to identify from simplex table an LPP has alternate optimum solution?

2) What is the use of artificial variable in LP solution?

11 a) Solve the following LPP:

Maximise $z = 10x_1 + 15x_2 + 20x_3$

S.t. $2x_1 + 4x_2 + 6x_3 \le 24$

$3x_1 + 9x_2 + 6x_3 \le 30$

$x_1, x_2, x_3 \ge 0$

11 b) Solve the following LPP using the result of its dual problem

Maximise $z = x_1 - x_2$

S.t. $2x_1 - x_2 \ge 2$

$-x_1 + x_2 \ge 1$

$x_1, x_2, \ge 0$

16 a) An advertising agency wishes to reach two types of audiences – Customers with annual income greater than ₹15,000 (target audience A) and customers with annual income less than ₹15,000 (target audience B). The total advertising budget is ₹2,00,000. One programme of T.V. advertising cost ₹50,000, one programme on radio advertising ₹20,000. For contract reasons, atleast three programmes ought to be on T.V. and the number of radio programmes must be limited to five. Surveys indicate that a single T.V. programme reaches 4,50,000 customers in target audience A and 50,000 in target audience B. One radio programme reaches 20,000 in target audience A and 80,000 in target audience B. Determine the media mix to minimise the total reach. Use simplex method to solve the problem.

Second Semester Examination – 2017

1) Define the following terms – Basic variables, artificial variables.

2) Write the dual of the following LP problem:

Maximise $z = 5x_1 + 6x_2$

Subject to $4x_1 + 7x_2 = 20$

$5x_1 + 2x_2 = 10$

$6x_1 + 8x_2 = 25$

$x_1, x_2 \ge 0$

11 a) i) Solve the following LPP using the Two-phase method.

Minimise $Z = 10x_1 + 6x_2 + 2x_3$

Subject to $-x_1 + x_2 + x_3 \ge 1$

$3x_1 + x_2 - x_3 \ge 2$

and $x_1, x_2, x_3 \ge 0$

11 a ii) Solve the following LPP using graphical method

Maximise $z = 100x_1 + 80x_2$

Subject to $5x_1 + 10x_2 \le 50$

$8x_1 + 2x_2 \ge 16$

$3x_1 - 2x_2 \ge 6$

and $x_1, x_2 \ge 0$

11 b) Use duality to solve the following LPP

Minimise $Z = 24x_1 + 30x_2$

Subject to $2x_1 + 3x_2 \ge 10$

$4x_1 + 9x_2 \ge 15$

$6x_1 + 6x_2 \ge 20$

and $x_1, x_2 \ge 0$

16) For a company engaged in the manufacture of three products X, Y and Z, the available data are given below. Determine the product mix to maximise the profit.

Operations	Time in Hours Required per Unit of Manufacturing			Total Available Hours per Month
	X	Y	Z	
1	1	2	2	200
2	2	1	1	220
3	3	1	2	180
Profit/Units (₹)	10	15	8	
Minimum sales requirements/month in units	10	20	30	

Second Semester Examination – 2018

1) How will you solve an LPP graphically?

2) What is the role of Slack variables and Surplus variables in the simplex method?

11 a) Use duality to solve the following LPP.

Minimise $Z = x_1 - x_2$

Subject to $2x_1 - x_2 \ge 2$

$-x_1 + x_2 \ge 1$

and $x_1, x_2 \ge 0.$

11 b) Solve the following LPP:

Maximise $Z = 1000x_1 + 4000x_2 + 5000x_3$

Subject to $3x_1 + 3x_3 \le 22$

$x_1 + 2x_2 + 3x_3 \le 14$

$3x_1 + 2x_2 \le 14$

and $x_1, x_2 \ge 0.$

16 a) ABC printing company is facing a tight financial squeeze and is attempting to cut costs wherever possible. At present it has only one printing contract and luckily, the book is selling well in both the handcover and the paperback editions. It has just

received a request to print more copies of this book in either the handcover or the paper back form. The printing cost for the handcover books is ₹600 per 100 books while that for paperback is only ₹500 per 100. Although the company is attempting to economise, it does not wish to lay-off any employee. Therefore, it feels obliged to run its two printing presses-I and II, at least 80 and 60 hours per week respectively. Press I can produce 100 handcover books in 2 hours or 100 paperback books in 1hour. Press II can produce 100 handcover books in 1 Hour or 100 paperbacks books in 2 hours. Determine how many books of each type should be printed in order to minimise cost. (Use Big M method).

Second Semester Examination – 2019

1) What is a feasible solution in LPP?

2) The dual of the dual problem is the primal problem. Why?

