Structural Analysis I (CE 8502)

UNIT I : STRAIN ENERGY METHOD.

Determination of static and kinematic Indeterminacies - Analysis of continuous beams, plane frames and indeterminate plane Vol.II trusses (P-403.) by strain energy method (upto two degree of redundancy).

Static indeterminacy of structures:

If the conditions of statics i.e $\Sigma H=0, \Sigma V=0$ and $\Sigma M=0$ alone are not sufficient to find either external reactions or internal forces in a structure, the structure is called statically indeterminate structure.

Kinematic indeterminacy of a structure:

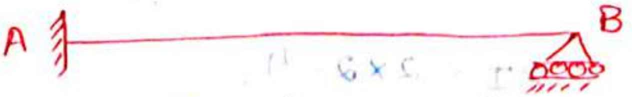
When a structure is subjected to loads, each joint will undergo displacements in the form of translations and rotations. KI of a structure means the number of unknown joint displacements in a structure.

Note: Kinematically determinate structure need not be statically determinate.

Kinematic indeterminacy refers to the number of possible jt displacements in a structures.

Type of support	Degrees of freedom (possible joint displacements)
Roller 	2
Hinges 	1
Fixity 	0

①



Support A = fixed. - no displacement.

Support B = rollers - 2 DOF (horizontal & rotational)

KI = 2 (Kinematically indeterminate to 2DOF).

Static and kinematic indeterminacy

In static method, we solve structure by [illegible] treating the reaction and internal forces as unknown.

In kinematics, we solve structure by finding the displacement and rotation as unknown.

Determinate: A structure is said to be determinate if total no. of unknown is equal to total no. of equilibrium equations.

Indeterminate: Total no. of unknown > total no. of equilibrium equations.

Unstable: Total no. of unknown < total no. of equilibrium eqn

D_r = Static indeterminacy.

I = Internal Static indeterminacy.

E = External " "

$$\boxed{D_r = I + E} \qquad \boxed{E = F - U - R}$$

$$\boxed{I = m * \text{no. of closed loops in structures.}}$$

For 2D, m = 3 , for 3D, m = 6.

F = no. of unknown reactions.

U = eqn. of static equilibrium available

R = any additional equation available.

2 Dimension Beams:

* For beams generally internal indeterminacy = 0. as no closed loop can be formed in beam.

$\therefore$ E = F − U − R.

* For calculating no. of unknown reactions: (1) Roller gives 1 reaction (2) Hinge gives two reactions (3) Fixed gives 3 rxs.

* U = Eqn. of Static equilibrium available

$$\Sigma F_x = 0, \Sigma F_y = 0, \Sigma M_z = 0.$$

Find the indeterminacy of the beams:

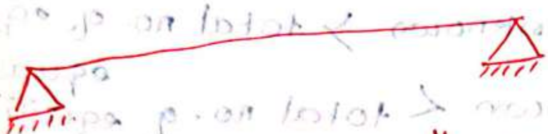
F = 2+1 = 3, U = 3, E = 3-3 = 0

statically determinate

F = 2+1+1 = 4, U = 3, E = 4-3 = 1

Beam is indeterminate of 1 degree.

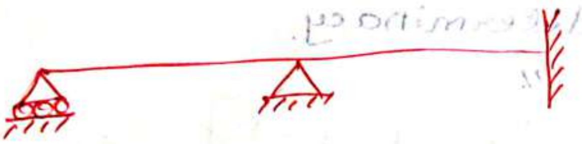
F = 3+1 = 4, U = 3, E = 4-3 = 1



F = 2+2 = 4, U = 3, E = 4-3 = 1.

F = 3+3 = 6, U = 3, E = 6-3 = 3

SI = 3.

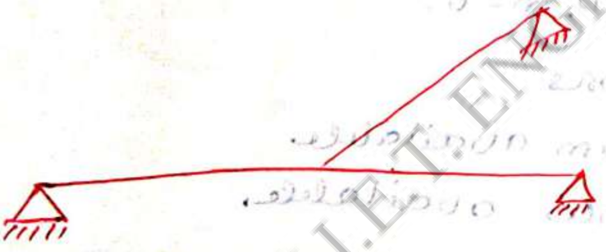


F = 1+2+3 ; U = 3 ; E = 6-3 = 3

SI = 3.

F = 2+1+1+2 ; U = 3

E = 6-3 = 3. , SI = 3.



F = 2+2+2 = 6 ; U = 3

E = 6-3 = 3 ; SI = 3.

2-Dimension Frame:

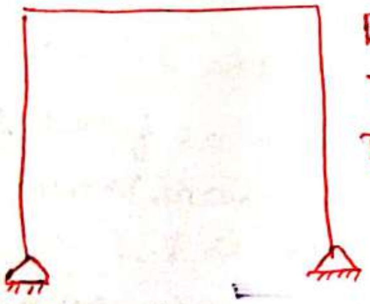
* A frame is a structure comprising of beam and column jointed by rigid jt.: can transfer axial load, shear force and BM without any change in angle between members.

* $D_r = I + E$, $E = F - U - R$, $I = m \times$ no. of closed loops in structure.

* Number of unknown reaction (1) Roller gives one reaction (2) Hinge gives two reaction. (3) Fixed support gives three reaction.

* $U =$ Equation of static equilibrium $= 3$, $\Sigma F_x = 0, \Sigma F_y = 0, \Sigma M_z = 0$.

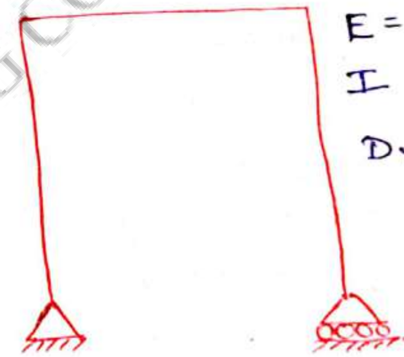
Find the Indeterminacy of the frames:



$E = 2+2-3 = 1$

$I = 0$

$D_r = 1+0 = 1.$



$E = 2+1-3 = 0$

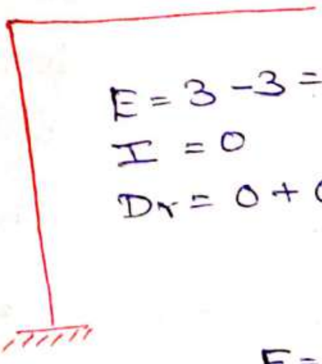
$I = 0$

$D_r = 0+0 = 0$

$E = 3+3-3 = 3$

$I = 0$

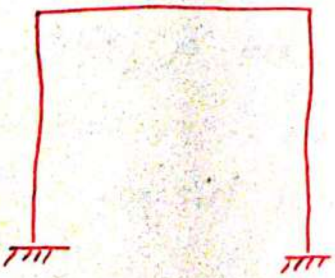
$D_r = 3+0 = 3$



$E = 3-3 = 0$

$I = 0$

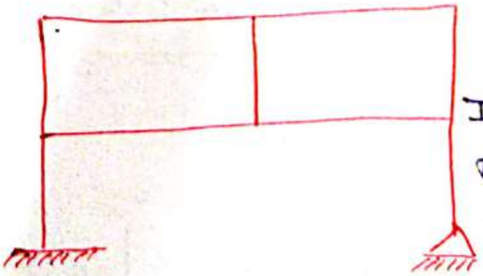
$D_r = 0+0 = 0.$



$E = 3+3-3 = \cancel{0}3$

$I = 0$

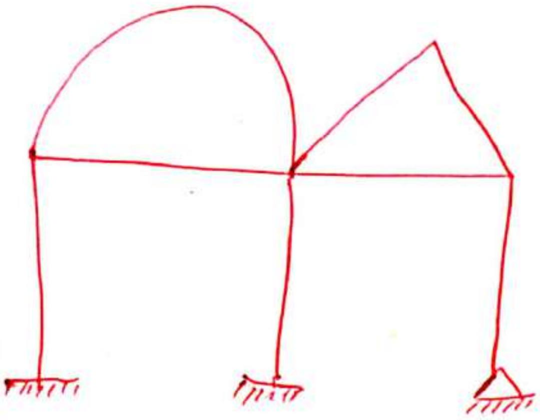
$D_r = 3+0 = 3.$



$E = 3+2-3$
$= 2.$

$I = 2 * 3 = 6$

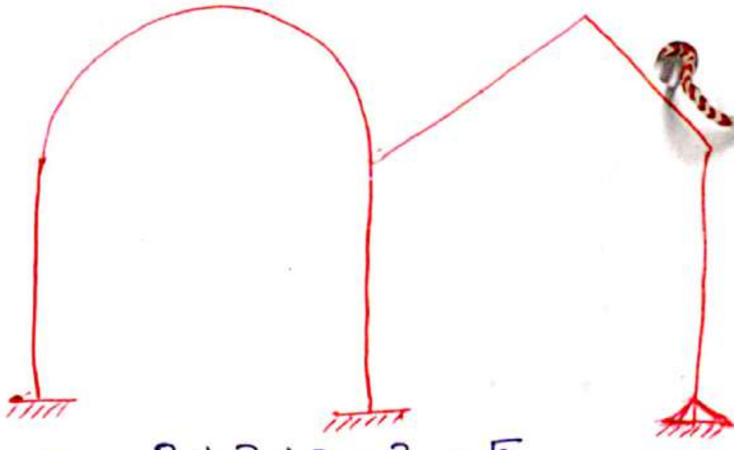
$D_r = 2+6 = 8$



$E = 3+3+2-3 = 5.$

$I = 3*2 = 6$

$D_r = 5+6 = 11.$



$E = 3+3+2-3 = 5$

$I = 3*0 = 0.$

$D_r = 5+0 = 5.$

Kinematic Indeterminacy (KI)

In KI we measure the total no. of DOF possible at joints.

For 2 dimension,

- * Fixed support gives 3 reaction viz one rotation and 2 translation hence degree of freedom at fixed support is 0. Hinge gives 2 reaction both translation and no rotation reaction ∴ DOF @ hinge is 1. Roller: gives 1 reaction ; DOF = 2, viz one rotation and one translation.
- * Similary for a two dimensional rigid joint or free end, the no. of DOF at a jt. is 3 (one rotation & 2 translation)
- * For internal hinge, the DOF is 4, two rotational and two translational.
- * Generally $\boxed{KI = 3*j - r + i}$ where j = no. of joint

r = no. of reactions.

i = no. of internal hinge.

$\boxed{DOF + Force = 3}$

For 3 dimension;

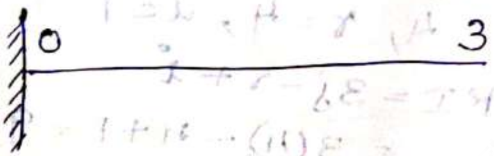
- * Fixed support gives 6 reaction : 3 rotation & 3 translation
  ∴ DOF = 0
- * Hinge gives 3 reaction : all in translation and no rotation
  ∴ DOF at hinge = 3
- * Roller gives 1 reaction in translation direction
  ∴ DOF at roller is 5 viz 3 rotation & 2 translation
- * For internal hinge, DOF is 9, 6 rotational and 3 translational.

$KI = 6j - r + 3i$

DOF + Force = 6

Ignoring axial deformation ⇒ KI will decrease

KI after ignoring axial deformation } = KI − no. of members in the structure.



$KI = 0 + 3 = 3$ Ignoring axial deformation $KI = 3 - 1 = 2$.

or

$\boxed{KI = 3j - r + i}$ $KI = 3 \times 2 - 3 = 3$; Ignoring axial deform

$j = 2,\ r = 3,\ i = 0$, r = no. of reactions $KI = 3 - 1 = 2$.

$KI = 1 + 2 = 3$ Ignoring axial deformation

or

$j = 2, r = 3$

$KI = 3j - r$
$= 3 \times 2 - 3 = 3$

$KI = 3 - 1 = 2$.

i = no. of internal hinge.

$KI = 0 + 2 = 2$ or $j = 2, r = 4$; $KI = 3j - r$

Ignoring axial deformation $= 3 \times 2 - 4 = 2$

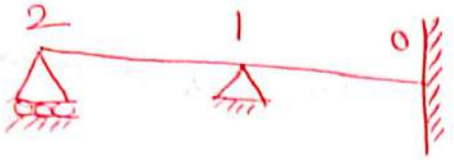
$KI = 2 - 1 = 1$.

$KI = 1 + 2 + 2 = 5$ or $j = 3, r = 4$

$KI = 3j - r = 3 \times 3 - 4 = 5$

Ignoring axial deformation

$KI = 5 - 2 = 3$.

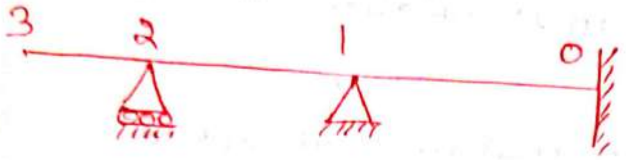


KI = 2+1 = 3 or j = 3, r = 6

KI = 3j − r = 9−6 = 3.

Ignoring axial deformation

KI = 3 − 2 = 1.

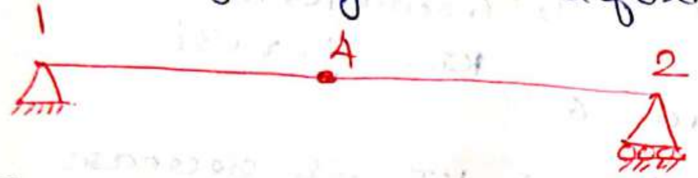


KI = 3+2+1+0 = 6

or j = 4, r = 6, KI = 3j − r

= 3×4 − 6 = 6.

Ignoring axial deformation KI = 6 − 3 = 3



KI = 1+4+2 = 7.

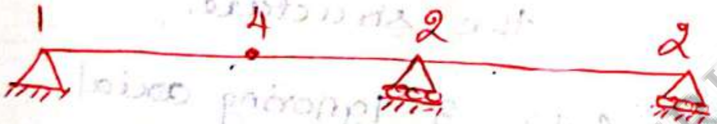
or j = 3, r = 3, i = 1.

KI = 3j − r + i

= 3×3 − 3 + 1 = 7.

Ignoring axial deformation

KI = 7 − 2 = 5.



KI = 1+4+2+2 = 9

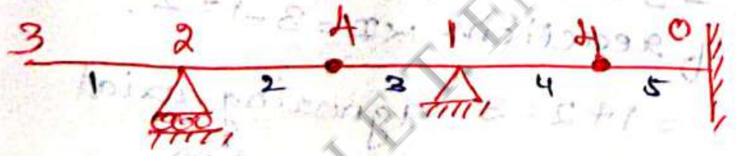
or

j = 4, r = 4, i = 1

Ignoring axial deformation, KI = 3j − r + i

= 3(4) − 4 + 1 = 9

KI = 9 − 3 = 6.



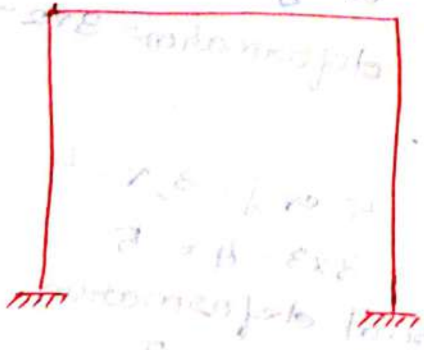
KI = 3+2+4+1+4+0 = 14

or j = 6, r = 6, i = 2

KI = 3j − r + i

Ignoring axial deformation, = 3(6) − 6 + 2 = 14

KI = 14 − 5 = 9

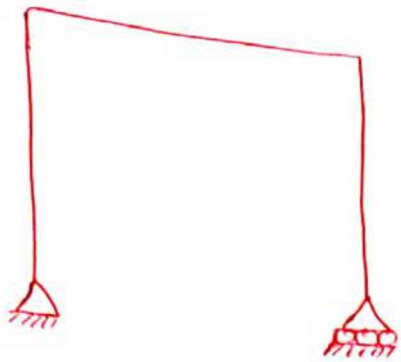


KI = 3j − r = 3(4) − 6 = 6.

j = 4, r = 6

Ignoring axial deformation,

KI = 6 − 3 = 3.

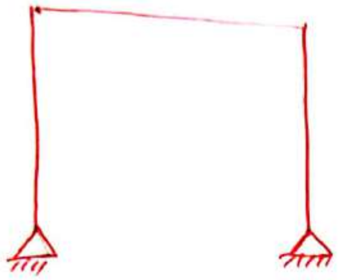


$j = 4,\ r = 3$

$KI = 3j - r = 3(4) - 3 = 9.$

Ignoring axial deformation,

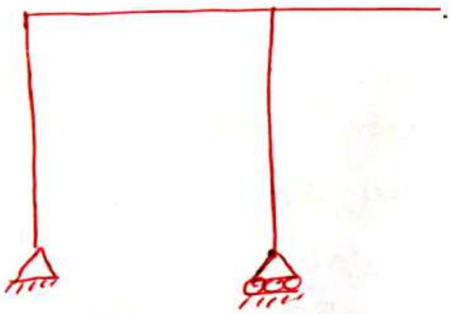
$KI = 9 - 3 = 6.$



$j = 4,\ r = 4,\ KI = 3j - r = 12 - 4 = 8.$

Ignoring axial deformation,

$KI = 8 - 3 = 5.$

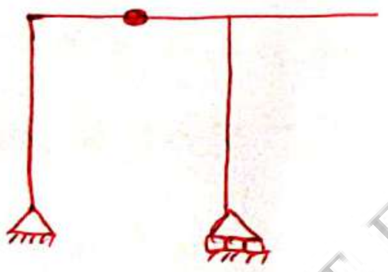


$j = 5,\ r = 3$

$KI = 3j - r = 15 - 3 = 12.$

Ignoring axial deformation,

$KI = 12 - 4 = 8.$

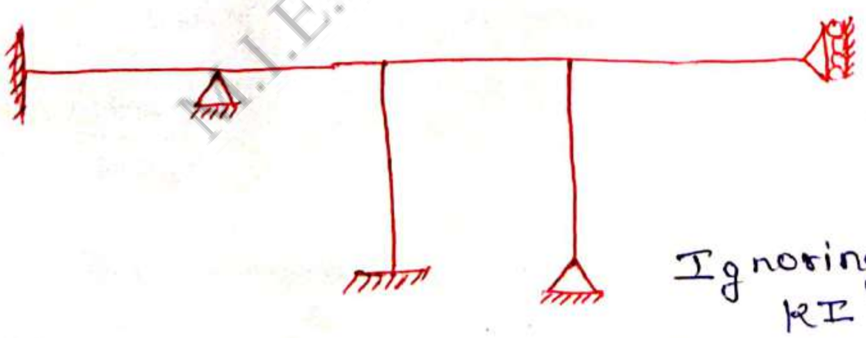


$j = 6,\ r = 3,\ i = 1$

$KI = 3j - r + i = 3(6) - 3 + 1 = 16.$

Ignoring axial deformation,

$KI = 16 - 5 = 11$



$j = 7,\ r = 11$

$KI = 3j - r$

$= 3(7) - 11 = 10$

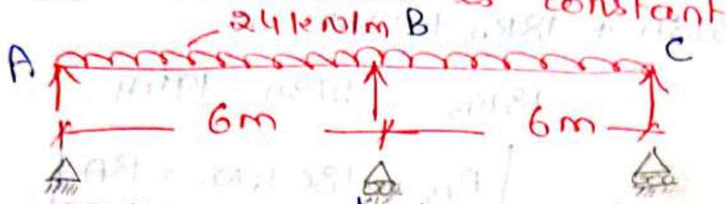
Ignoring axial deformation,

$KI = 10 - 6 = 4.$

continuous beam.