11 a) Solve the following LPP

Maximise $Z = 2x_1 + x_2$

Subject to: $x_1 + 2x_2 \le 10$,

$x_1 + x_2 \le 6$,

$x_1 - x_2 \le 2$,

$x_1 - 2x_2 \le 1$,

$x_1, x_2 \le 0$.

11 b) Solve the following LPP

Minimise: $Z = 3x_1 + 2x_2$

Subject to: $5x_1 + x_2 \ge 10$,

$x_1 + x_2 \ge 6$,

$x_1 + 4x_2 \ge 12$,

$x_1, x_2 \ge 0$.

Second Semester Examination – 2020

1) Define the following terms – feasible region, optimal feasible solution of Linear Programming Problem.

2) Find the dual problem of the following problem.

Minimise $z = 3x_1 - 2x_2 + 4x_3$

Subject to $3x_1 + 5x_2 + 4x_3 \ge 7$

$6x_1 + x_2 + 3x_3 \ge 4$

$7x_1 - 2x_2 - x_3 \le 10$

$x_1 + 2x_2 + 5x_3 \ge 3$

$4x_1 + 7x_2 - 2x_3 \ge 2$

$x_1, x_2, x_3 \ge 0$

11 a) Use the Simplex Method to solve the following LPP: Maximise $Z = 2x_1 + x_2$. Subject to the constraints $2x_1 - x_2 \le 8$; $2x_1 + x_2 \le 12$; $-x_1 + x_2 \le 3$ and $x_1, x_2 \ge 0$. Find also an alternative optimal solution, if it exists.

UNIT – II

Second Semester Examination – 2015

3) What is an unbalanced transportation problem?

4) Which cell will be the first basic variable in case of North-West Corner Method and Least Cost Method?

12 a) Solve the following transportation problem to minimise the total transportation cost for shifting the goods from factories (A, B and C) to warehouses (P, Q and R) where unit transportation cost, availability and demand, at factories and warehouses respectively are given in the following matrix:

		Warehouse			
		P	Q	R	Availability
Factory	A	1	2	0	30
	B	2	3	4	35
	C	1	5	6	35
	Demand	30	40	30	

Find the allocation so that the total transportation cost is minimum.

12 b) A company has 4 territories and four salesmen for assignment. The territories are not equally rich in their sales potential. It is estimated that a typical salesman operating in each territory would bring the following annual sales:

Territory	I	II	III	IV
Annual Sales in (₹)	60,000	50,000	40,000	30,000

The four salesmen are also considered to differ in their ability; it is estimated that working under same condition, their yearly sales would be proportionately as follows:

Salesman	A	B	C	D
Proportion	0.1	0.2	0.3	0.4

If the criteria is to maximise expected sales, what is your intuitive answer and verify your answer with Hungarian method.

Second Semester Examination – 2016

3) What is unbalanced transportation problem?

4) What is the Travelling Salesman problem?

12 a) Obtain an optimal solution to the following transportation problem by U-V method. Use VAM to get the starting BFS.

From \ To	I	II	III	IV	Supply
A	19	30	50	10	7
B	70	30	40	60	9
C	40	8	70	20	18
Demand	5	8	7	14	

12 b) Solve the following assignment problem:

Cost Matrix Table

		Machines				
		A	B	C	D	E
Jobs	1	11	17	8	16	20
	2	9	7	12	6	15
	3	13	16	15	12	16
	4	21	24	17	28	26
	5	14	10	12	11	13

Second Semester Examination – 2017

3) Write a linear programming model of the transportation problem.

4) What is/are the optimality criterion of the assignment problem?

12 a) Consider the following transhipment problem with two sources and three destinations. The unit cost of transportation between different possible nodes is given in the following table. Find the optimal shipping plan such that the total cost is minimised.

		Destination					
		S_1	S_2	D_1	D_2	D_3	Supply
Source	S_1	0	3	12	4	12	800
	S_2	5	0	3	6	10	700
	D_1	8	10	0	4	20	–
	D_2	20	12	5	0	15	–
	D_3	8	10	30	8	0	–
	Demand	–	–	500	400	600	

12 b) The following table represents the factory capacities, store requirements an unit cost (in rupee) of shipping from each factory to each store. Find the optional transportation plan so as to minimise the transportation cost.

		Stores							
		S_1	S_2	S_3	S_4	S_5	S_6	S_7	Factory Capacity
Factory	F_1	5	6	4	3	7	5	4	7000
	F_2	9	4	3	4	3	2	1	4000
	F_3	8	4	2	5	4	8	3	10000
	Store Demand	1500	2000	4500	4000	2500	3500	3000	

Second Semester Examination – 2018

3) Find the basic feasible solution of the following transportation problem.

	1	2	3	4	Supply
1	2	3	11	7	6
2	1	0	6	1	1
3	5	8	15	9	10
Demand	7	5	3	2	

4) Write the mathematical formulation of travelling salesman problem.