③

Analyse the continuous beam loaded as shown by the strain energy method. EI is constant.



$E = F-U-R$

$SI = 2+1+1-3$

$= 4-3 = 1$

For beams with vertical loads only,

$SI = 3+0-2 = 1.$

Hence beam is statically indeterminate to first degree.

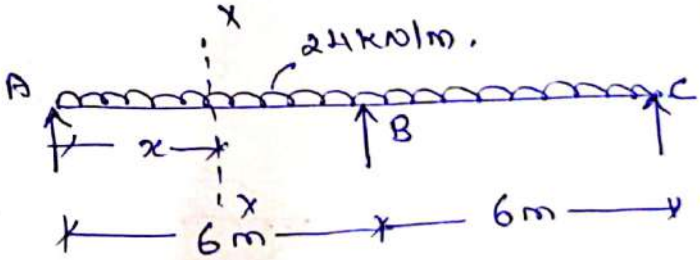
Let us treat R_B as redundant.

$$R_A = R_C = \frac{\text{Total load} - R_B}{2} = \frac{(24\times12) - R_B}{2} = 144 - 0.5R_B.$$

The partial derivative of the total strain energy U in the beam AC w.r.t. R_B is zero since $\delta_B = 0$.

$$\frac{\partial U_{AC}}{\partial R_B} = 0 \quad \text{ie} \quad \frac{1}{EI}\int M\frac{\partial M}{\partial R_B}\,dx = 0.$$

consider section xx,



$$M_x = R_A\cdot x - wx\cdot\frac{x}{2}$$

$$M_x = (144 - 0.5R_B)x - 24\cdot\frac{x^2}{2}$$

$$M_x = 144x - 0.5R_B\cdot x - 12x^2 \text{ ------- (i)}$$

$$\frac{\partial M_x}{\partial R_B} = -0.5x.$$

The integration limits are 0m and 6m.

$$\frac{\partial U}{\partial R_B} = \left\{\frac{1}{EI}\int_0^6 [144x - 0.5R_Bx - 12x^2][-0.5x]\,dx\right\}\times 2 = 0$$

$$= \left\{\frac{1}{EI}\int_0^6 [-72x^2 + 0.25R_Bx^2 + 6x^3]\,dx\right\}\times 2 = 0.$$

$$\left[-72\frac{x^3}{3} + 0.25R_B\frac{x^3}{3} + \frac{6x^4}{4}\right]_0^6 = 0.$$

$$\left[-72\times\frac{6^3}{3}+0.25\times R_B\frac{6^3}{3}+\frac{6(6)^4}{4}\right]_0^6=0.$$

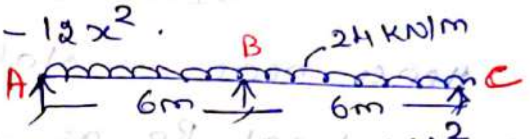
$$-5184+18R_B+1944=0.$$

$$18R_B = 5184-1944$$

$$\boxed{R_B = 180\text{ kN.} = R_A}$$

Substituting in equ. ①: $\boxed{M_x = 144x-\overset{0.5}{R_B}x-12x^2}$

$M_x = 144x-0.5\times180x-12x^2.$



when $x=0$, $\boxed{M_A=0}$

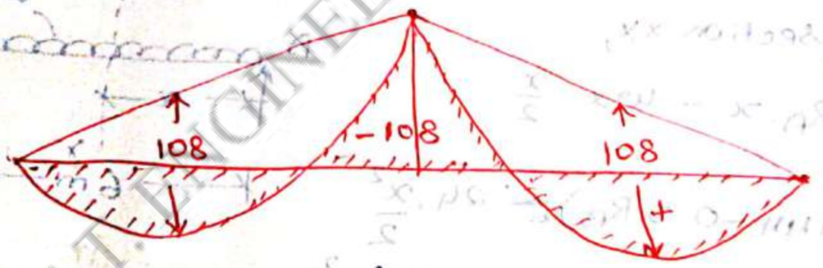
$x=6\text{m}$, $M_B=(144\times6)-(0.5\times180\times6)-12(6)^2$

$$\boxed{M_B=-108\text{ kNm.}}$$

$$\boxed{M_C=0}$$

s/s beam moment in AB and BC $=\frac{wl^2}{8}=\frac{24\times6^2}{8}$

$=108$ kNm.



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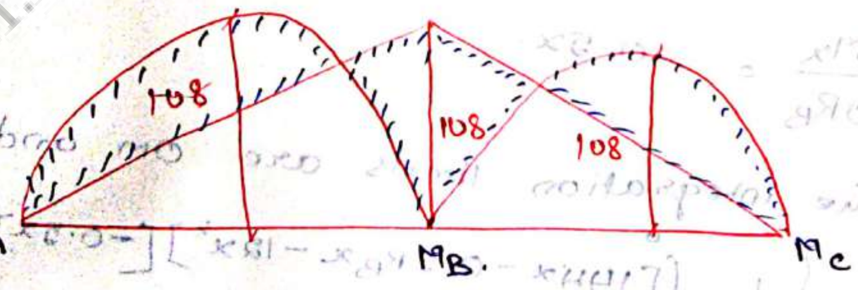
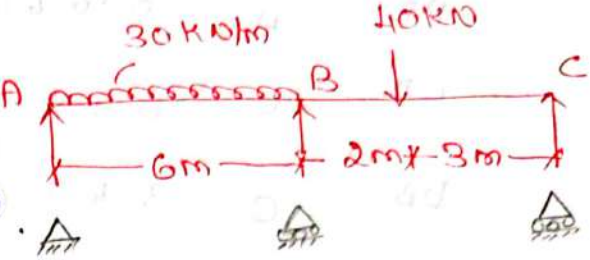


② Analyse the continuous beam shown by strain energy method and determine the reactions.



SI = vertical reactions + end moment − 2 (for vertical loads only)

$= 3+0-2 = 1$ | $SI = 2+1+1-3 = 1$

Treating R_B as redundant and taking moment @ C,

$R_A \times 11 + R_B \times 5 - 30\times 6\times(5+\frac{6}{2}) - 40\times 3 = 0.$

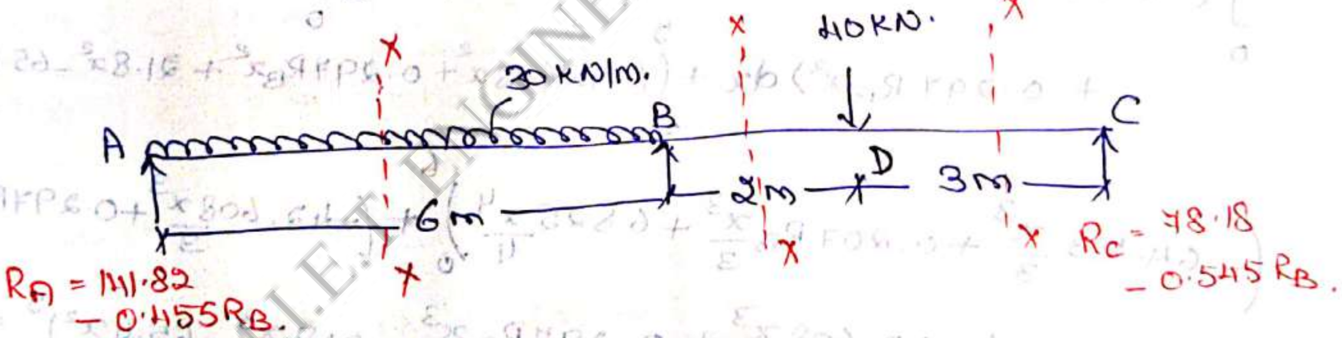
$11R_A + 5R_B = 1440 + 120 = 1560.$

$R_A = \frac{1560 - 5R_B}{11} = 141.82 - 0.455R_B.$

$R_c =$ Total load $- R_A - R_B.$

$= (30\times 6) + 40 - (141.82 - 0.455R_B) - R_B.$

$= 78.18 - 0.545R_B.$



$\boxed{\frac{\partial U}{\partial R_B} = 0}$ since $\delta_B = 0$. ——①

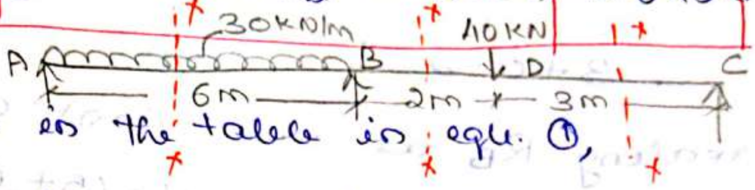
$\frac{1}{EI}\int M \frac{\partial M}{\partial R_B} dx = 0.$, EI is constant and can be removed.

The integration will be done seperately for 3 zones, AB, CD and DB.

Portion	Origin.	Limits. (m)	M_x.	$\frac{\partial M_x}{\partial R_B}$.
AB	A	0 to 6	$(141.82 - 0.455R_B)x - 30x^2/2$	$-0.455x$
CD	C	0 to 3	$(78.18 - 0.545R_B)x$	$-0.545x$.
DB	C	3 to 5	$(78.18 - 0.545R_B)x - 40(x-3)$	$-0.545x$.

$R_A = 141.82 - 0.455 R_B$

$R_C = 78.18 - 0.545 R_B$.



Substitution the values in the table in equ. ①,

$$\int_0^6 (141.82x - 0.455R_Bx - 15x^2)(-0.455x)\,dx. +$$

$$\int_0^3 (78.18x - 0.545R_Bx)(-0.545x) +$$

$$\int_3^5 (78.18x - 0.545R_Bx - 40x + 120)(-0.545x)dx = 0.$$

$$\int_0^6 (-64.53x^2 + 0.207R_Bx^2 + 6.825x^3)dx + \int_0^3 (-42.608x^2 + 0.297R_Bx^2)dx + \int_3^5 (-42.608x^2 + 0.297R_Bx^2 + 21.8x^2 - 65.4x)dx = 0$$

$$\left(-64.53\frac{x^3}{3} + 0.207R_B\frac{x^3}{3} + 6.825\frac{x^4}{4}\right)_0^6 + \left(-42.608\frac{x^3}{3} + 0.297R_B\frac{x^3}{3}\right)_0^3$$

$$+ \left(-42.608\frac{x^3}{3} + 0.297R_B\frac{x^3}{3} + 218\frac{x^3}{3} - 65.4\frac{x^2}{2}\right)_3^5 = 0.$$

$$-4646.16 + 141.904R_B + 22118.3 - 383.47 + 2.673R_B$$

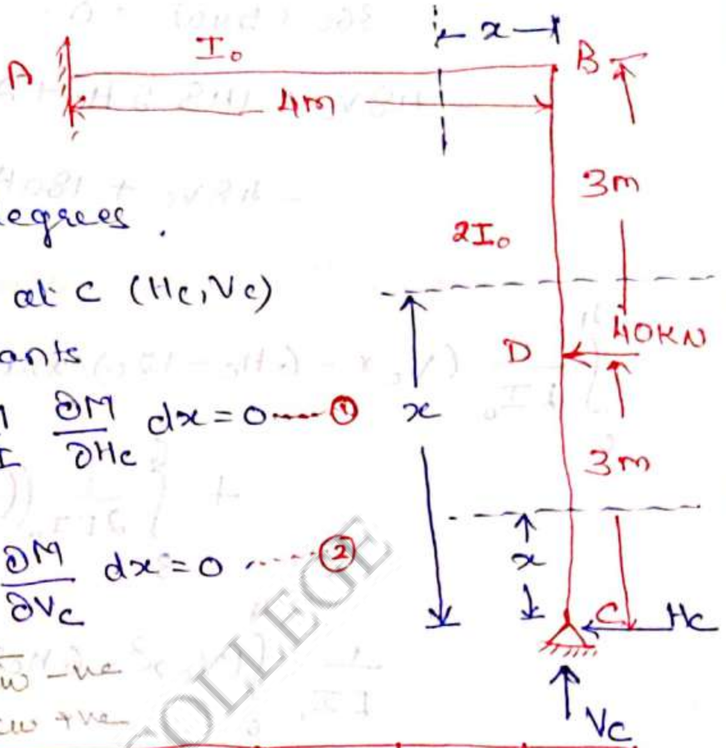
$$-1391.88 + 9.702R_B + 712.13 - 523.2 = 0.$$

$$27.729R_B - 4021.26 = 0.$$

Hence, $R_B = 147.41$ KN, $R_A = 74.75$ KN, $R_C = -2.16$ KN.

⑧ Analyse the structure shown by strain energy method

Sketch the BMD.



a) Finding Redundants: $SI = 3+2-3$
$= 2$

The given structure is

Statically indeterminate to 2 degrees.

The horz. and vertical reactions at C (H_c, V_c) will be treated as two redundants

Since C is hinged, $\frac{\partial U}{\partial H_c} = 0$. i.e. $\int \frac{M}{EI} \frac{\partial M}{\partial H_c} dx = 0$ ---- ①

and $\frac{\partial U}{\partial V_c} = 0$ i.e. $\int \frac{M}{EI} \frac{\partial M}{\partial V_c} dx = 0$ ...... ②

$\rightarrow$ C.W +ve
anticw -ve

$\leftarrow$ cw -ve
acw +ve

Portion	origin	Limits (m)	M_x. anticlock +ve clockwise -ve	$\frac{\partial M}{\partial H_c}$	$\frac{\partial M}{\partial V_c}$	I
BA	B	0 to 4	$V_c x - H_c \times 6 - 40 \times 3$ sign?	-6	x	I_0
CD	C	0 to 3	$-H_c \cdot x$.	$-x$	0	$2I_0$
DB	C	3 to 6	$-H_c \cdot x - 40(x-3)$	$-x$	0	$2I_0$

Sub. the above values in equ. ①,

$$\int_0^4 \frac{1}{EI_0}(V_c * x - 6H_c - 120)(-6)dx + \int_0^3 \frac{1}{E \times 2I_0}(-H_c \cdot x)(-x)dx$$

$$+ \int_3^6 \frac{1}{E \times 2I_0}(-H_c x - 40x + 120)(-x)dx = 0.$$

$$\int_0^4 (-6V_c x + 36H_c + 720)dx + \int_0^3 \frac{1}{2} H_c x^2 dx + \int_3^6 \frac{1}{2}(H_c x^2 + 40x^2 - 120x)dx = 0$$

$$\left[-6V_c \frac{x^2}{2} + 36H_c x + 720x\right]_0^4 + \left[\frac{1}{2} H_c \frac{x^3}{3}\right]_0^3 + \left[\frac{1}{2}\left(H_c \frac{x^3}{3} + 40\frac{x^3}{3} - 120\frac{x^2}{2}\right)\right]_3^6 = 0$$

$$-48V_c + 144H_c + 2880 + 4.5H_c + \frac{1}{2}\left[\left(H_c \times \frac{6^3}{3} + 40 \times \frac{6^3}{3} - 120 \times \frac{6^2}{2}\right) - \left(H_c \times \frac{3^3}{3} + 40 \times \frac{3^3}{3} - 120 \times \frac{3^2}{2}\right)\right] = 0.$$

$-48V_c + 148.5H_c + 2880 + \frac{1}{2}(72H_c + 2880 - 2160 - 9H_c -$

$360 + 540) = 0.$

$-48V_c + 148.5H_c + 2880 + 31.5H_c + 450 = 0.$

$-48V_c + 180H_c = -3330$. - - - - - ③

Substituting the table values in equ. ②.

$$\int_0^4 \frac{1}{EI_0}(V_c x - 6H_c - 120)(x)dx + \int_0^3 \frac{1}{2EI_0}(-H_c x)(0)\,dx.$$

$$+ \int_3^6 \frac{1}{2EI_0}((-H_c x - 40(x-3))(0)\,dx = 0.$$

$$\frac{1}{EI_0}\int_0^4 (V_c x^2 - 6H_c x - 120x)\,dx = 0.$$

$$\left(V_c\frac{x^3}{3} - 6H_c\frac{x^2}{2} - 120\frac{x^2}{2}\right)_0^4 = 0.$$

$21.33V_c - 48H_c - 960 = 0$

$21.33V_c - 48H_c = 960$. - - - - - ④

Solving equ ③ and ④

$V_c = 8.44$ KN. ; $H_c = -16.25$ KN.

(b) Determining bending moments:

AB: $M_x = V_c x - H_c \times 6 - 40 \times 3$

$= 8.44x - (-16.25) \times 6 - 40 \times 3$

$= 8.44x - 22.5$

@ $x = 0$, $M_B = -22.5$ KNm.

@ $x = 4$m, $M_A = 8.44 \times 4 - 22.5 = 11.26$ KNm.

CD: $M_x = -H_c\,x = -(-16.25)x$

$M_x = 16.25x.$

@ $x = 0$; $M_c = 0$

$x = 3$; $M_D = 16.25 \times 3 = 48.75$ KNm.