12 a) A company manufacturing air-coolers has two plants located at X and Y places with a capacity of 200 units and 100 units per week respectively. The company supplies the air-coolers to its four showrooms situated at A B, C and D which have a maximum demand of 75, 100, 100 and 30 units respectively. Due to the differences in raw material cost and transportation costs, the profit per unit in rupees differs which is shown in the table below. Plan the production programme so as to maximise the profit. The company may have its production capacity at both plants partly or wholly unused.

	A	B	C	D
X	90	90	100	110
Y	50	70	130	85

12 b) i) A company has a team of four salesman and there are four districts where the company wants to start its business. After taking into account the capabilities of salesman and the nature of districts, the company estimates that the profit per day in rupees for each salesman in each district is as below:

		District			
		1	2	3	4
Salesman	A	16	10	14	11
	B	14	11	15	15
	C	15	15	13	12
	D	13	12	14	15

Find the assignment of salesman to various districts which will yield maximum profit.

12 b) ii)Explain stepping stone methods for checking the solution for optimality in transportation problems.

Second Semester Examination – 2019

3) What is an unbalanced assignment problem?

4) What is a pure transient node in a transshipment problem?

12 a) Minimise the transportation cost.

		Destination				Supply
		A	B	C	D	
Source	I	19	30	50	10	7
	II	70	30	40	60	9
	III	40	8	70	20	18
Demand		5	8	7	14	34

12 b) Assign Jobs to Men

		Men				
		I	II	III	IV	V
Jobs	A	2	9	2	7	1
	B	6	8	7	6	1
	C	4	6	5	3	1
	D	4	2	7	3	1
	E	5	3	9	5	1

16 a) The Owner of a bus company is planning to provide accommodation for his crew. He has five buses which ply between Chennai and Coimbatore with three crew members in each return trip. The seating capacity in each bus is 50. The crew can either stay in Chennai or in Coimbatore. Suggest an appropriate decision model for this case where the crew can have a home to reside or a temporary place to stay during a trip. Show an illustration with hypothetical data. Make and state the assumptions regarding the time schedules of the trips.

Second Semester Examination – 2020

3) Explain the following in the context of transportation problem. Degenerate transportation problem.

4) Give the Mathematical formulation of an assignment problem.

12 a) Solve the following transportation problem. Use VAM to find the initial basic feasible solution.

	1	2	Supply
A	1	0	20
B	2	4	10
C	5	2	15
D	6	0	10
Demand	10	15	

Or

12 b) i) A car hire company has one car at each of 6 depots A, B, C, D, E, F. A customer in each of 6 towns 1, 2, 3, 4, 5, 6 requires a car. The distance (in kms.) between the depots and the towns is given in the following distance matrix. How should the cars be assigned to the towns so as to minimise the distance travelled? (05)

	1	2	3	4	5	6
A	20	41	50	30	30	72
B	23	20	16	29	30	19
C	33	27	42	38	40	35
D	22	22	41	50	29	17
E	40	71	55	39	19	12
F	41	15	42	29	62	31

12 b) ii) ABC Ice-Cream Company has a distribution depot in greater Chennai for distributing ice cream in Madurai. There are four vendors located in different parts of Madurai who have to be supplied ice-cream everyday. The following matrix displays the distances (in kilometers) between the depot and the four vendors.

	To				
	Debot Vendor	A	B	C	D
Debot From Vendor	–	3.5	3	4	2
A	3.5	–	4	2.5	3
B	3	4	–	4.5	3.5
C	4	2.5	4.5	–	4
D	2	3	3.5	4	–

What route should the company van follow so that the total distance travelled is minimised? (08)

16 b) A company has 3 factories manufacturing the same product and 5 sale agencies in different parts of the country. Production costs differ from factory to factory and the sales prices from agency to agency. The shipping cost per unit product from each factory to each agency is known. Given the following data, find the production and distribution schedules most profitable to the company.

Factory i	Production Cost/Unit ₹	Man. Capacity (No. of Units)
1	18	140
2	20	190
3	16	115

	Agency					
Factory I/j	1	2	3	4	5	
1	2	2	6	10	5	
2	10	8	9	4	7	Shipping Cost (₹)
3	5	6	4	3	8	
Demand	74	94	69	39	119	
Sales Price (₹)	35	37	36	39	34	

UNIT – III

Second Semester Examination – 2015

8) What do the terms 'uncertainty' and 'risk' refer?

14 b) Concisely explain the criterions used to assist decision-making under uncertainty.

6) Differentiate between pure and mixed strategies.

13 b) i) State the rules of dominance.

13 b) ii) Solve the following game:

$$\text{Player A} \overset{\text{Player B}}{\begin{pmatrix} 1 & 7 & 2 \\ 6 & 2 & 7 \\ 5 & 1 & 6 \end{pmatrix}}$$

Second Semester Examination – 2016

6) Define two person zero sum game.

7) What is a decision tree?

13 b) Consider the payoff matrix with respect to the player A as shown in table. Solve this game optimally using graphical method.

Table

	1	2	3	4	5
1	4	2	1	7	3
2	2	7	8	1	5

14 b) A newspaper boy has the probability of selling a magazine as shown in table:

No. of Copies Sold	9	10	11	12	13	14
Probability	0.05	0.1	0.15	0.3	0.25	0.15

The cost of a copy sold is ₹30 and the sale price of the magazine is ₹40. The unsold copies fetch a salvage value of ₹5 in the second sale market. How many copies should be ordered to maximise the gain? Use EOL criterion to solve the problem.

16 b) A person has two independent investment – A and B available to him but he can undertake only one at a time due to certain constraints. He can choose A first and then stop or if A is successful, then take B or *vice versa*. The probability of success of A is 0.6 while for B it is 0.4. Both the investments require an initial capital outlay of ₹10,000 and both return nothing if the venture is unsuccessful. Successful completion A will return ₹20,000 (over cost) and successful completion of B will return ₹24,000 (over cost). Draw decision tree and determine the best strategy.

Second Semester Examination – 2017

7) List the various approaches for decision under uncertainty.

14 b) Consider the following data in daily not profit, find the best order size based on the:

i) Maximum criterion

ii) Savage minimax regret criterion and

iii) Hurwicz criterion Take ($\alpha = 0.4$)

		Demand				
		50	100	150	200	250
Order Size	75	950	1200	575	–675	–1425
	150	50	1700	2000	2250	1600
	225	–850	850	2550	3550	4525
	300	–1800	600	1800	2000	5000

6) Find the value of the following game:

		B	
		1	2
A	1	6	9
	2	8	4

13 b) Consider the payoff matrix of player A as shown in table below and solve it optimally using the graphical method.

		Player B				
		1	2	3	4	5
Player A	1	3	6	8	4	4
	2	–7	4	2	10	2

Second Semester Examination – 2018

8) Differentiate between Decision Making under conditions of uncertainty and risk.

14 a) i) A steel manufacturing company is concerned with the possibility of a strike. It will cost an extra ₹20,000 to acquire an adequate stockpile. If there is a strike and the company has not stockpiled, management estimates an additional expense of ₹60,000 on account of lost sales should the company stockpile or not if it is to use (08)

a) Optimistic criterion

b) Savage criterion

c) Hurwicz criterion for a = 0.4

d) Laplace criterion.

6) Find the optimal strategies and game value of the following game.

		Player B	
		H	T
Player A	H	2	−1
	T	−1	0

13 b) i) Solve the following game by graphical method.

		Player B		
		1	2	3
Player A	1	6	4	3
	2	2	4	8

13 b) ii)Reduce the following game by dominance rule and hence find the game value.

		Player B			
		1	2	3	4
Player A	1	3	2	4	0
	2	3	4	2	4
	3	4	2	4	0
	4	0	4	0	8

Second Semester Examination – 2019

8) What is risk?

16 b) Mr. Senthil has ₹10,000 invest in one of the three options. A, B or C. The return on his investment depends on whether the economy experiences inflation, recession or no change at all. His possible returns under each economic condition are given below:

	State of Nature		
Strategy	Inflation	Recession	No change
A	2000	1200	1500
B	3000	800	1000
C	2500	1000	1800

What should he decide using the:
i) Maximax Criterion,
ii) Maximin Criterion,
iii) Regret Criterion,
iv) Hurwicz Criterion ($\alpha = 0.5$), and
v) Laplace Criterion?

5) Why is it called a zero sum game in game theory?

13 b) Solve the following game for the payoff matrix shown below:

		Player B		
		B1	B2	B3
Player A	A1	20000	30000	60000
	A2	45000	45000	30000

Second Semester Examination – 2020

7) Write short notes on Laplace criterion.

6) The payoff matrix of a game is given below. Find the solution of the game.

	B				
A	−4	−2	−2	3	1
	1	0	−1	0	0
	−6	−5	−2	−4	4
	3	1	−6	0	−8

13 b) i) Solve the following game after reducing it to a 2 × 2 game. (05)

	B_1	B_2	B_3
A_1	1	7	2
A_2	6	2	7
A_3	5	1	6

13 b) ii) Solve the following game graphically. (08)

	A_1	A_2	A_3	A_4	A_5
B_1	1	4	9	−3	2
B_2	2	5	−7	−4	1

14 a) A businessman has two independent investment portfolios A and B, available to him but he lacks the capital to undertake both of them simultaneously. He can either choose A first and then stop or if A is not successful then take B or *vice versa*. The probability of success of A is 0.6 while for B it is 0.4. Both investment schemes require an initial capital outlay of ₹10,000 and both return nothing if the venture proves to be unsuccessful. Successful completion of A will return ₹20,000 (over cost) and successful completion of B will return ₹24,000 (over cost). Draw a decision tree in order to determine the best strategy.