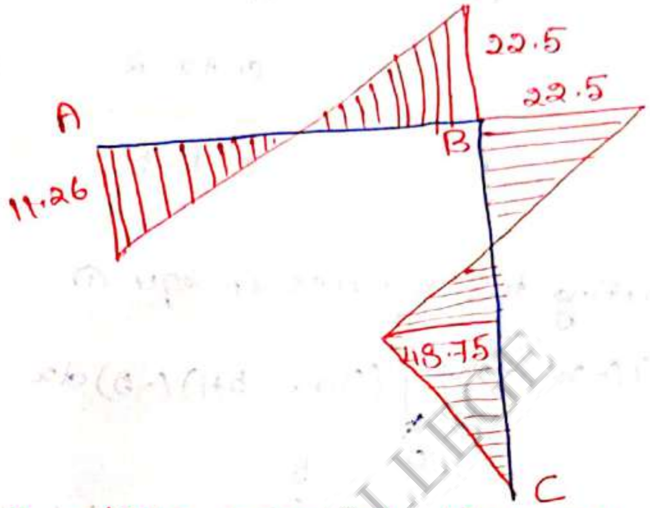
DB:

$M_x = -H_c \cdot x - 40(x-3) = -(-16.25x) - 40(x-3).$

$M_x = -23.75x + 120.$

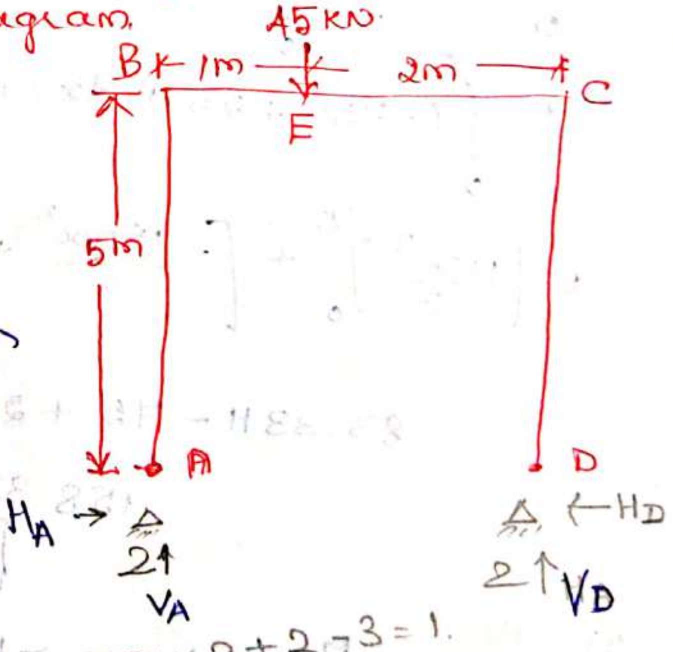
@ $x = 3m$, $M_D = -23.75 \times 3 + 120 = 48.75$ KNm.

@ $x = 6m$, $M_B = -23.75 \times 6 + 120 = -22.5$ KNm.



Bending moment diagram.

④ The simple portal frame shown is asymmetrically loaded. EI is constant. Analyse the frame by strain energy method. Sketch BM diagram.



a) Finding the redundant force!

Degree of static indeterminacy

$= [illegible] = 1.$

Let us treat the horz. reaction at D as redundant. Since there is no other horz. force,

$H_A = -H_D = H.$

$SI = 2+2-3 = 1.$

Since D is hinged, $\delta D = 0.$

$$\frac{\partial U}{\partial H_D} = 0.$$

$$\frac{1}{EI}\int M \frac{\partial M}{\partial H}\, dx = 0$$

$\Sigma M_D = 0.$

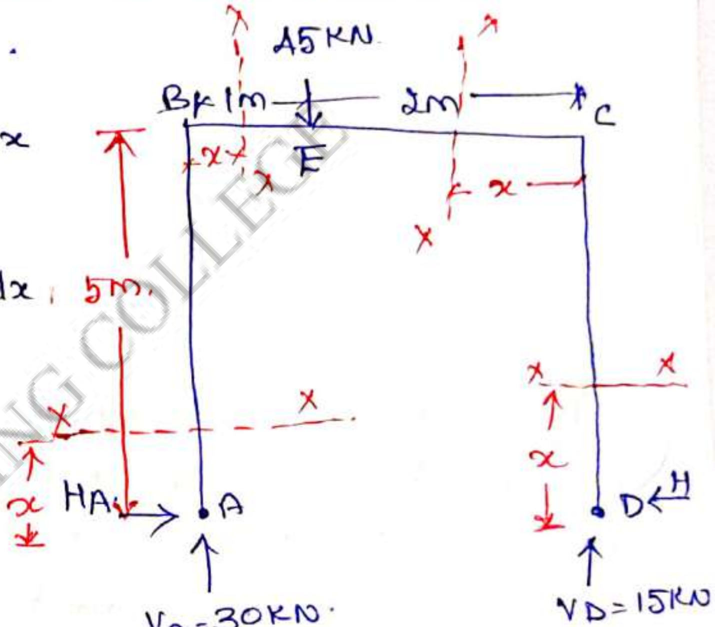
$(V_A \times 3) - (45 \times 2) = 0.$

$$V_A = \frac{45 \times 2}{3} = 30 \text{ KN}.$$

$V_D = 45 - 30 = 15$ KN.

Portion.	Origin.	Limits (m)	M_x or M clockwise +ve	$\frac{\partial M}{\partial H}$.
AB	A	0 to 5	$-H.x$.	$-x$
BE	B	0 to 1	$30x - H.5$	-5
CE	C	0 to 2	$15x - H\times5$	-5
DC	D	0 to 5	$-H.x$.	$-x$.

Substituting the values in equ. ①.



$$\frac{1}{EI}\int_0^5 (-Hx)(-x)\,dx + \int_0^1 (30x-5H)(-5)\,dx$$

$$+ \int_0^2 (15x-5H)(-5)\,dx + \int_0^5 (-H.x)(-x)\,dx = 0.$$

$$2\int_0^5 Hx^2\,dx + \int_0^1 (-150x + 25H)\,dx$$

$$+ \int_0^2 (-75x + 25H)\,dx = 0.$$

$$2\left[H\frac{x^3}{3}\right]_0^5 + \left[-\frac{150x^2}{2} + 25Hx\right]_0^1 + \left[-\frac{75x^2}{2} + 25H.x\right]_0^2 = 0.$$

$$83.33H - 75 + 25H - 150 + 50H = 0$$

$$158.33H = 225.$$

$$\boxed{H = 1.421\text{ KN}}$$

$\overrightarrow{\ }$ CW+ ACW−. $\overleftarrow{\ }$ CW− ACW+

b) AB: $M_x = -Hx = -1.421x$

when $x=0$, $M_A = 0$.

$x = 5m$, $M_B = -7.105$ KN.

BE: $M_x = 30x - 5H = 30x - 5\times1.421 = 15x - 7.105$

@ $x=0$, $M_B = -7.105$ KNm.

@ $x=1m$, $M_E = 30\times1 - 7.105 = 22.895$ KNm.

CE:

$M_x = 15x - 5H = 15x - 5\times1.421 = 15x - 7.105$

@ $x=0$, $M_c = -7.105$

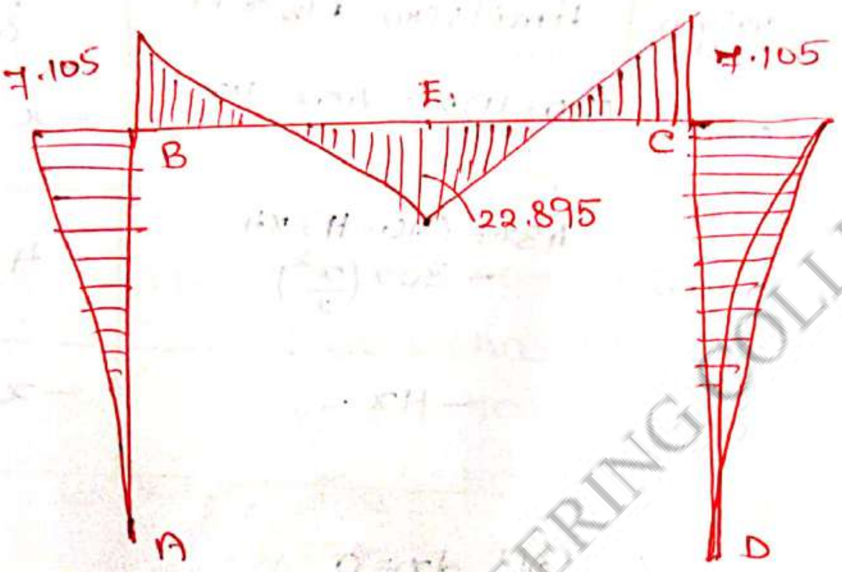
@ $x=2m$, $M_E = (15\times2) - 7.105 = 22.895$ kNm.

DC:

$M_x = -Hx = -1.421x$

@ $x=0$, $M_D = 0$

@ $x=5m$, $M_c = -1.421\times5 = -7.105$ kNm.



$M_A = 0$

$M_B = -7.105$

$M_E = 22.895$

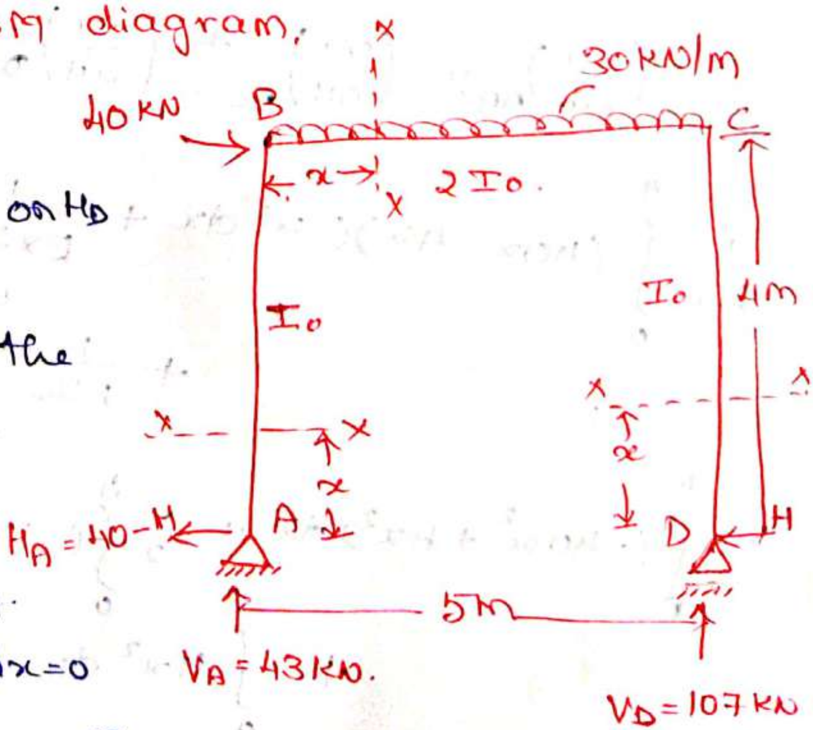
$M_c = -7.105$

$M_E = 22.895$

$M_D = 0$

$M_c = -7.105$

⑤ Analyse the portal frame with hinged base, by strain energy method. Sketch BM diagram.



$SI = 1$

Let us treat the horz. reacn on H_D @ D as redundant.

Since end D does not yield, the partial derivative of strain energy of whole frame w.r.t H_D or H will be zero

consider H_D as H - redundant.

$\frac{\partial U}{\partial H} = 0$; $\frac{\partial U}{\partial H} = \frac{1}{EI}\int M \frac{\partial M}{\partial H} \cdot dx = 0$ ---- ①.

Reactions $\Sigma H = 0$ gives

$H_A = 40 - H$.

Taking moments about D,

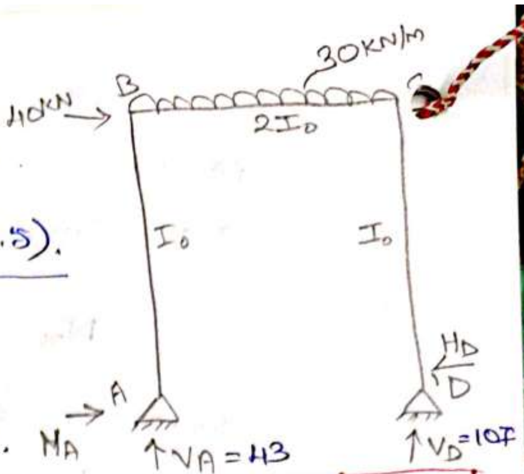
$$(V_A \times 5) + (40 \times 4) - 30 \times 5 \times \frac{5}{2} = 0$$

$$V_A = \frac{(-40 \times 4) + (30 \times 5 \times 2.5)}{5}$$

$$\boxed{H_D = 40 - H}$$

$$\boxed{V_A = 43 \text{ kN}}$$

$$V_D = 30 \times 5 - 43 = 107 \text{ kN}.$$



S.No.	Portion	Origin	Limits (m)	Mx or M	$\frac{\partial M}{\partial H}$	I
1.	AB	A	0 to 4	$(40-H)x = 40x - Hx.$	$-x$	I_0
2.	BC	B	0 to 5	$43x + (40-H) \times 4 - 30 \times \left(\frac{x^2}{2}\right)$	-4	$2I_0$
3.	DC	D	0 to 4	$-Hx.$	$-x$	I_0

$$\frac{\partial U}{\partial H} = \frac{1}{EI} \int M \frac{\partial M}{\partial H} dx = 0$$

$$\left(\frac{\partial U}{\partial H}\right)_{AB} + \left(\frac{\partial U}{\partial H}\right)_{BC} + \left(\frac{\partial U}{\partial H}\right)_{DC} = 0$$

$$\frac{1}{EI} \int_0^4 (40x - 4x)(-x)\,dx + \frac{1}{E \times 2I_0} \int_0^5 [43x + (40-H)4 - 15x^2](-4)\,dx$$

$$+ \frac{1}{EI_0} \int_0^4 (-Hx)(-x)\,dx = 0.$$

$$\int_0^4 (-40x^2 + Hx^2)\,dx + \frac{1}{2} \int_0^5 (-172x - 640 + 16H + 60x^2)\,dx$$

$$+ \int_0^4 Hx^2\,dx = 0$$

$$\left[-40\frac{x^3}{3} + H\frac{x^3}{3}\right]_0^4 + \frac{1}{2}\left[172\frac{x^2}{2} - 640x + 16Hx + 60\frac{x^3}{3}\right]_0^5$$

$$+ \left[H\frac{x^3}{3}\right]_0^4 = 0.$$

$$-853.33 + 21.33H - 1075 - 1600 + 40H + 1250 + 21.33H = 0.$$

$$82.66H - 2258.33 = 0$$

$$H = \frac{2258.33}{82.66} = 27.32 \text{ kN}.$$

$$\therefore H_A = 40 - H = 40 - 27.32 = 12.68 \text{ kN}.$$

b) Final moments:

AB: $M_x = 40x - Hx = 40x - 27.32x$
$= 12.68x.$

@ $x = 0$, $M_A = 0$.

@ $x = 4$, $M_B = 50.72$ kNm

BC: $M_x = 43x + (40 - H) \times 4 - 15x^2$
$= 43x + (40 - 27.32)4 - 15x^2$
$= 43x + 50.72 - 15x^2.$

@ $x = 0$, $M_B = 50.72$ kNm.

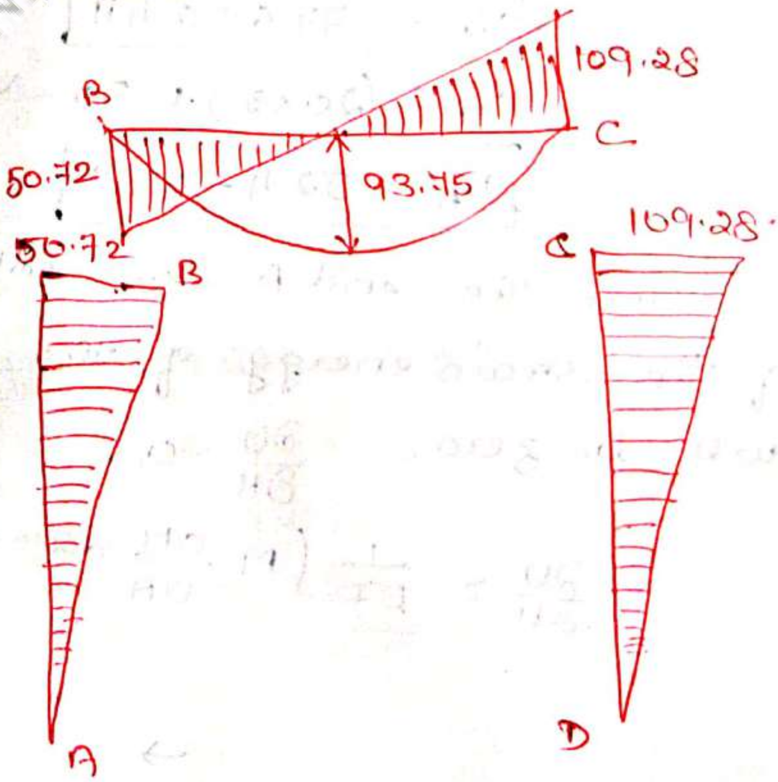
@ $x = 5$m, $M_C = 43 \times 5 + 50.72 - 15 \times 5^2.$
$= -109.38$ kNm.

DC:

$M_x = -Hx = -27.32x.$

@ $x = 0$, $M_D = 0$

@ $x = 4$m, $M_C = -27.32 \times 4$
$= -109.28$ kNm.



Final moments:

$M_A = 0$

$M_B = +50.72$ kNm.

$M_B = +50.72$ kNm.

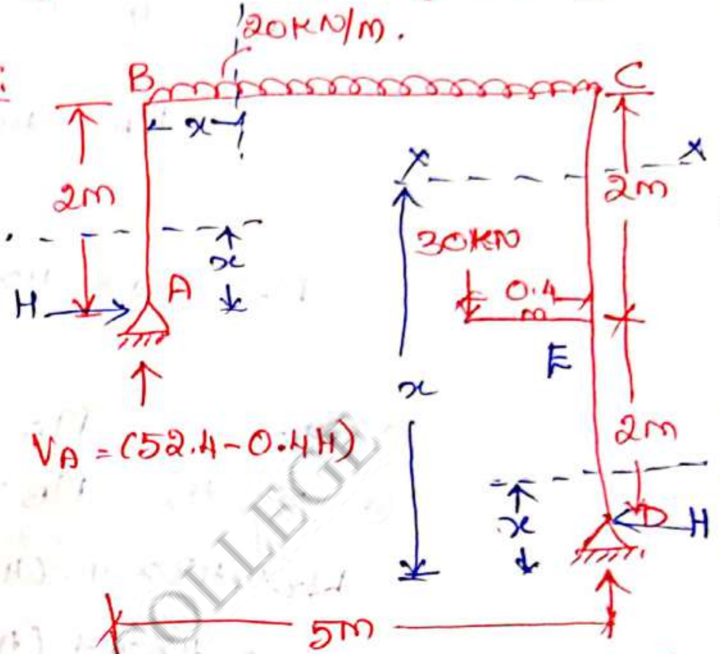
$M_C = -109.38$ kNm.

$M_D = 0$.

$M_C = -109.28$ kNm.

⑥ Analyse the frame shown by the strain energy method. Sketch the BM diagram. All the members of the frame have uniform flexural rigidity throughout.

a) Finding redundant forces:



The frame is statically indeterminate to first degree. Let horz. reaction H @ A be the redundant. Since there is no other horz. force, H @ D will be equal and opp. to H @ A.