16 a) Solve the following Game by Linear Programming Method.

	B_1	B_2	B_3
A_1	1	7	2
A_2	6	2	7
A_3	5	1	6

UNIT – IV

Second Semester Examination – 2015

7) List the elements of carrying cost.

10) If the money carries an interest rate of 10% per year, what will be the value of one rupee after two years?

14 a) i) Derive EOQ formula for simple inventory model with no shortages and instantaneous replenishment.

14 a) ii) Find the optimum order quantity for a product for which the price break is given below:

Quantity	Unit Cost (in ₹)
$0 \le q1 < 100$	20 per unit
$100 \le q2 < 200$	18 per unit
$200 \le q3$	16 per unit

The monthly demand for the product is 400 units. The storage cost is 20% of the unit cost of the product and the cost of ordering is ₹25.

15 b) A machine costs ₹15,000 and its running costs for different years are given below. Find optimum replacement period if the capital is worth 10% and the machine has no salvage value.

Year	1	2	3	4	5	6	7
Running Cost (₹)	2,500	3,000	4,000	5,000	6,500	8,000	10,000

Second Semester Examination – 2016

8) Find EOQ if annual demand 15000 units, ordering cost ₹125/order and carrying cost ₹15/unit/year.

14 a) Annual demand for an item is 6000 units. Ordering cost is ₹600 per order. Inventory carrying cost is 18% of the purchase price/unit/year. The price break-ups are as shown below:

Quantity	Price (in ₹) per Unit
$0 \le Q_1 \le 2000$	20
$2000 \le Q_2 < 4000$	15
$4000 \le Q_3$	9

Find the optional order size.

10) Distinguish between breakdown maintenance and preventive maintenance.

15 b) An electronic equipment contains 500 resistors. When any resistor fails, it is replaced. The cost of replacing a resistor individually is ₹20. If all the resistors are replaced at the same time, the cost per resistor is ₹5.

The percentage of surviving S(i) at the end of month i is given in table.

Table: Per cent Survival Rate

Month i:	0	1	2	3	4	5
S(i):	100	90	75	55	30	0

What is the optimum replacement plan?

Second Semester Examination – 2017

8) Define the following terms – Lead time, shortage costs.

14 a) The demand for an item is 6,000 units per year. Its production rate is 1,000 units per months. The carrying cost is ₹50/unit/year and the set-up cost is ₹2,000 per set-up. The shortage cost is ₹1,000 per unit per year. Find economic batch quantity, maximum inventory, maximum stock out and the period of shortage.

10) Distinguish between breakdown maintenance and preventive maintenance.

15 b) A factory has a large number of bulbs, all of which must be in working condition. Mortality rate is given below:

Week	1	2	3	4	5	6
Proportion which Failed	0.10	0.15	0.25	0.35	0.12	0.03

If a bulb fails in service, it costs ₹3.50 to replace; if all bulbs are replaced at a time, it costs ₹1.20 each. Find optimum replacement policy.

Second Semester Examination – 2018

7) Explain the following terms in inventory management.
i) Carrying Cost
ii) Shortage Costs.

14 a) ii)Compute the EOQ and the total variable cost for the following: (05)
Annual Demand: 25 units
Unit Price: ₹2.50
Order Cost: ₹4.00
Storage Rate: 1% per year
Interest Rate: 12% per year
Obsolescence Rate: 7% per year.

14 b) Find the optimal order quantity for a product for which the price breaks are as follows:

Quantity (q)	Unit Cost (₹)
$0 < q < 500$	10
$500 \leq q < 750$	9.25
$750 \leq q$	8.75

The monthly demand for the product is 200 units. Storage costs are 2% of the unit cost and cost of ordering is ₹100.

10) Distinguish between Individual Replacement and Group Replacement policies.

15 a) A manufacturer is offered two machines A and B. A has cost price of ₹2,500 its running cost is ₹400 for each of the first 5 years and increases by ₹100 every subsequent year. Machine B, having the same capacity as A, cost ₹1,250, has running cost of ₹600 for 6 years, increasing by ₹100 per year thereafter. If money is worth 10% per year, which machine should be purchased?

Second Semester Examination – 2019

7) What is meant by a quantity discount?

10) What is discounted operating cost?

14 a) Find the optimum order quantity of an item for which the price breaks are as follows. The monthly demand for the item is 400 units, the cost of storage is 20% of the unit cost and ordering cost is Rupees 50 per order.

Quantity	Purchasing Cost
0 – 100	200
101 – 200	180
Above 200	160

15 b) A pipeline is due for repairs. The repair would cost ₹10,000 and would last for three years, alternatively, a new pipeline can be laid at a cost of ₹30,000 which would last for 10 years. Assuming the interest rate to be 10% and ignoring salvage value, which is better alternative?

Second Semester Examination – 2020

8) A particular item has a demand of 9000 units/year. The cost of one procurement is ₹100 and the holding cost per unit is ₹2.40 per year. The replacement is instantaneous and no shortages are allowed. Determine the economic lot size.

9) Distinguish between Individual Replacement and Group Replacement policies.

14 b) A dealer supplies you the following information with regard to a product that he deals in:
Annual demand = 10000 units
Ordering cost = ₹10 per order
Price = ₹20 per unit
Inventory carrying cost = 20 per cent of the value of inventory per year.

The dealer is considering the possibility of allowing some backorder to occur. He has estimated that the annual cost of backordering will be 25 per cent of the value of inventory.

14 b) i) What should be the optimum number of units of the product he should buy in one lot? (03)

14 b) ii) What quantity of the product should be allowed to be backordered, if any? (03)

14 b) iii) What would be the maximum quality of inventory at any time of the year? (03)

14 b) iv) Would you recommend to allow backordering? If so, what would be the annual cost saving by adopting the policy of backordering. (04)

15 b) A computer contains 10000 resistors. When any resistor fails, it is replaced. The cost of replacing a resistor individually is ₹1 only. If all the resistors are replaced at the same time, the cost per resistor would be reduced to 35 paise. The percentage of servicing resistors say S(t) at the end of month t and the probability of failure P(t) during the month t are as follows:

t:	0	1	2	3	4	5	6
S(t):	100	97	90	70	30	15	0
P(t):	–	0.03	0.07	0.20	0.40	0.15	0.15

What is the optimal replacement plan?

UNIT – V

Second Semester Examination – 2015

9) In a store with one cashier, nine customers arrive on the average of every five minutes and the cashier can serve them ten in five minutes. Find utilisation factor.

15 a) i) A T.V. repairman finds that the time he spent on his job has an exponential distribution with mean 30 minutes. If he repairs the set in the order it arrives, and the arrival rate is approximately Poisson, with an average rate of 10 per 8 hour day, what is the expected idle time of repairman each day? How many jobs are ahead of average before the job just brought in?

15 a) ii) A telephone exchange has two long distance operators. The telephone company finds that during the peak load long distance calls arrive in a Poisson fashion at an average rate of 15 per hour.

The length of service on these calls is approximately distributed with mean 5 minutes.

1) What is the probability that subscriber will have to wait for his long distance call during the peak hours of the day?

2) If the subscriber will wait and be serviced in turn, what is the expected waiting time in queue?

Second Semester Examination – 2016

9) Find P_0 for (M/M/C): (GD/∞/∞) model if

λ = Arrival rate = 18/hour

μ = Service rate = 6/hour and C = 4

15 a) There are four booking counters in a railway station, the arrival rate of customers follows Poission distribution and it is 30 per hour. The service rate also follows Poisson distribution and it is 10 customers per hour.

Find the following:

i) Average number of customers waiting in the system.

ii) Average waiting time of a customer in the system.

Second Semester Examination – 2017

9) State the Little's formula in queuing theory.

15 a) Ships arrive at a port at the rate of one in every 4 hours with exponential distribution of interarrival times. The time a ship occupies a berth for unloading has exponential distribution with an average of 10 hours. If the average delay of ships waiting for berths is to kept below 14 hours, how many berths should be provided at the port?

Second Semester Examination – 2018

9) In a bank, 20 customers on the average, are served by a cashier in an hour. If the service time has exponential distribution, what is the probability that it will take more than 10 minutes to serves a customer?

15 b) i) Semi-finished components arrive at a workstation of an assembly line at an average rate of 2 per minute, poisson distributed. A machine is to be installed at this workstation for the specific operation. Three alternative machine P, Q and R are available.

The characteristics of the machines are given below. Whenever a component is idle awaiting the machine to get free, the cost is estimated at ₹18 per minute.

Using the concept of single-channel queue system and considering all relevant costs, recommend the machine that would be the best for this work station.

Machine	P	Q	R
Fixed Costs (₹/min.)	36	60	90
Variable Costs (₹/min.)	18	15	8
Processing Rate (Unit/min.)	3	6	12

15 b) ii) Derive the steady state probabilities for (M/M/C): (G_D/N/∞), where C > 1.

16 b) An insurance company has three claim adjusters in its branch office. People with claims against the company are found to arrive in a Poisson fashion, at an average rate of 20 per 8-hour day. The amount of time that an adjuster spends with a claimant is found to have exponential distribution with a mean service time of 40 minutes. Claimants are processed in the order of their appearance.

i) How many hours a week can an adjuster expect to spend with claimants?

ii) How much time, on an average, does a claimant spend in the branch office?