Let $H_A = H$, $H_A = -H_D = H$

Taking moments about A, clockwise -ve, anticlockwise +ve

$V_D = 77.6 + 0.4H$

$$V_D\times5 - 30\times(5-0.4) - 20\times5\times\left(\frac{5}{2}\right) - H_D\times2 = 0$$

$$5V_D = (30\times4.6) + (10\times25) + 2H$$

$$\boxed{V_D = 77.6 + 0.4H}$$

$$V_A = (20\times5) + 30 - V_D = 130 - (77.6 + 0.4H)$$

$$\boxed{V_A = 52.4 - 0.4H.}$$

Since the end A does not yield, the partial derivative of the strain energy of the whole frame w.r.t. H will be zero. $\frac{\partial U}{\partial H} = 0$.

$$\frac{\partial U}{\partial H} = \frac{1}{EI}\int M\frac{\partial M}{\partial H}\,dx = 0.$$

→	←
CW +ve	CW -ve
ACW -ve	ACW +ve

S.No	Portion	Origin.	Limits (m)	M	$\frac{\partial M}{\partial H}$.
1.	AB	A	0 to 2	$-Hx.$	$-x$
2.	BC	B	0 to 5	$(52.4-0.4H)x - H\times2 - 20(\frac{x^2}{2})$	$-0.4x-2$
3.	DE	D	0 to 2	$-H.x$	$-x$
4.	EC	D	2 to 4	$-Hx. + 30\times0.4$	$-x.$

$$\left(\frac{\partial U}{\partial H}\right)_{AB} + \left(\frac{\partial U}{\partial H}\right)_{BC} + \left(\frac{\partial U}{\partial H}\right)_{DE} + \left(\frac{\partial U}{\partial H}\right)_{EC} = 0$$

$\rightarrow$ CW+ ACW− | $\leftarrow$ CW− ACW+

$$\frac{1}{EI}\left\{\int_0^2 (-Hx)(-x)\,dx + \int_0^5 [52.4x - 0.4Hx - 2H - 10x^2](-0.4x-2)\,dx + \int_0^2 (-Hx)(-x)\,dx \int_2^4 (-Hx+12)(-x)\,dx\right\} = 0.$$

$$\int_0^2 Hx^2\,dx + \int_0^5 (-20.96x^2 + 0.16Hx^2 + 0.8Hx + 4x^3 - 104.8x + 0.8Hx + 4H + 20x^2)\,dx + \int_0^2 (+Hx^2)\,dx + \int_2^4 (-Hx^2 - 12x)\,dx = 0.$$

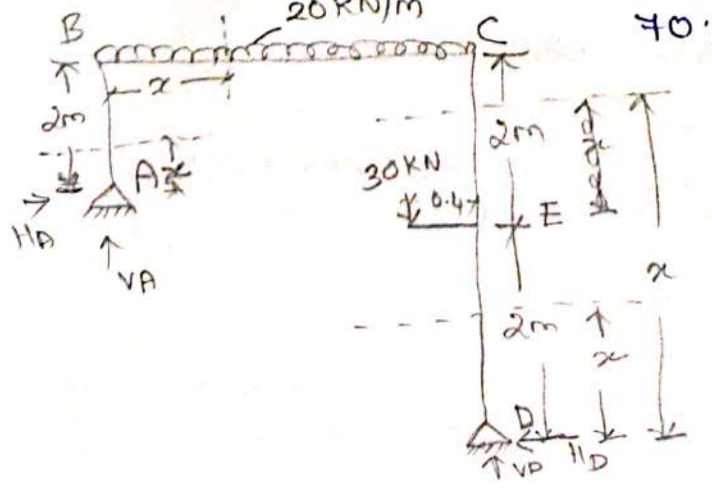
$$\left[\frac{Hx^3}{3}\right]_0^2 + \left[-20.96\frac{x^3}{3} + 0.16H\frac{x^3}{2} + 0.8H\frac{x^2}{2} + 4\frac{x^4}{4} - 104.8\frac{x^2}{2} + 0.8H\frac{x^2}{2} + 4Hx + 20\frac{x^3}{3}\right]_0^5 + \left[H\frac{x^3}{3}\right]_0^2 + \left[H\frac{x^3}{3} - 12\frac{x^2}{2}\right]_2^4 = 0.$$

$2.667H - 873.33 + 6.667H + 10H + 625 - 1310 + 10H + 20H$

$+ 833.33 + 2.667H + 18.667H + 72 = 0.$

$70.668H - 797 = 0.$

$$\boxed{H = 11.278\,KN.}$$



b) Final moments :

AB: $M_x = -Hx = -11.278x$

when $x = 0$, $M_A = 0$

$x = 2$, $M_B = -22.556$ kNm.

BC: $M_x = (52.4 - 0.4H)x - 2H - 10x^2$

$M_x = (52.4 - 0.4 \times 11.278)x - 2 \times 11.278 - 10x^2$

$= 47.889x - 22.556 - 10x^2$

@ $x = 0$, $M_B = -22.556$ kNm

@ $x = 5m$, $M_C = 47.889 \times 5 - 22.556 - 10 \times 5^2 = -33.111$ kNm.

S/S BM in BC $= \frac{20 \times 5^2}{8} = 62.5$ kNm.

DE : $M_x = -H.x$

$M_D(x = 0) = 0$

$M_E = (x = 2) = -11.278 \times 2$

$= -22.556$

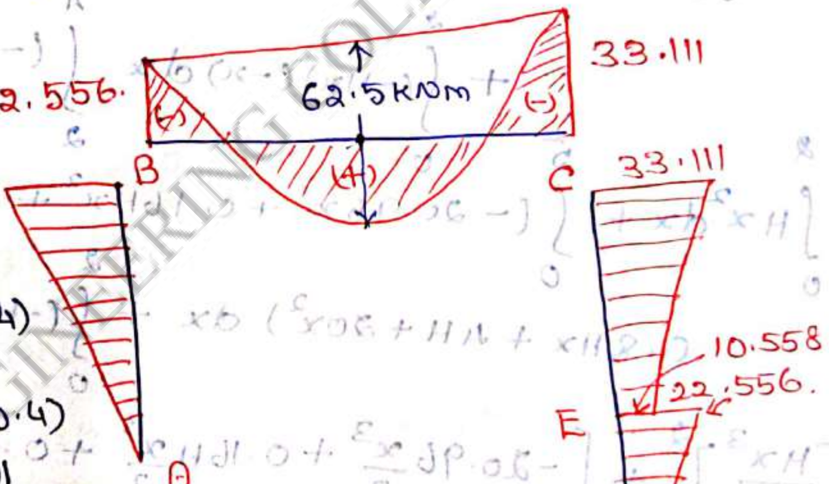
EC: $M_x = -H.x + (30 \times 0.4)$

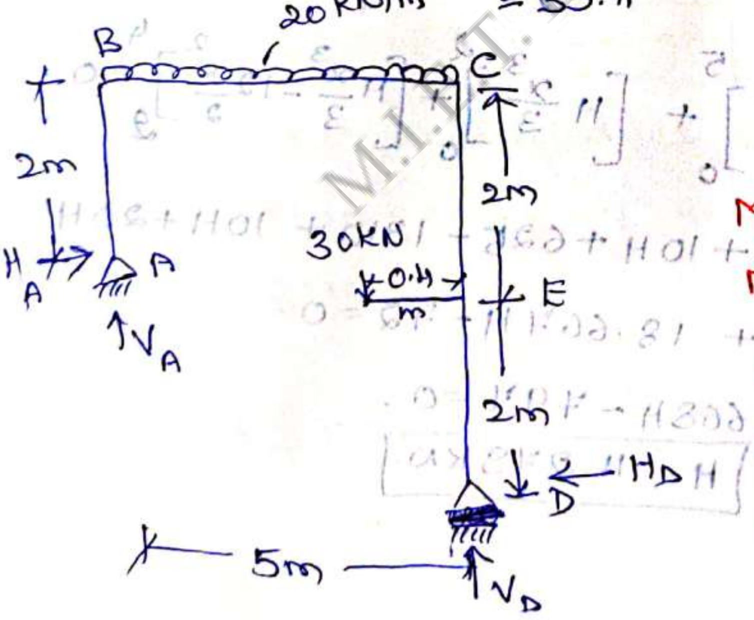
$M_E(x=2) = (-11.278 \times 2) + (30 \times 0.4)$

$= -10.558$

$M_C(x = 4) = (-11.278 \times 4) + (30 \times 0.4)$

$= 33.11$





$M_A = 0$

$M_B = -22.556$ kNm

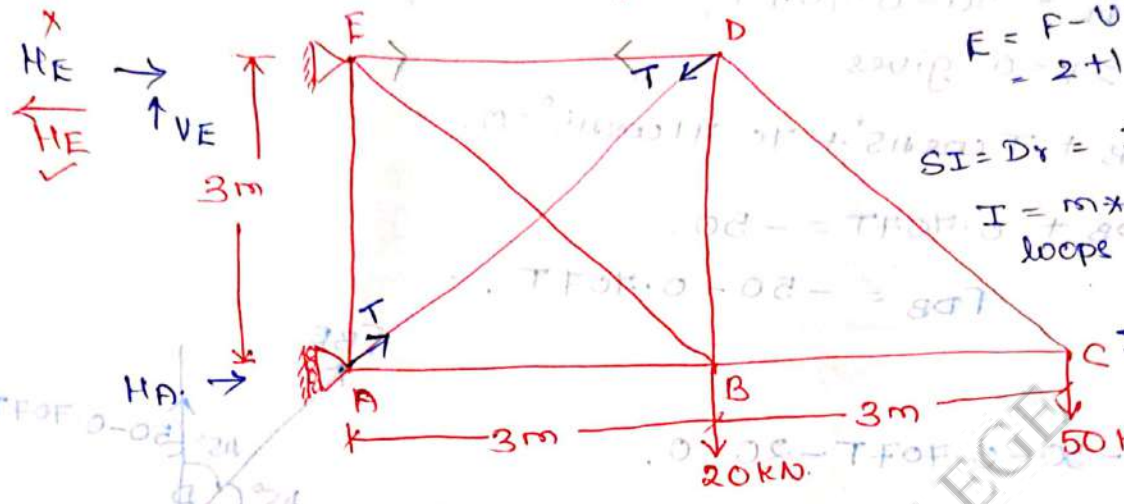
$M_B = -22.556$ kNm

$M_C = -33.111$ kNm.

$M_D = 0$

$M_E = -22.556$ kNm.

A braced cantilever truss is loaded as shown. All the members are of the same material and have the same C.S. area. Find the axial force in the member AD.



$E = F - U - R$
$= 2+1-3 = 0.$

$SI = D_r = I + E.$

$I = m \times$ no. of closed loops for beams & frames

$I = m-(2j-3)$ for trusses.

Degree of static indeterminacy $I = m-(2j-3)$

$m = 8;\ j = 5.$

$= 8-(2\times5-3)$

$= 8-7 = 1.$

The truss is externally determinate and internally indeterminate to the first degree

Treating member AD as redundant, remove AD and apply tensile force at A and D.

Taking moment about A, $\Sigma M_A = 0.$

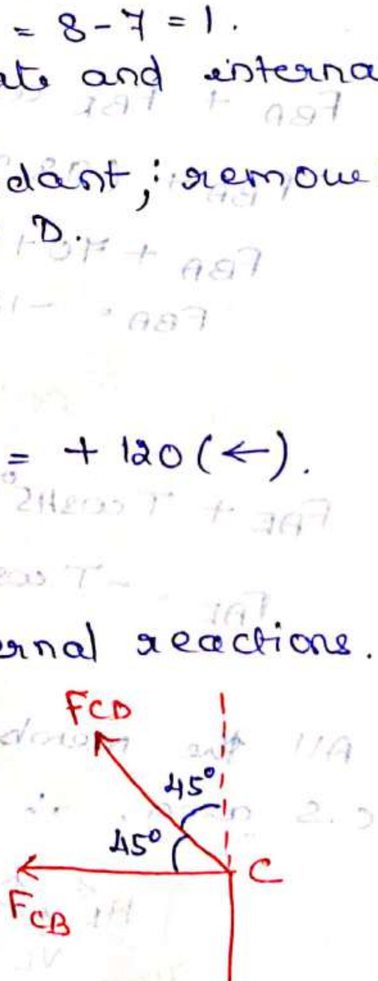
$$(+H_E \times 3) + (50\times6) + (20\times3) = 0.$$

$$H_E = -\left(\frac{300+60}{3}\right) = 120\ KN(\rightarrow) = +120(\leftarrow).$$

$$H_A = 120(\rightarrow)$$

T-T will not create and external reactions.

Joint C: $\Sigma V = 0$ gives,



$F_{CD}\cos45^\circ - 50 = 0$

$F_{CD} = \frac{50}{\cos45} = 70.71.$ (tensile)

$\Sigma H = 0$ gives

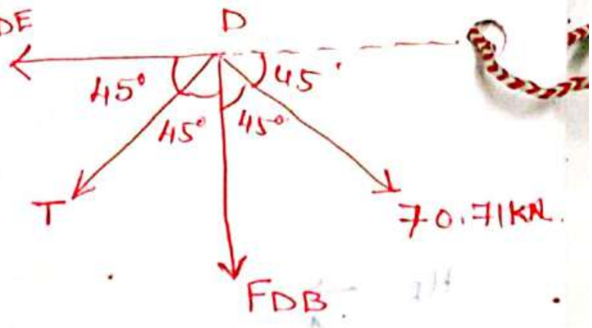
$-F_{CB} \overline{\oplus} F_{CD}\cos45^\circ = 0.$

$-F_{CB} \overline{\oplus} 70.71\cos45^\circ = 0;\ F_{CB} = -50\ KN;\ F_{CB} = 50\ KN$ (comp.).

2'DOF).

Joint D: $\Sigma H = 0$ gives.

$-F_{DE} - T\cos45^\circ + 70.71\cos45^\circ = 0.$



$F_{DE} + 0.707T = 50$

← −ve
→ +ve

$\boxed{F_{DE} = 50 - 0.707T}$

$\Sigma V = 0$ gives

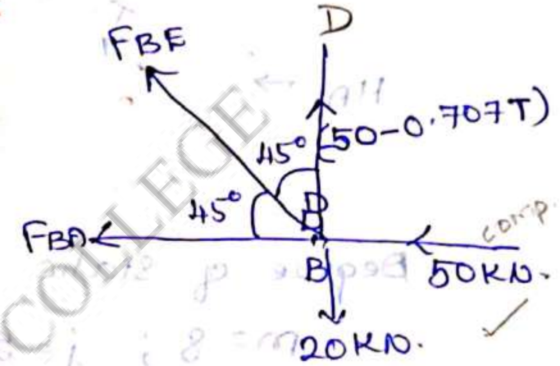
$-(F_{DB} + T\cos45^\circ + 70.71\cos45^\circ) = 0.$

↑ +ve
↓ −ve

$F_{DB} + 0.707T = -50.$

$\boxed{F_{DB} = -50 - 0.707T}.$

Joint B: $\Sigma V = 0$ gives,



$F_{BE}\cos45^\circ + (-50 - 0.707T) - 20 = 0,$

$F_{BE}\cos45^\circ = 70 + 0.707T$

$\boxed{F_{BE} = 98.99 + T}$

$\Sigma H = 0$ gives.

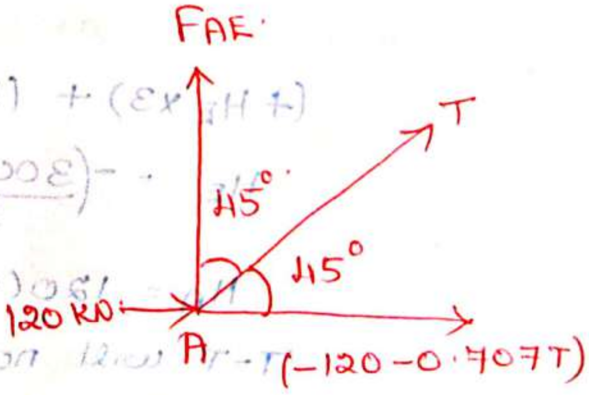
$-(F_{BA} + F_{BE}\cos45^\circ + 50) = 0$

$F_{BA} + (98.99 + T)\cos45^\circ + 50 = 0.$

$F_{BA} + 70 + 0.707T + 50 = 0$

$\boxed{F_{BA} = -120 - 0.707T}$

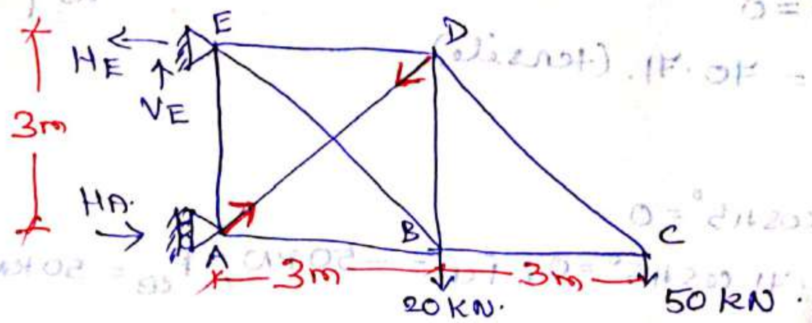
Joint A: $\Sigma V = 0$ gives,



$F_{AE} + T\cos45^\circ = 0$

$F_{AE} = -T\cos45^\circ = -0.707T$

All the members are of ~~of~~ same material and same c.S. area. ∴ AE = constant.



S.No.	Member	F(KN)	$\frac{\partial F}{\partial R}$ or $\frac{\partial F}{\partial T}$	l(m)	$F\frac{\partial F}{\partial R}l$ or $F\frac{\partial F}{\partial T}l$.
1.	AB	$-120-0.707T$	-0.707	3	$(120+0.707T)\times0.707\times3$
2.	BC	-50	0	3	0
3.	CD	70.71	0	4.243	0
4.	DE	$50-0.707T$	-0.707	3	$(50-0.707T)\times(-0.707)\times3$
5.	EA	$-0.707T$ ~~98.99+T~~	-0.707	3	$0.707T\times0.707\times3$
6.	EB	$98.99+T$	1	4.243	$(98.99+T)\times4.243.$
7.	BD	$-50-0.707T$	-0.707	3	$(50+0.707T)\times0.707\times3$
8.	AD	T	1	4.243	$4.243T.$

$$\frac{\partial U}{\partial R} = 0 = \Sigma F\frac{\partial F}{\partial R}\cdot\frac{l}{AE}.$$

$$= \frac{1}{AE}[254.52 + 1.5T - 106.05 + 1.5T + 1.5T + 420.01 + 4.243T + 106.05 + 1.5T] = 0.$$

$$674.53 + 14.486T = 0.$$

$$\boxed{T = -46.56 \text{ KN.}}$$

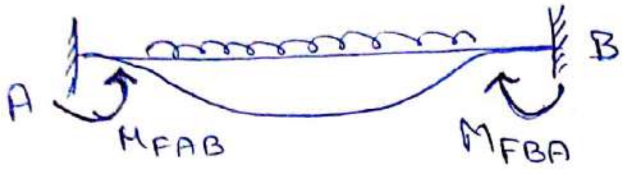
∴ Force in the redundant member AD = 46.56 KN (comp.)