Second Semester Examination – 2019

9) Mention any two queuing rules used.

15 a) A car washing unit has two cleaning bays manned by a three-man crew.

Cars arrive at an average rate of 10 cars per hour and the arrival rate is Poisson distributed. The under chassis cleaning of a car takes 4 minutes on an average and can be assumed to be exponentially distributed. Determine the:

i) Probability that a customer has to wait before being served.

ii) Expected percentage of idle time for each bay, and

iii) What is the expected waiting time for a car?

14 b) The demand for a bakery product is tabulated below based on previous data.

Daily Demand	0	10	20	30	40	50
Probability	0.01	0.20	0.15	0.50	0.12	0.02

Using the following random numbers simulate the demand for 10 days. Random numbers: 25, 39, 65, 76, 12, 05, 73, 89, 19, 49.

Second Semester Examination – 2020

10) Workers come to tool store room to receive special tools for accomplishing a particular project assignment to them. The average time between two arrivals is 60 seconds and the arrivals are assumed to be in Poisson distribution. The average service time is 40 seconds. Determine the mean waiting time of an arrival.

15 a) A Petroleum Company is considering expansion of its one unloading facility at its refinery. Due to random variation in weather, loading delays and other factors – Ships arriving at the refinery to unload crude oil arrive at a rate of 5 ships per week. The service rate is 10 ships per week. Assume arrivals follow a Poisson process and the service time is exponential.

15 a) i) Find the average time a ship must wait before beginning to deliver its cargo to the refinery. (03)

15 a) ii) If a second berth is rested what will be the average number of ships waiting before being unloaded? (07)

15 a) iii) What would be the average time a ship would wait before being unloaded with two berths? (03)

IMPORTANT QUESTIONS FOR FORTHCOMING EXAMINATION

UNIT-I

Ques 1) Discuss the assumptions and limitations of linear programming. Which are the essential components of a LPP?

Ques 2) What is sensitivity analysis? Explain with example.

Ques 3) A paint company produces both interior and exterior paints from two raw materials, M1 and M2. The following table provides the basic data of the problem – the daily demand for interior paint cannot exceed that of extension point by more than 1 tonne. Also the maximum daily demand interior paint is 2 tonnes. Determine the optimum product mix of interior and exterior paint to maximise the total profit and determine the optimal range of the coefficients, if the objective function is changed to $Z = c_1x_1 + c_2x_2$

	Tonnes of Raw Material Per Tonne on		Maximum Daily Availability (Tonnes)
	Exterior Paint	Interior Paint	
Raw Material M_1	6	4	24
Raw Material M_2	1	2	6
Profit per Tonne (₹1,000)	5	4	

Ques 4) Solve graphically the following LPP:

Maximise $Z = 8x_1 + 16x_2$

Subject to $x_1 + x_2 \le 200$

$x_2 \le 125$

$3x_1 + 6x_2 \le 900$

$x_1, x_2 \ge 0$

Ques 5) Solve the following problem using simplex method

Maximize $Z = 21x_1 + 15x_2$

Subject to $-x_1 - 2x_2 \ge -6$

$4x_1 + 3x_2 \le 12$

$x_1, x_2 \ge 0$

UNIT-II

Ques 1) Use Vogel's Approximation Method to obtain an initial basic feasible solution of the transportation problem:

	D	E	F	G	Available
A	11	13	17	14	250
B	16	18	14	10	300
C	21	24	13	10	400
Demand	200	225	275	250	

Ques 2) Solve the following transportation problem using Vogel's approximation method and check for optimality using MODI method.

Destination / Machine	D_1	D_2	D_3	D_4	Supply
O_1	21	16	25	13	11
O_2	17	18	14	23	13
O_3	32	17	18	41	19
Demand	6	10	12	15	43

Ques 3) Find optimal solution to the following transportation problem:

		To			
		x	y	z	Demand
From	P	18	12	3	150
	Q	9	24	21	120
	R	12	12	6	180
Supply		60	285	105	

Ques 4) Five Salesmen are to be assigned to five Territories based on the past performance; the following table shows the annual sales (in rupees of lac) that can be generated by each salesman in each territory. Find the optional assignment.

	Territory				
Salesmen	T_1	T_2	T_3	T_4	T_5
S_1	26	14	10	12	9
S_2	31	27	30	14	16
S_3	15	18	16	25	30
S_4	17	12	21	30	25
S_5	20	19	25	16	10

Ques 5) A department has five employees with five jobs to be performed. The time (in hours) each men will take to perform each job is given in the effectiveness matrix. How should the jobs be allocated, so as to minimise the total man-hours?