UNIT II

Slope Deflection method.

continuous beams and rigid frames (with and without sway) - symmetry and antisymmetry - simplification of hinged end - support displacements.

Moment M_A in a typical member AB is made up of 4 parts.



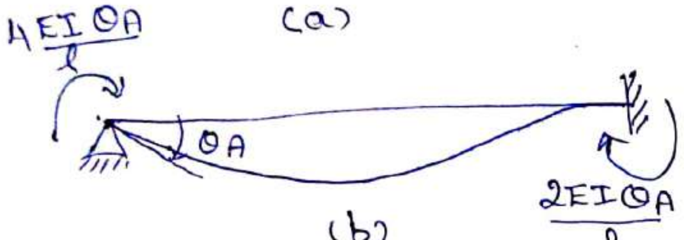
(a)

Sign convention.

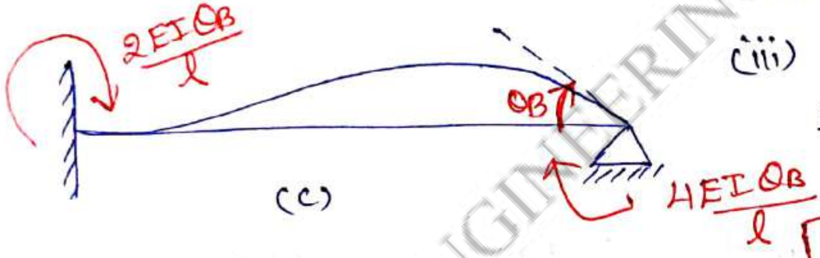
(i) Fig. shows +ve signs for forces, moments, deflections & rotations.

(ii) For moments & rotations clockwise is +ve

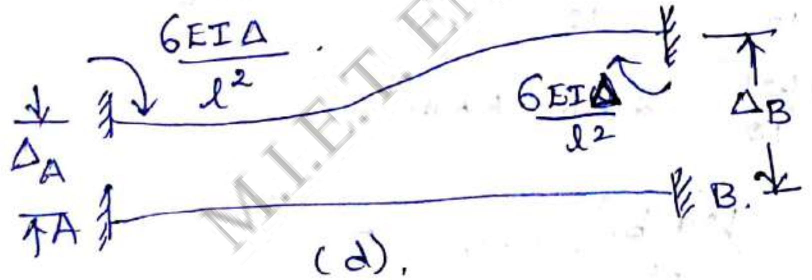
(iii) for differential sinking, right upward movement is positive.



(b)



(c)



(d).

Fixed end moments
udl - $\frac{wl^2}{12}$
pt load - $wl/8$.

$\Delta = \Delta_B - \Delta_A$

Slope deflection equations: comprises of :

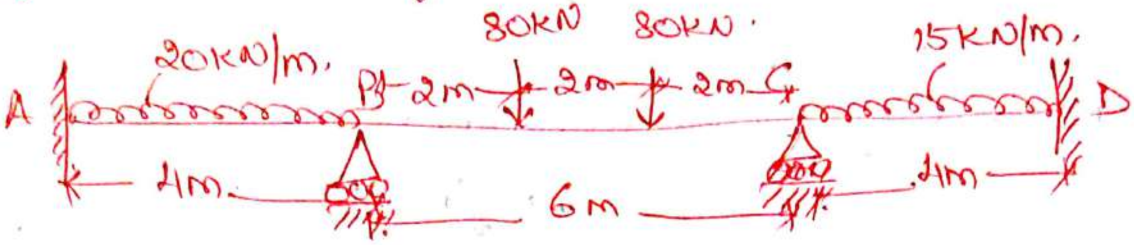
1) Fixed end moment due to external load
2) Moment due to rotation at A.
3) Moment due to rotation at B.
4) Moment due to differential transverse displacement of B above A.

$$M_{AB} = M_{FAB} + \frac{4EI\theta_A}{l} + \frac{2EI\theta_B}{l} + \frac{6EI\Delta}{l^2}, \text{ if } \frac{I}{l} = K,$$

$$\boxed{M_{AB} = M_{FAB} + 2E\frac{I}{l}\left(2\theta_A + \theta_B + \frac{3\delta}{l}\right)} ; \boxed{M_{BA} = M_{FBA} + 2E\cancel{K}\left(\theta_A + 2\theta_B + \frac{3\delta}{l}\right)}$$

Analysis of continuous Beam:

① Analyse a continuous beam loaded as shown in fig by the slope deflection method and sketch BM diagram.



$$2I_{AB} = I_{BC} = 2I_{CD} = 2I.$$

$I_{AB} = I_{CD} = I$, $I_{BC} = 2I$, $\theta_A = \theta_D = 0$ (A and D are fixed)

Unknowns θ_B and θ_C

a) Fixed end moments:

b) Slope defln equations.

c) Equilibrium equations.

d) Final Moments.

Fixed end moments:

For udl $= \frac{-wl^2}{12}$

Pt. load $= \frac{-Wab^2}{l^2}$ MFAB

MFBA $= +\frac{Wa^2b}{l^2}$

MFAB $= -\frac{Wl}{8}$

MFBA $= +\frac{Wl}{8}$

a) Fixed end moments:

Span AB:

$$M_{FAB} = -\frac{wl^2}{12} = \frac{-20\times4^2}{12} = -26.67\text{ kNm}.$$

$$M_{FBA} = +\frac{wl^2}{12} = 26.67\text{ kNm}.$$

Span BC:

$$M_{FBC} = -\left[\frac{Wa_1b_1^2}{l^2} + \frac{Wa_2b_2^2}{l^2}\right]$$

$$= -\left[\frac{80\times2\times4^2}{6^2} + \frac{80\times4\times2^2}{6^2}\right] = -106.67\text{ kNm}.$$

$$MFCB = \left[\frac{Wa_1^2b_1}{l^2} + \frac{Wa_2^2b}{l^2}\right] = +106.67\text{ kNm}.$$

Span CD: $M_{FCD} = -\frac{wl^2}{12} = \frac{-15\times4^2}{12} = -20\text{ kNm},$

$$M_{FDC} = \frac{wl^2}{12} = 20\text{ kNm}.$$

b) Slope deflection equations.

$$M_{AB} = M_{FAB} + \frac{2EI}{l}\left[2\cancelto{0}{\theta_A} + \theta_B + \cancelto{0}{\frac{3\delta}{l}}\right]$$; $\delta = 0$ since no settlement of supports.

$$= -26.67 + \frac{2EI}{4}\left[0 + \theta_B + 0\right]$$

$$= -26.67 + 0.5EI\,\theta_B.$$

$$M_{BA} = M_{FBA} + \frac{2EI}{l}\left(2\theta_B + \cancelto{0}{\theta_A} + \cancelto{0}{\frac{3\delta}{l}}\right)$$

$$= 26.67 + \frac{2EI}{4}\left(2\theta_B + 0 + 0\right)$$

$$= 26.67 + EI\,\theta_B.$$

$$M_{BC} = M_{FBC} + \frac{2EI}{l}\left(2\theta_B + \theta_C + \cancelto{0}{\frac{3\delta}{l}}\right)$$

$$= -106.67 + \frac{2E \times 2I}{6}\left(2\theta_B + \theta_C + 0\right).$$

$$= -106.67 + EI\left(1.333\theta_B + 0.666\theta_C\right).$$

$$M_{CB} = M_{FCB} + \frac{2EI}{l}\left(2\theta_C + \theta_B + \cancelto{0}{\frac{3\delta}{l}}\right).$$

$$= 106.67 + \frac{2E \times 2I}{6}\left(2\theta_C + \theta_B + \cancelto{0}{\frac{3\delta}{l}}\right)$$

$$= 106.67 + EI\left(1.333\theta_C + 0.666\theta_B\right).$$

$$M_{CD} = M_{FCD} + \frac{2EI}{l}\left(2\theta_C + \cancelto{0}{\theta_D} + \cancelto{0}{\frac{3\delta}{l}}\right)$$

$$= -20 + \frac{2EI}{A}\left(2\theta_C + 0 + 0\right)$$

$$= -20 + \cancel{2}EI\theta_C.$$

$$M_{DC} = M_{FDC} + \frac{2EI}{l}\left(2\cancelto{0}{\theta_D} + \theta_C + \cancelto{0}{\frac{3\delta}{l}}\right)$$

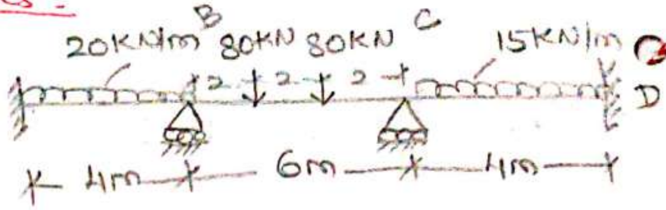
$$= 20 + \frac{2EI}{4}\left(\cancel{\theta_C}\ 0 + \theta_C + 0\right).$$

$$= 20 + EI\left(0.5\theta_C\right).$$

c) Equilibrium equations:

$M_{BA} + M_{BC} = 0$

$M_{CB} + M_{CD} = 0$



(1) $M_{BA} + M_{BC} = 0$

$$26.67 + EI\,\theta_B - 106.67 + EI(1.333\,\theta_B + 0.666\,\theta_C) = 0$$

$$2.333\,\theta_B + 0.666\,\theta_C = \frac{80}{EI} \quad \cdots\cdots ①$$

(2) $M_{CB} + M_{CD} = 0.$

$$106.67 + EI(0.666\,\theta_B + 1.333\,\theta_C) - 20 + EI\,\theta_C = 0$$

$$0.666\,\theta_B + 2.333\,\theta_C = \frac{-86.67}{EI} \quad \cdots ②$$

solving equ. ① and ②,

$$\begin{bmatrix} 2.333 & 0.666 \\ 0.666 & 2.333 \end{bmatrix} \begin{Bmatrix} \theta_B \\ \theta_C \end{Bmatrix} = \frac{1}{EI} \begin{Bmatrix} 80 \\ -86.67 \end{Bmatrix}$$

$$\begin{Bmatrix} \theta_B \\ \theta_C \end{Bmatrix} = \frac{1}{EI} \begin{Bmatrix} 80 \\ -86.67 \end{Bmatrix} \begin{bmatrix} 2.333 & 0.666 \\ 0.666 & 2.333 \end{bmatrix}^{-1}$$

$$\frac{1}{2.333^2 - 0.666^2} \ast \frac{1}{EI} \begin{Bmatrix} \overrightarrow{16} \\ -17.34 \end{Bmatrix} \begin{bmatrix} 2.333 & -0.666 \\ -0.666 & 2.333 \end{bmatrix}\downarrow$$

$$= \frac{1}{5}$$

$$\begin{Bmatrix} \theta_B \\ \theta_C \end{Bmatrix} = \frac{1}{EI} \begin{Bmatrix} 48.88 \\ -51.11 \end{Bmatrix} = \begin{Bmatrix} 2.333 \times 16 + 17 \times 0.666 \\ -17.34 \times 0.666 - 17.34 \times 2.333 \end{Bmatrix}$$

d) Final Moments:

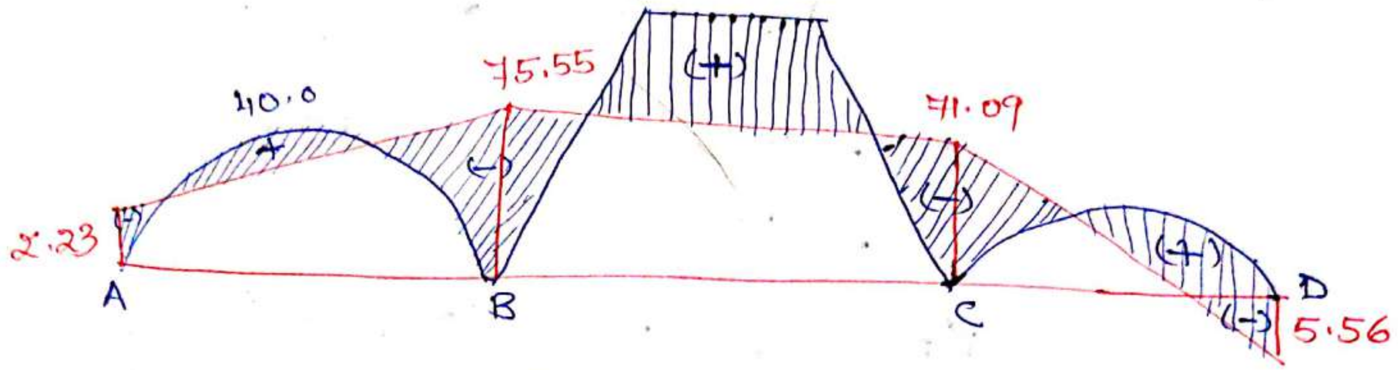
Sub. the values of θ_B and θ_C in all the moment equations,

$$M_{AB} = -26.67 + 0.5\,EI\left(\frac{48.88}{EI}\right) = -2.23 \text{ KNm.}$$

$M_{BA} = 75.55$ KNm ; $M_{BC} = -75.55$ KNm.

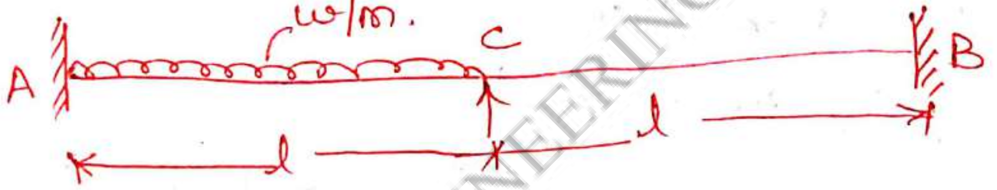
$M_{CB} = +71.09$ KNm ; $M_{CD} = -71.1$ KNm.

$M_{DC} = -5.56$ KNm.

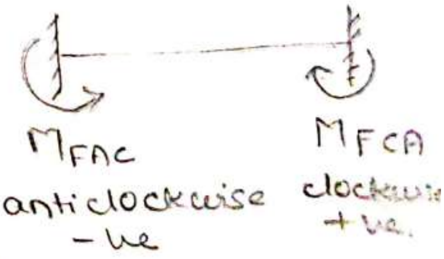


BMD.

② Analyse the two span continuous beam loaded as shown in fig. by the slope deflection method and sketch BMD and SFD.



a) Fixed end moments:



Span AC: $M_{FAC} = -\frac{wl^2}{12}$

$M_{FCA} = \frac{wl^2}{12}$

Span CB: $M_{FCB} = 0$, $M_{FBC} = 0$

b) Slope deflection equations:

Span AC: M_{AC} = Final moment at A.

$$M_{AC} = M_{FAC} + \frac{2EI}{l}\left(2\cancelto{0}{\theta_A} + \theta_C + \cancelto{0}{\frac{3\delta}{l}}\right).$$

$$= -\frac{wl^2}{12} + \frac{2EI}{l}\left[2(0) + \theta_C + 0\right]$$

($\theta_A = 0$ ∵ A is fixed and δ = settlement = 0)

$$M_{AC} = \frac{-wl^2}{12} + \frac{2EI\theta_C}{l}$$

$$M_{CA} = M_{FCA} + \frac{2EI}{l}\left(2\theta_C + \overset{0}{\cancel{\theta_A}} + \overset{0}{\cancel{\frac{3\delta}{l}}}\right)$$

$$= \frac{wl^2}{12} + \frac{4EI\theta_C}{l}$$

Span CB: $M_{CB} = M_{FCB} + 2\frac{EI}{l}\left[2\theta_C + \overset{0}{\cancel{\theta_B}} + \overset{0}{\cancel{\frac{3\delta}{l}}}\right]$

$$= 0 + \frac{2EI}{l}(2\theta_C + 0 + 0)$$

$$= \frac{4EI\theta_C}{l}.$$

$$M_{BC} = M_{FBC} + \frac{2EI}{l}\left(2\overset{0}{\cancel{\theta_B}} + \theta_C + \overset{0}{\cancel{\frac{3\delta}{l}}}\right)$$

$$= 0 + \frac{2EI}{l}\left[2(0) + \theta_C + 0\right]$$

$$= \frac{2EI\theta_C}{l}.$$

c) Joint equilibrium equations:

Joint c: $M_{CA} + M_{CB} = 0.$

$$\frac{wl^2}{12} + \frac{4EI\theta_C}{l} + \frac{4EI\theta_C}{l} = 0.$$

$$\frac{wl^2}{12} + \frac{8EI\theta_C}{l} = 0 \Rightarrow \frac{8EI\theta_C}{l} = -\frac{wl^2}{12}$$

$$\theta_C = \frac{-wl^3}{12\times 8EI} \Rightarrow \boxed{\theta_C = \frac{-wl^3}{96EI}.}$$

d) Final Moments:

$$M_{AC} = -\frac{wl^2}{12} + \frac{2EI\theta_C}{l} = -\frac{wl^2}{12} + \frac{2EI}{l}\left(-\frac{wl^3}{96EI}\right)$$

$$= -\frac{wl^2}{12} - \frac{wl^2}{48} = -wl^2\left(\frac{4+1}{48}\right)$$

$$= -\frac{5wl^2}{48}$$ anticlock

$$M_{CA} = \frac{wl^2}{12} + \frac{4EI\theta_C}{l} = \frac{wl^2}{12} + \frac{4EI}{l}\left(-\frac{wl^3}{96EI}\right).$$

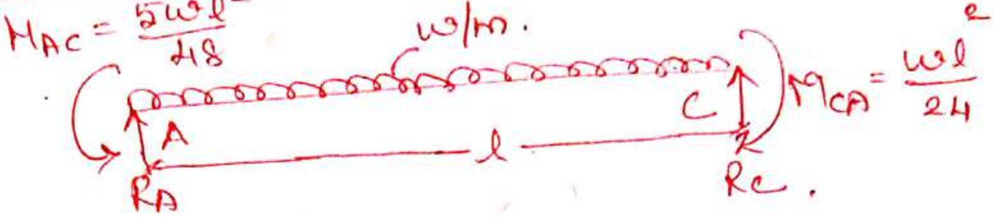
$$= \frac{wl^2}{12} - \frac{wl^2}{24} = \frac{wl^2}{24}$$ clock

$$M_{CB} = \frac{4EI\theta_c}{l} = \frac{4EI}{l}\left(-\frac{wl^3}{96EI}\right) = -\frac{wl^2}{24}$$ anticlock

$$M_{Bc} = \frac{2EI\theta_c}{l} = \frac{2EI}{l}\left(-\frac{wl^3}{96EI}\right) = -\frac{wl^2}{48}$$ anticlock

e) To draw shear force diagram:

Span AC:



Taking moment about A,

$$-R_c\times l + \frac{w.l.l}{2} - \frac{5w.l^2}{48} + \frac{wl^2}{24} = 0.$$

anti cw -ve
cw +ve.

$$\therefore -R_c\times l + wl^2\left(\frac{1}{2} - \frac{5}{48} + \frac{1}{24}\right) = 0.$$

$$R_c\times l = wl^2\left(\frac{24-5+2}{48}\right) = wl^2\left(\frac{21}{48}\right).$$

$R_c = \frac{9wl}{}$

$$\boxed{R_c = \frac{7wl}{16}}.$$

R_A = total load − R_c = $wl - \frac{7wl}{16}$

$$\boxed{R_A = \frac{9wl}{16}}$$

Span CB:

↑ ✓

$$\boxed{R_B = \frac{-wl}{16}}$$ ↓

Taking moments about C,

$$-R_B\times l - \frac{wl^2}{48} - \frac{wl^2}{24} = 0 \qquad R_B = \frac{-wl}{16}\ (\uparrow)$$

$R_c + R_B = 0$ ✓

$$\boxed{R_c = +\frac{wl}{16}}$$ ↑

or $R_B = \frac{wl}{16}$ (↓) ; $R_c = \frac{wl}{16}$ (↑)

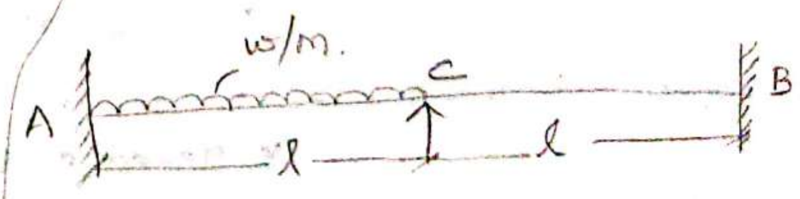
∴ ↑ ✗

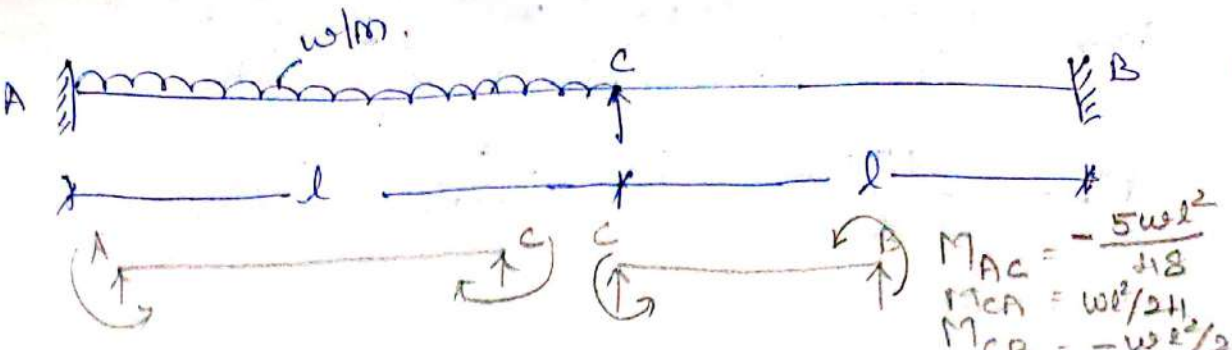
↓ ✓

taking moment about B,

$$R_c\times l - \frac{wl^2}{24} - \frac{wl^2}{48} = 0.$$

$$\boxed{R_c = \frac{wl}{16}\ (\uparrow)}$$



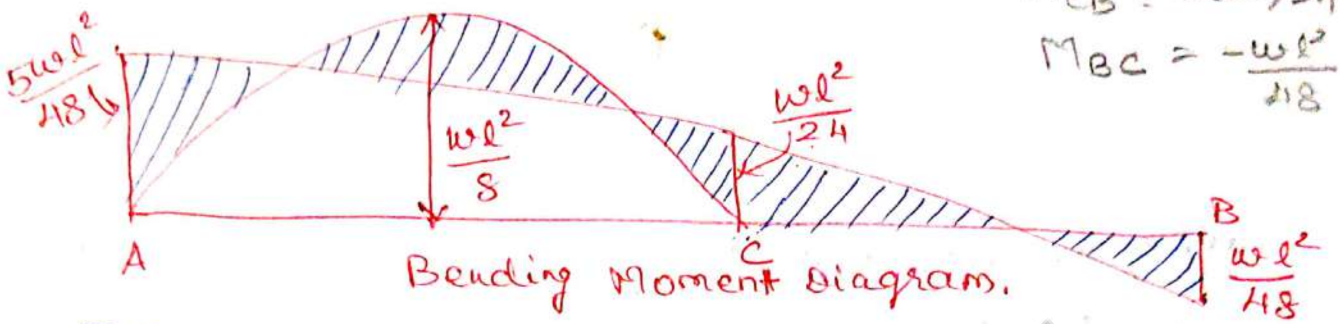


$M_{AC} = -\frac{5wl^2}{48}$

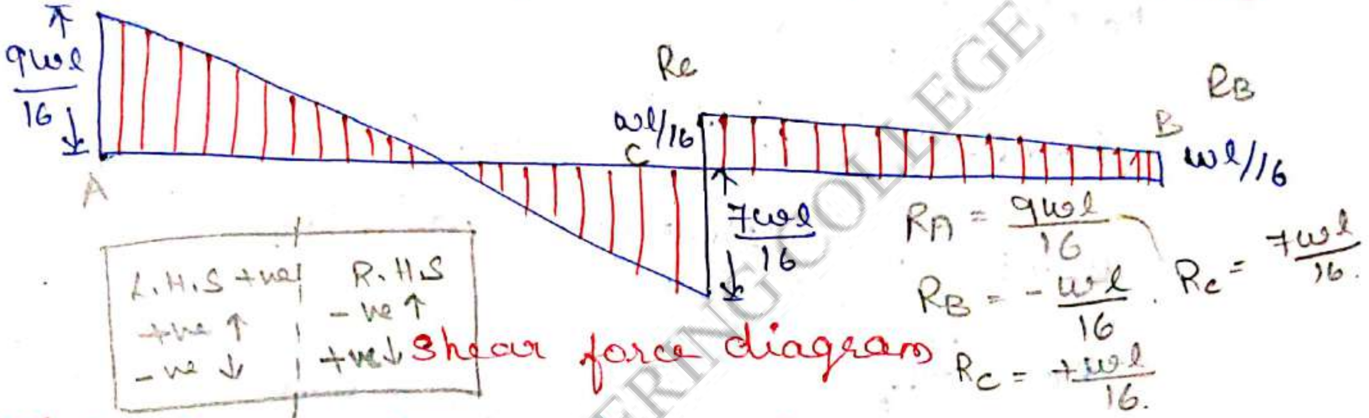
$M_{CA} = wl^2/24$

$M_{CB} = -wl^2/24$

$M_{BC} = -\frac{wl^2}{48}$



Bending Moment Diagram.



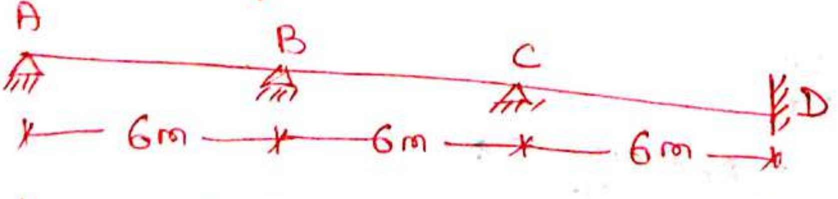
Shear force diagram

$R_A = \frac{9wl}{16}$ $R_c = \frac{7wl}{16}$

$R_B = -\frac{wl}{16}$

$R_C = +\frac{wl}{16}$

③ Analyse the continuous beam shown in fig. by the slope defln. method. The supports B and C sink 10mm and 5mm respectively and the support D rotates through an anti-clockwise angle of 0.1 radian. There are no loads on the beam. Values of E and I are constants thro'out the length of the beam. Take $E = 2 \times 10^5$ MPa, $I = 4 \times 10^7 mm^4$. Sketch BM diagram.



Fixed end moments:* Since all spans are not loaded, fixed end moments in all spans = 0. due to loading.

* Fixed end moments due to settlement of B and C and rotation of end D will be included in slope defln. equation.

Slope deflection equations:

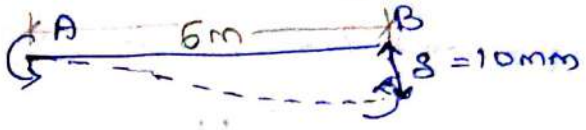
$\boxed{M_{AB}} = M_{FAB} + \frac{2EI}{l}\left(2\theta_A + \theta_B + \frac{3\delta}{l}\right)$

∵ no load

$= \cancel{0}^{0} + \frac{2EI}{l}\left(2\theta_A + \theta_B - \frac{3\times10}{6000}\right).$

$\left[\frac{2EI}{l} = \frac{2\times2\times10^5\times4\times10^7}{6000} = \frac{16}{6}\times10^9 = \frac{8\times10^9}{3}\right]$

R.H.S support sinking -ve.



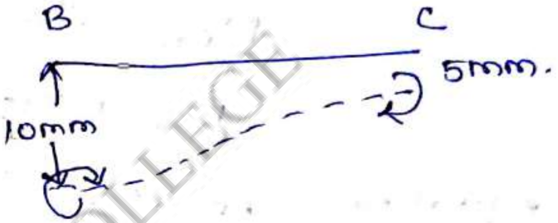
$M_{AB} = \frac{2EI}{l}\left(2\theta_A + \theta_B - \frac{1}{200}\right)$

no load

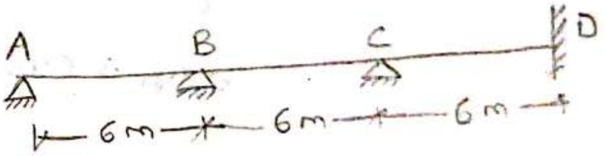
$\boxed{M_{BA}} = \cancel{M_{FBA}} + \frac{2EI}{l}\left(2\theta_B + \theta_A + \frac{3\delta}{l}\right)$

$= 0 + \frac{2EI}{l}\left(2\theta_B + \theta_A - \frac{3\times10}{6000}\right)$

$= \frac{2EI}{l}\left(2\theta_B + \theta_A - \frac{1}{200}\right)$



$\boxed{M_{BC}} = M_{FBC} + \frac{2EI}{l}\left(2\theta_B + \theta_C + \frac{3\delta}{l}\right)$



Net $\delta = 5mm$ (↑ at B)

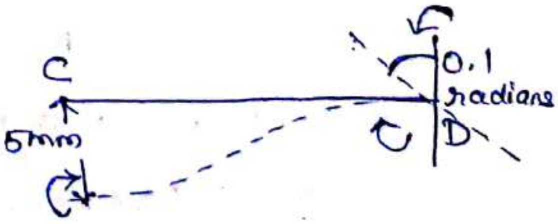
$M_{BC} = \cancel{0} + \frac{2EI}{l}\left(2\theta_B + \theta_C + \frac{3\times5}{6000}\right)$ L.H.S support sinking hence +ve

$= \frac{2EI}{l}\left(2\theta_B + \theta_C + \frac{1}{400}\right)$

$\boxed{M_{CB}} = M_{FCB} + \frac{2EI}{l}\left(2\theta_C + \theta_B + \frac{3\delta}{l}\right)$

$= 0 + \frac{2EI}{l}\left(2\theta_C + \theta_B + \frac{3\times5}{6000}\right)$

$= \frac{2EI}{l}\left(2\theta_C + \theta_B + \frac{1}{400}\right)$



$\boxed{M_{CD}} = M_{FCD} = \frac{2EI}{l}\left(2\theta_C + \theta_D + \frac{3\delta}{l}\right)$

$= \frac{2EI}{l}\left(2\theta_C + \theta_D + \frac{3\times5}{6000}\right).$ L.H.S sinking +ve

$= \frac{2EI}{l}\left(2\theta_C - 0.1 + \frac{1}{400}\right)$ anticlockwise θ ∴ -ve

$$\boxed{M_{DC}} = M_{FDC} + \frac{2EI}{l}\left(2\theta_D + \theta_C + \frac{3\delta}{l}\right).$$

$$= 0 + \frac{2EI}{l}\left(-0.2 + \theta_C + \frac{3 \times 5}{6000}\right)$$

$$= \frac{2EI}{l}\left(-0.2 + \theta_C + \frac{1}{400}\right)$$

Equilibrium equations: There are three unknowns θ_A, θ_B and θ_C. The equilibrium eqns. are

$$M_{AB} = 0 \quad (\text{at } A)$$

$$M_{BA} + M_{BC} = 0 \quad (\text{at } B) \text{ ..... } ①$$

$$M_{CB} + M_{CD} = 0 \quad (\text{at } C) \text{. . . . } ②.$$

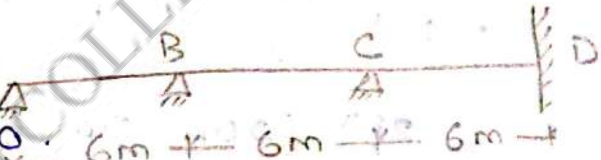
$\theta_A = 0$

$M_{AB} \neq 0$

$M_{AB} = 0$

$\theta_A \neq 0$

equating $M_{AB} = 0$,



$$\frac{2EI}{l}\left(2\theta_A + \theta_B - \frac{1}{200}\right) = 0$$

$$2\theta_A + \theta_B - \frac{1}{200} = 0$$

$$2\theta_A + \theta_B = 0.005 \text{ ------- } ⑤$$

Sub. ~~equ.~~ M_{BA} ~~&~~ M_{BC} in equ. ①

$$\frac{2EI}{l}\left(2\theta_B + \theta_A - \frac{1}{200}\right) + \frac{2EI}{l}\left(2\theta_B + \theta_C + \frac{1}{400}\right) = 0$$

$$\frac{2EI}{l}\left(4\theta_B + \theta_A + \theta_C - \frac{1}{400}\right) = 0$$

$$\theta_A + 4\theta_B + \theta_C - 0.0025 = 0$$

$$\theta_A + 4\theta_B + \theta_C = 0.0025 \text{ . . . . } ③$$

Sub. M_{CB} and M_{CD} in equ. ②,

$$\frac{2EI}{l}\left(2\theta_C + \theta_B + \frac{1}{400}\right) + \frac{2EI}{l}\left(2\theta_C - 0.1 + \frac{1}{400}\right) = 0.$$

$$\frac{2EI}{l}\left(4\theta_C + \theta_B - 0.1 + 0.005\right) = 0.$$

$$4\theta_C + \theta_B = 0.095 \text{ ------- } ④$$

$$③ \times 4 \Rightarrow 4\theta_A + 16\theta_B + 4\theta_C = 0.010$$

$$④ \Rightarrow \theta_B + 4\theta_C = 0.095 \quad \cdots\cdots ⑤$$

(−) (−) (−) (−)

$$4\theta_A + 15\theta_B = -0.085$$

$$(-)\ ⑤ \times 2 \Rightarrow 4\theta_A + 2\theta_B = 0.010$$

(−) (−) (−)

$$13\theta_B = -0.095.$$

$$\boxed{\theta_B = -0.0073}$$

$$4\theta_A + 15\theta_B = -0.085$$

$$4\theta_A + 15(-0.0073) = 0.085$$

$$4\theta_A = 0.085 + 0.1095$$

$$4\theta_A = 0.1945$$

$$\theta_A = \frac{0.1945}{4} = 0.0486$$

Sub. in ⑤,

$$\theta_B + 4\theta_C = 0.095$$

$$-0.0073 + 4\theta_C = 0.095.$$

$$4\theta_C = 0.1023$$

$$\boxed{\theta_C = 0.025575}$$

Final moments: Substituting the values of θ_A, θ_B and θ_C in the fixed moment equations,

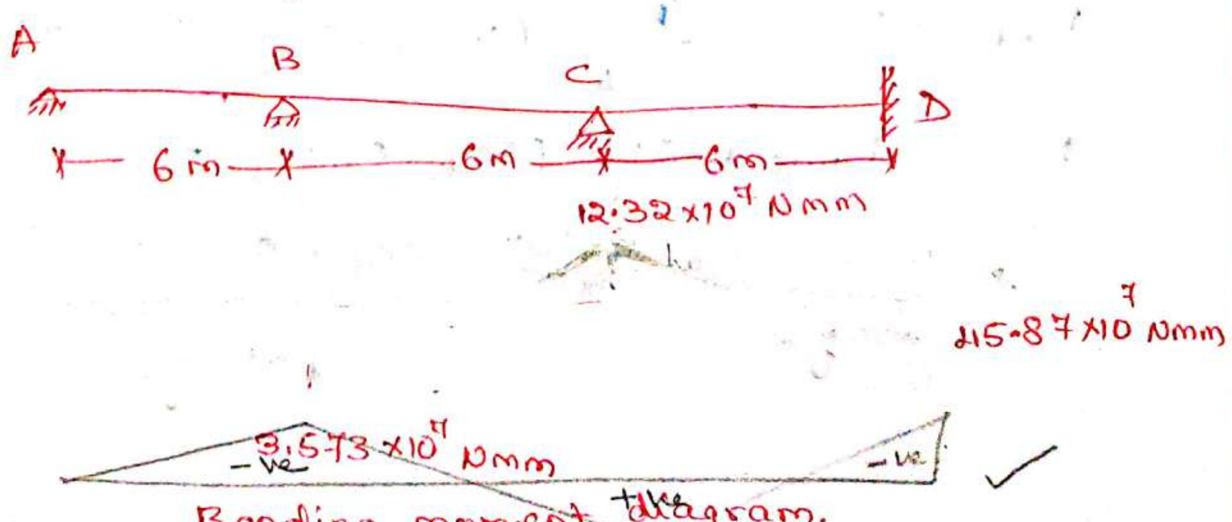
$M_{AB} = 0.027 \times 10^7$ Nmm. , $M_{BA} = -3.573 \times 10^7$ Nmm.

$M_{BC} = 3.573 \times 10^7$ Nmm, $M_{CB} = 12.32 \times 10^7$ Nmm.

−45.87

$M_{CD} = -12.40 \times 10^7$ Nmm , $M_{DC} =$ ~~[illegible]~~ $\times 10^7$ N mm.