		Employees				
		I	II	III	IV	V
Jobs	A	10	05	13	15	16
	B	03	09	18	13	06
	C	10	07	02	02	02
	D	07	11	09	07	12
	E	07	09	10	04	12

UNIT-III

Ques 1) The research department of a company has recommended to the marketing department to launch a product of three different types. The marketing manager has to decide one of the types of the product to be launched under the following estimated pay-offs for various level of sales:

Type of Product	Estimated Level of Sales (Units)		
	15,000	10,000	5,000
A	30	10	10
B	40	15	5
C	55	20	3

What will be the marketing manager's decision if:

i) Maximin,

ii) Maximax,

iii) Laplace, and

iv) Regret criteria is applied.

Ques 2) A food product company is contemplating the introduction of a revolutionary new product with new product with new packaging to replace the existing product at much higher price (s1) or a moderate change in the composition of the existing product with a new packaging at a small increase in price (s2) or a small change in the composition of the existing except product the word 'new' with a negligible increase in price (s3). The three possible states of nature of events are:

i) High increase in sales (n1),
ii) No change in sales (n2), and
iii) Decrease in sales (n3).

The marketing department of the company worked out the payoffs in terms of yearly net profits for each course of action for these events (expected sales). This is represented in the following table:

States of Nature	Courses of Action		
	S_1	S_2	S_3
N_1	7,00,000	5,00,000	3,00,000
N_2	3,00,000	4,50,000	3,00,000
N_3	1,50,000	0	3,00,000

Which strategy should the company choose on the basis of:

a) Maximin criterion,
b) Maximax criterion,
c) Minimax regret criterion, and
d) Laplace criterion.

Ques 3) Reduce the following two person zero sum game to 2x2 order and obtain the optimal strategies for each player and the value of the game.

Player B

Player A		B_1	B_2	B_3	B_4
	A_1	3	2	4	0
	A_2	3	4	2	4
	A_3	4	2	4	0
	A_4	0	4	0	8

Ques 4) Solve the game whose pay-off matrix is given by:

Player B

Player B	1	2	1
	0	−4	−1
	1	3	−2

Ques 5) Find the saddle point (or points) and hence solve the following game:

$$\begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \end{array}\begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{bmatrix} 3 & 0 & -3 \\ 2 & 3 & 1 \\ -4 & 2 & -1 \end{bmatrix} \end{array}$$

UNIT-IV

Ques 1) Discuss the individual and group replacement with example.

Ques 2) The annual demand for an automobile component is 36,000 units. The carrying cost is ₹0.50/unit/year, the ordering cost is ₹25.00 per order and the shortage cost is ₹15/unit/year.

Find the optimal values of the following:

i) Economic order quantity
ii) Maximum inventory
iii) Maximum shortage quantity
iv) Cycle time
v) Inventory period (t1)
vi) Shortage period (t_2)

Ques 3) The following failure rates have been observed for a certain type of transistors in a digital computer:

End of Week	1	2	3	4	5	6	7	8
Probability of Failure to Date	0.05	0.13	0.25	0.43	0.68	0.88	0.96	1

Total transistors in one computer are 1000. The cost of replacing an individual failed transistor is ₹1.25, while the cost of group replacement is 30 paise per transistor. Determine the best interval between group replacements. Compare group individual replacements.

Ques 4) A truck owner finds from his past records that the maintenance costs per year of a truck whose purchase price is ₹8,000 are as given below:

Year	01	02	03	04	05	06	07	08
Maintenance Cost (₹)	1,000	1,300	1,700	2,000	2,900	3,800	4,800	6,000
Resale Price (₹)	4,000	2,000	1,200	600	500	400	400	400

Determine at which time it is profitable to replace the truck.

Ques 5) A Bus owner from his past experience estimates that the maintenance cost per year of a bus whose purchase price is ₹1,50,000 and the resale value of the Bus will be:

Year	Maintenance	Resale Value
1	10,000	1,30,000
2	15,000	1,20,000
3	20,000	1,15,000
4	25,000	1,05,000
5	30,000	90,000
6	40,000	75,000
7	45,000	60,000
8	50,000	50,000

Determine at which time it is profitable to replace the bus.

UNIT-V

Ques 1) What is queuing theory? Discuss the elements of Queuing system.

Ques 2) Workers come to tool store room to receive special tools (required by them) for accomplishing a particular project assigned to them. The average time between two arrivals is 60 seconds and the arrivals are assumed to be in Poisson distribution. The average service time (of the tool room attendant) is 40 seconds. Determine:

i) Average queue length.
ii) Average length of non-empty queues.
iii) Average number of workers in system.
iv) Mean waiting time of an arrival.
v) Average waiting time of an arrival (worker) who waits.