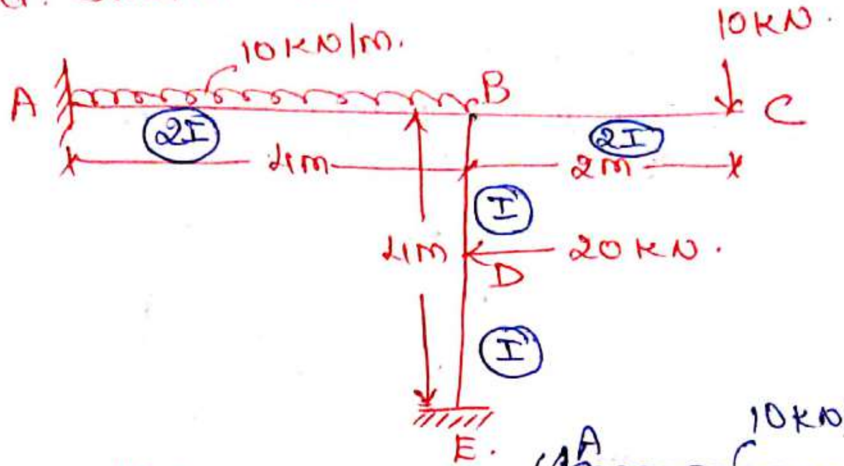
Sign convention



Bending moment ~~the~~ diagram.

✓

④ Analyse the structure shown in figure by the slope deflection method. Sketch the BMD and SFD.

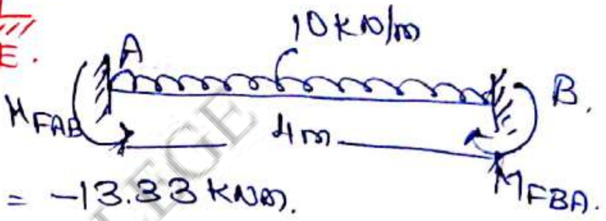


$\theta_A = 0$

$M_{AB} \neq 0$

$\theta_A \neq 0,\ M_{AB} = 0$

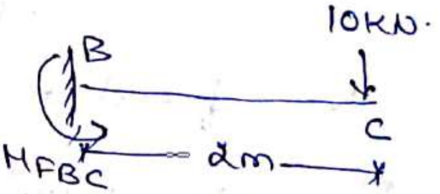
a) Fixed end moments:



Span AB: $M_{FAB} = -\frac{wl^2}{12} = \frac{-10\times16}{12} = -13.33$ KNm.

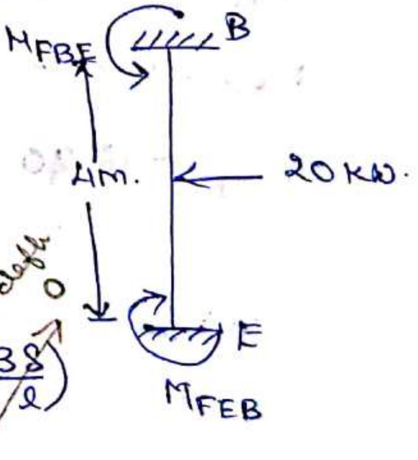
$M_{FBA} = \frac{wl^2}{12} = \frac{10\times16}{12} = 13.33$ KNm.

Span BC:



$M_{FBC} = -10\times2 = -20$ KNm.

Span B.E:



$M_{FBE} = \frac{-Wl}{8} = \frac{-20\times4}{8} = -10$KNm.

$M_{FEB} = \frac{Wl}{8} = 10$ KNm.

b) slope deflection equations:

Span AB: $M_{AB} = M_{FAB} + \frac{2EI}{l}\left(2\overset{\text{fixed } 0}{\cancel{\theta_A}} + \theta_B + \overset{\text{no defl } 0}{\cancel{\frac{3\delta}{l}}}\right)$

$M_{AB} = -13.33 + \frac{2E(2I)}{4}(0+\theta_B+0) = -13.33 + EI\theta_B.$

$M_{BA} = M_{FBA} + \frac{2EI}{l}\left(2\theta_B + \overset{0}{\cancel{\theta_A}} + \overset{0}{\cancel{\frac{3\delta}{l}}}\right) -$

$= 13.33 + \frac{2E(2I)}{4}(2\theta_B + 0 + 0) = 13.33 + 2EI\theta_B.$

Span BE: $M_{BE} = M_{FBE} + \frac{2EI}{l}\left(2\theta_B + \overset{\text{fixed}}{\cancel{\theta_E}} + \overset{\text{no defln}}{\cancel{\frac{3\delta}{l}}}\right)$

$= -10 + \frac{2EI}{4}(2\theta_B + 0 + 0)$

$= -10 + EI\theta_B$.

$M_{EB} = 10 + \frac{2EI}{l}\left(\overset{\text{fixed, 0}}{\cancel{2\theta_E}} + \theta_B + \overset{\text{no defln}}{\cancel{\frac{3\delta}{l}}}\right)$

$= 10 + \frac{2EI}{4}(\theta_B + 0)$

$= 10 + \frac{EI}{2}\theta_B$.

c) Joint equilibrium:

At jt B, $\Sigma M = 0$

$M_{BA} + M_{BC} + M_{BE} = 0$.

$13.33 + 2EI\theta_B - 10 + EI\theta_B - 20 = 0$

$3EI\theta_B - 16.67 = 0$

$\theta_B = \frac{5.557}{EI}$.

d) Final moments:

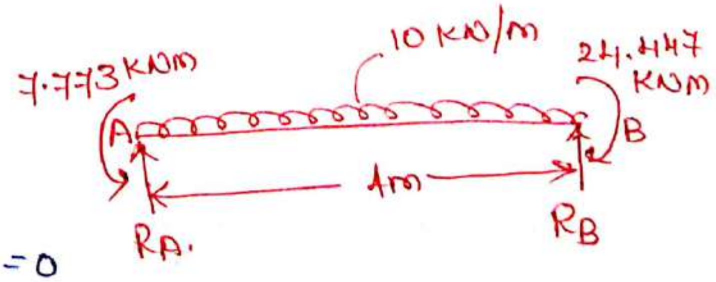
substituting the value of θ_B in moment equations,

$M_{AB} = -7.773$ kNm , $M_{BA} = 24.447$ kNm.

$M_{BC} = -20$ kNm , $M_{BE} = -4.443$ kNm.

$M_{EB} = 12.778$ kNm

e) To draw SFD:



Span AB:

Taking moment about B,

$R_A \times 4 - 10 \times 4\left(\frac{4}{2}\right) - 7.773 + 24.447 = 0$

$\boxed{R_A = 15.832\text{ kN}}$

$R_{BA} = 10 \times 4 - R_A = \boxed{24.168\text{ kN} = R_{BA}}$

CW +ve

ACW −ve.

Span BE: Taking moments at B,

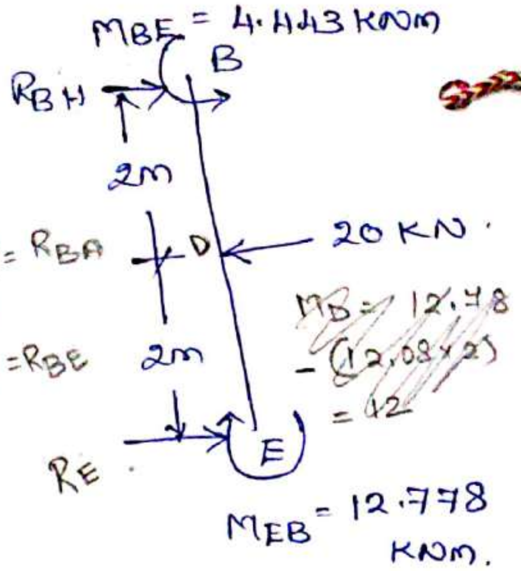
$-R_E(4) - 4.443 + 12.778 + 20(2) = 0$

$\boxed{R_E = 12.084 \text{ KN.}}$

R_{BH} = Total load − R_E
= 20 − 12.084

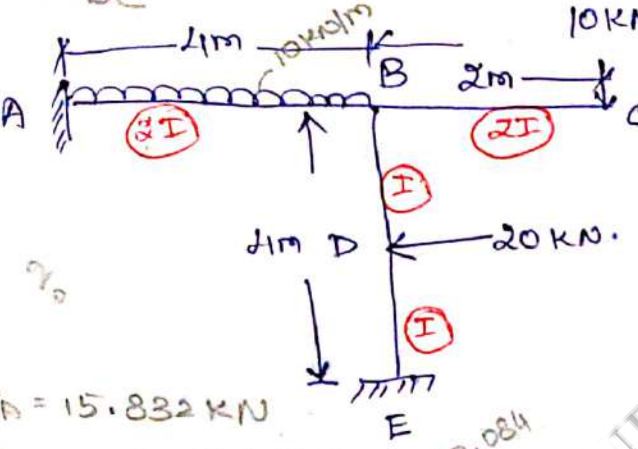
$\boxed{R_{\underset{BE}{BH}} = 7.916 \text{ KN}}$

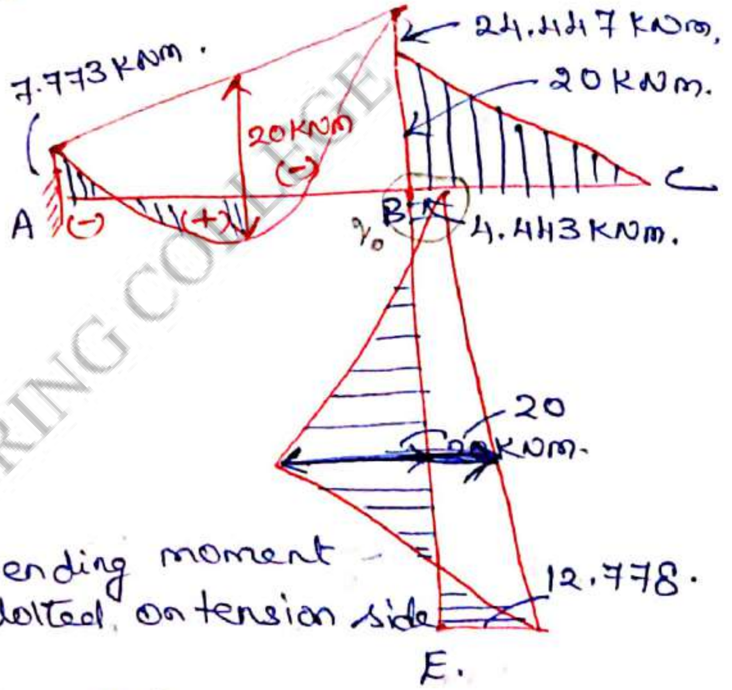
$R_A = 15.832$
$R_{BH} = 24.168 = R_{BA}$
$R_E = 12.084$
$R_{BH} = 7.916 = R_{BE}$



~~$M_B = 12.78 - (12.08 \times 2) = 12$~~

$M_{AB} = -7.773$; $M_{BE} = -4.443$
$M_{BA} = 24.447$; $M_{EB} = +12.778$
$M_{BC} = -20$;

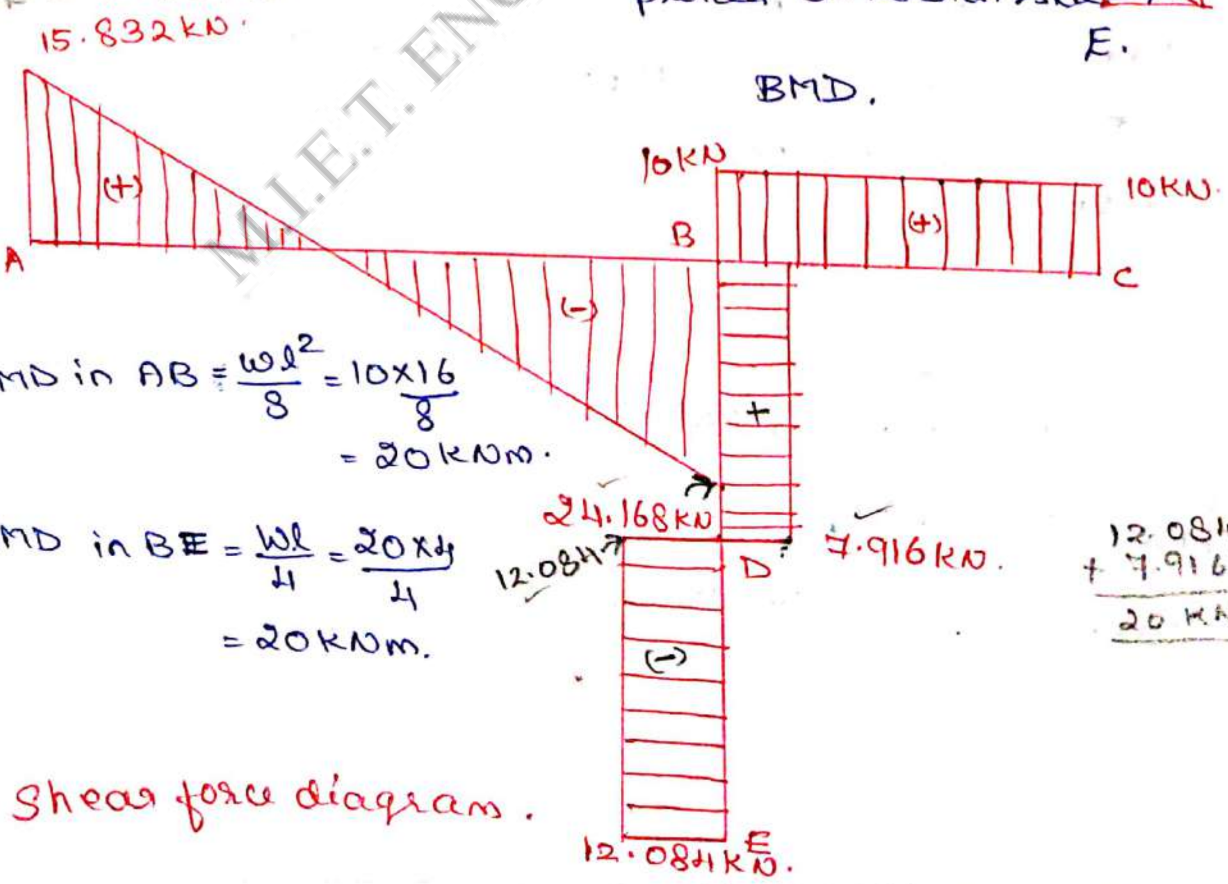




Bending moment plotted on tension side

BMD.

$R_A = 15.832$ KN
$R_{BA} = 24.168$ KN.
$R_{BE} = 7.916$ KN.
$R_E = 12.084$



S/S BMD in AB $= \frac{wl^2}{8} = \frac{10 \times 16}{8}$
= 20 KNm.

S/S BMD in BE $= \frac{Wl}{4} = \frac{20 \times 4}{4}$
= 20 KNm.

12.084
+ 7.916
20 KN

Shear force diagram.

Portal frames with side sway.

The most general equation:

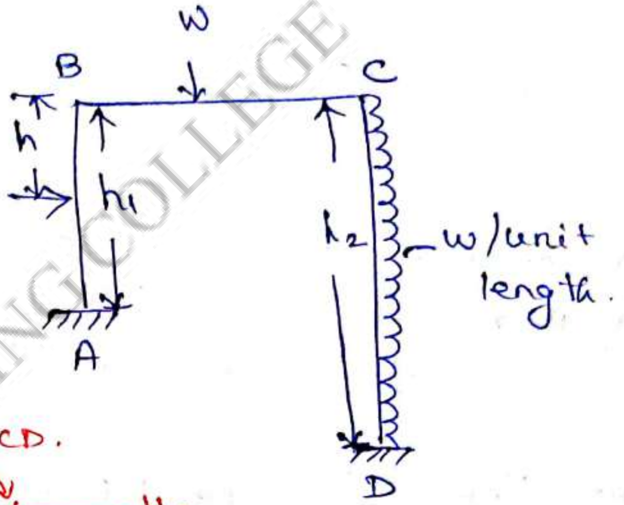
case 1: $\theta_A + \theta_D = 0$ ∵ it is fixed.

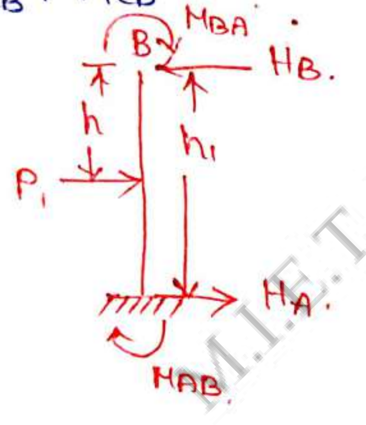
unknowns are θ_B, θ_C and δ.

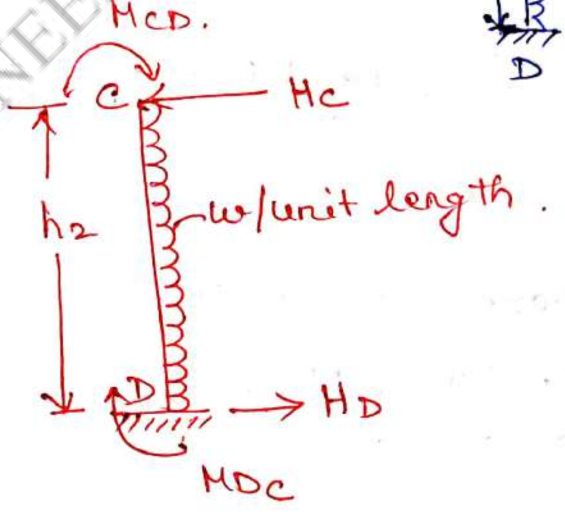
For solutions, equilibrium eqns are

$M_{BA} + M_{BC} = 0$

$M_{CB} + M_{CD} = 0.$



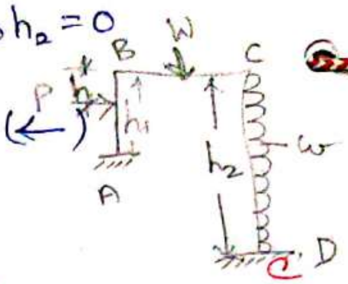




Case 1:

$$\frac{M_{AB}+M_{BA}-P\times h}{h_1}+\frac{M_{DC}+M_{CD}+\frac{wh_2^2}{2}}{h_2}+P-wh_2=0$$

(→) (→) (→) (←)



Case 2:

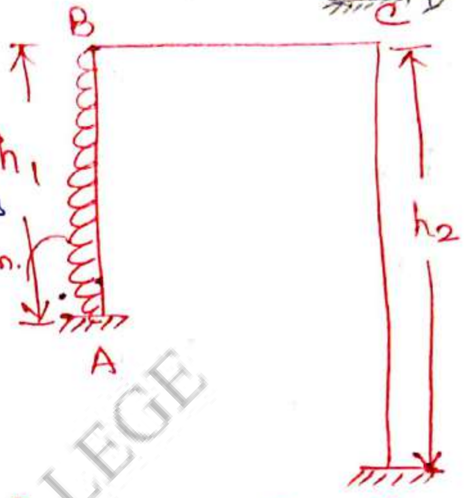
In column AB, udl is acting instead of P and column CD is free from udl.

∴ The shear equation gets modified as

$$\frac{M_{AB}+M_{BA}-\frac{wh_1^2}{2}}{h_1}+\frac{M_{DC}+M_{CD}}{h_2}+w\times h_1=0.$$

0 no load

→



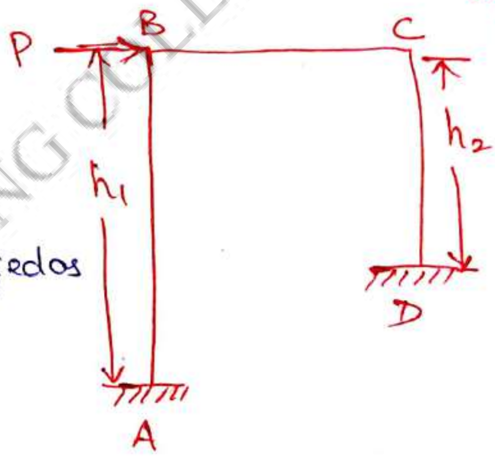
Case 3:

Here lateral load acts at the top of AB. (∴ h = 0)

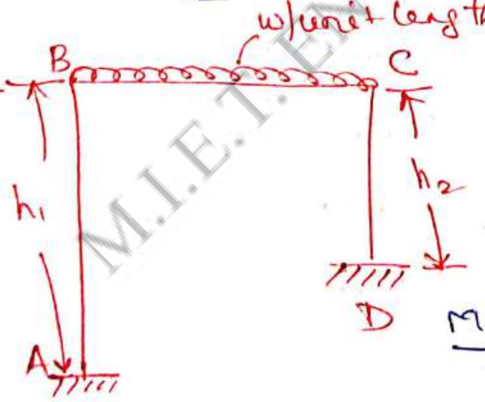
There is no udl of CD,

∴ The shear equation gets modified as

$$\frac{M_{AB}+M_{BA}}{h_1}+\frac{M_{DC}+M_{CD}}{h_2}+P=0.$$



Case 4:



Here there is no lateral load on AB and CD.

Hence shear equation:

$$\frac{M_{AB}+M_{BA}}{h_1}+\frac{M_{DC}+M_{CD}}{h_2}=0.$$

S

Case 5:

Here there is no lateral loading on AB. There is a pt. load P, hinge support D

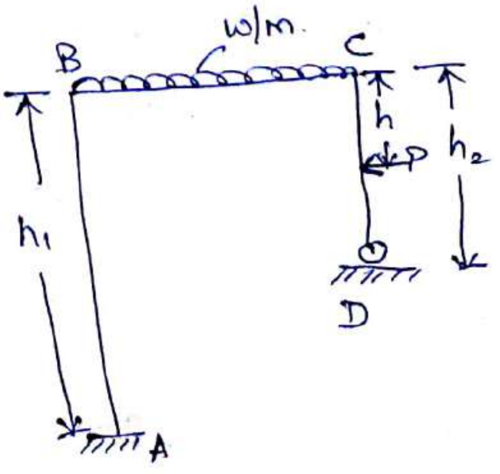
S

∴ $M_{DC}=0$.

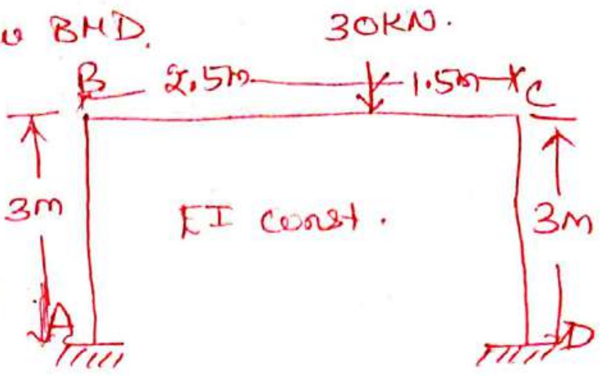
shear equation:

$$\frac{M_{AB}+M_{BA}}{h_1}+\frac{M_{CD}+Ph}{h_2}-P=0.$$

(←)



⑤ Analyse the portal frame loaded as shown in figure by slope defln. method and draw BMD.



a) Fixed end moments:

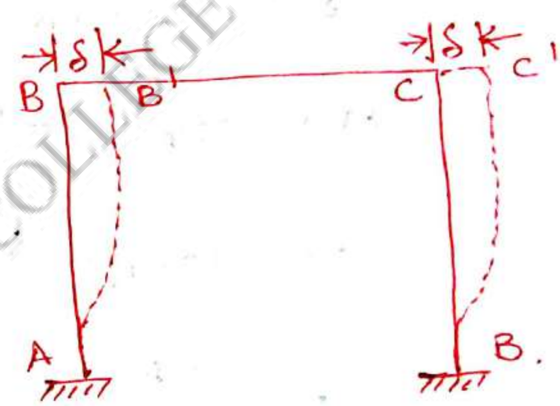
$M_{FAB} = M_{FBA} = M_{FCD} = M_{FDC} = 0.$

[∵ No lateral load on spans AB and CD]

$$M_{FBC} = -\frac{Wab^2}{l^2} = -\frac{30\times 2.5\times 1.5^2}{4^2} = -10.547 \text{ KNm.}$$

$$M_{FCB} = +\frac{Wa^2b}{l^2} = +\frac{30\times 2.5^2\times 1.5}{4^2} = 17.578 \text{ KNm.}$$

b) Slope deflection equations:



The frame will sway, if there is eccentric or unsymmetrical loading.

Span AB:

$$M_{AB} = M_{FAB} + \frac{2EI}{l}\left(2\theta_A + \theta_B - \frac{3\delta}{l}\right)$$

$$= 0 + \frac{2EI}{3}\left(0 + \theta_B - \frac{3\delta}{3}\right).$$ ($\theta_A = 0$, since support A is fixed)

$$= \frac{2EI}{3}(\theta_B - \delta).$$

$$M_{BA} = M_{FBA} + \frac{2EI}{l}\left(2\theta_B + \theta_A - \frac{3\delta}{l}\right)$$

$$= 0 + \frac{2EI}{3}\left(2\theta_B + 0 - \frac{3\delta}{3}\right) = \frac{2EI}{3}(2\theta_B - \delta).$$

Span BC: $M_{BC} = M_{FBC} + \frac{2EI}{l}\left(2\theta_B + \theta_C + \cancel{\frac{3\delta}{l}}^{0}\right)$

$$= -10.547 + \frac{2EI}{4}(2\theta_B + \theta_C + 0).$$

$$= -10.547 + \frac{EI}{2}(2\theta_B + \theta_C).$$

$$M_{CB} = M_{FCB} + \frac{2EI}{l}\left(2\theta_C + \theta_B + \frac{3\delta}{l}\right)$$

$$= 17.578 + \frac{2EI}{4}(2\theta_C + \theta_B + 0)$$

$$= 17.578 + \frac{EI}{2}(2\theta_C + \theta_B).$$

Span CD:

$$M_{CD} = M_{FCD} + \frac{2EI}{l}\left(2\theta_C + \theta_D + \frac{3\delta}{l}\right)$$

$$= 0 + \frac{2EI}{3}\left(2\theta_C + 0 - \frac{3\delta}{3}\right) \qquad [\theta_D = 0, \text{ D is fixed}]$$

$$= \frac{2EI}{3}(2\theta_C - \delta).$$

$$M_{DC} = M_{FDC} + \frac{2EI}{l}\left(2\theta_D + \theta_C + \frac{3\delta}{l}\right)$$

$$= 0 + \frac{2EI}{3}\left(\overset{0\ +}{\cancel{2\theta_D +}}\ \theta_C - \frac{3\delta}{3}\right) = \frac{2EI}{3}(\theta_C - \delta)$$

c) Joint equilibrium equations:

jt B, $\quad M_{BA} + M_{BC} = 0$.

$$\frac{2EI}{3}(2\theta_B - \delta) - 10.547 + \frac{EI}{2}(2\theta_B + \theta_C) = 0$$

$$\frac{2EI}{3}(2\theta_B - \delta) + \frac{EI}{2}(2\theta_B + \theta_C) = 10.547$$

$$EI\left(\frac{4}{3}\theta_B - \frac{2}{3}\delta + \theta_B + \frac{\theta_C}{2}\right) = 10.547.$$

$$\frac{7}{3}\theta_B + \frac{\theta_C}{2} - \frac{2\delta}{3} = \frac{10.547}{EI} \quad \text{------} ①$$

jt C: $\quad M_{CB} + M_{CD} = 0$

$$17.578 + \frac{EI}{2}(2\theta_C + \theta_B) + \frac{2EI}{3}(2\theta_C - \delta) = 0.$$

$$EI\left(\frac{7}{3}\theta_C + \frac{\theta_B}{2} - \frac{2\delta}{3}\right) = -17.578.$$

$$\frac{\theta_B}{2} + \frac{7}{3}\theta_C - \frac{2\delta}{3} = -\frac{17.578}{EI} \quad \text{------} ②$$

d) Shear equation:

only for vertical members.

$$\frac{M_{AB}+M_{BA}}{l_{AB}} + \frac{M_{CD}+M_{DC}}{l_{CD}} = 0.$$

$$M_{AB}+M_{BA}+M_{CD}+M_{DC}=0$$

$$\frac{2EI}{3}(\theta_B-\delta)+\frac{2EI}{3}(2\theta_B-\delta)+\frac{2EI}{3}(2\theta_C-\delta)+\frac{2EI}{3}(\theta_C-\delta)=0$$

$\times 3$

$=0.$

$$\frac{2EI}{3}(\theta_B-\delta+2\theta_B-\delta+2\theta_C-\delta+\theta_C-\delta)=0$$

$$3\theta_B+3\theta_C-4\delta=0 \quad \text{.....} ③$$

$$① - ② \Rightarrow \frac{11\theta_B}{6}-\frac{11}{6}\theta_C=\frac{28.125}{EI} \quad \text{.....} ④$$

$① \times 2 \Rightarrow$ ~~$\frac{14\theta_B}{6}-\frac{11}{6}\theta_C=$~~ [illegible] $\frac{14\theta_B}{3}+\theta_C-\frac{4\delta}{3}=\frac{21.094}{EI}$

$③ \div 3 \Rightarrow$

$$\theta_B+\theta_C-\frac{4\delta}{3}=0.$$

(−) (−) (+) (−)

$$\frac{11}{3}\theta_B=\frac{21.094}{EI}$$

$$\boxed{\theta_B=\frac{5.753}{EI}}$$

sub θ_B in equ ④,

$$\frac{11}{6}\left(\frac{5.753}{EI}\right)-\frac{11}{6}\theta_C=\frac{28.125}{EI}$$

$$\boxed{\theta_C=-\frac{9.588}{EI}}$$

sub. θ_B and θ_C in equ ③,

$$3\left(\frac{3.753}{EI}\right)-4\delta+3\left(-\frac{9.588}{EI}\right)=0$$

$$4\delta=\frac{17.259}{EI}-\frac{28.764}{EI} \Rightarrow \boxed{\delta=\frac{-2.876}{EI}}$$

e) Final moments:

Sub. θ_B, θ_C and δ in moment equations,

M_{AB}= 5.753 kNm. , M_{BA}= 9.588 kNm

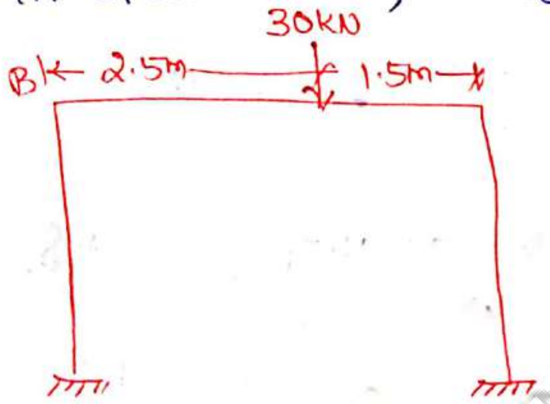
M_{BC} = $-$9.588 kNm , M_{CB}= 10.866 kNm

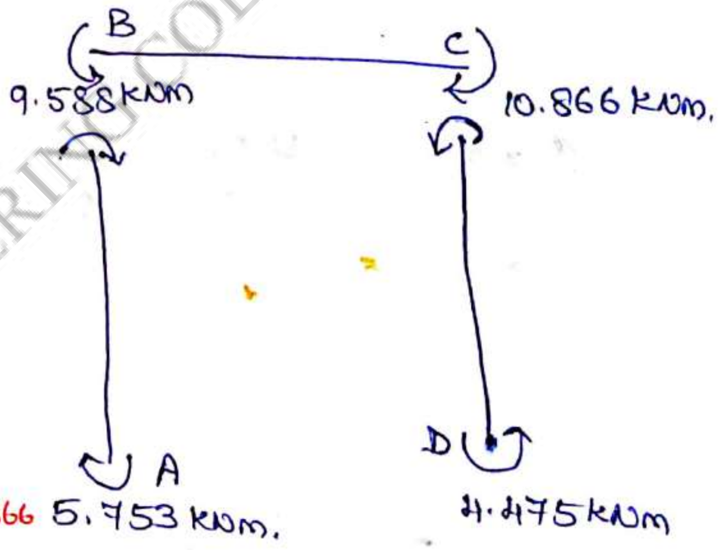
M_{CD} = $-$10.8666 kNm , M_{DC} = $-$4.475 kNm.

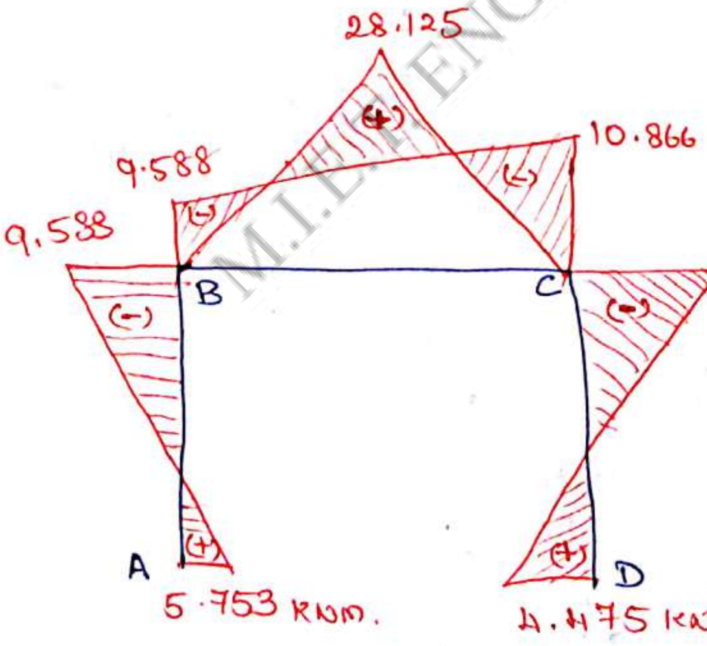
f) Simply supported bending moments:

In span AB, CD $\mu_{AB} = \mu_{CD} = 0$.

In span BC, $\mu_{BC} = \frac{Wab}{l} = \frac{30\times2.5\times1.5}{4} = 28.125$ kNm.

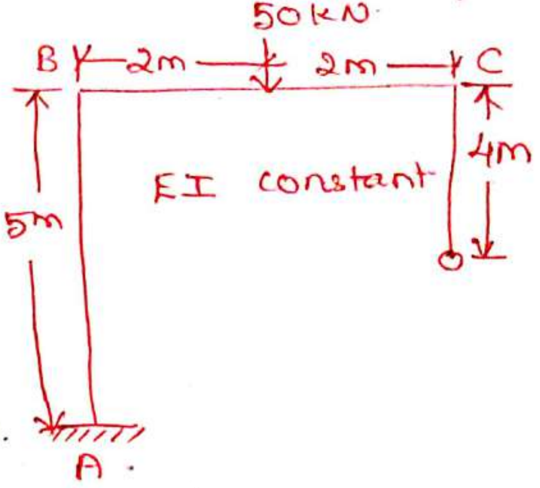






Bending Moment diagram.

⑥ Analyse the portal frame loaded as shown by the slope deflection method and sketch BM and SF diagram.



a) Fixed end moments:

Span AB, CD:

$M_{FAB} = M_{FBA} = M_{FDC} = M_{FCD} = 0$

(∵ no load on the span).

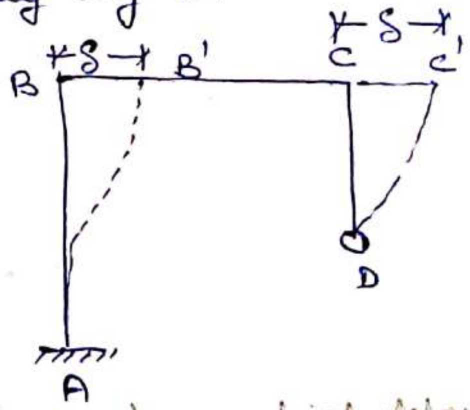
Span BC:

$$M_{FBC} = -\frac{Wl}{8} = \frac{-50\times4}{8} = -25\ \text{KNm}.$$

$$M_{FCB} = +\frac{Wl}{8} = \frac{+50\times4}{8} = +25\ \text{KNm}.$$

b) Slope deflection equation:

The frame is unsymmetrical, Hence the frame will sway. Let B and C move horizontally by δ.



Span AB:

$$M_{AB} = M_{FAB} + \frac{2EI}{l}\left[2\theta_A + \theta_B + \frac{3\delta}{l}\right]$$

$$= 0 + \frac{2EI}{5}\left[2(0) + \theta_B - \frac{3\delta}{5}\right]$$

$$= \frac{2EI}{5}\left[\theta_B - \frac{3\delta}{5}\right] \quad \text{----} ①$$

A member which deforms ex AB & DC, δ should be -ve. BC is +ve.

$$M_{BA} = M_{FBA} + \frac{2EI}{l}\left[2\theta_B + \theta_A + \frac{3\delta}{l}\right]$$

$$= \frac{2EI}{5}\left[2\theta_B - \frac{3\delta}{5}\right] \quad \text{---} ②$$

Span BC: $M_{BC} = M_{FBC} + \frac{2EI}{l}\left[2\theta_B + \theta_C + \frac{3\delta}{l}\right]$

$$= -25 + \frac{2EI}{4}\left[2\theta_B + \theta_C + 0\right] = -25 + \frac{EI}{2}(2\theta_B + \theta_C) \quad \text{----} ③$$

$$M_{CB} = M_{FCB} + \frac{2EI}{l}\left[2\theta_C + \theta_B + \frac{3\delta}{l}\right] = 25 + \frac{2EI}{4}\left[2\theta_C + \theta_B + 0\right]$$

$$M_{CB} = 25 + \frac{EI}{2}\left[2\theta_C + \theta_B\right] \quad \text{------} ④$$

Span CD: $M_{CD} = M_{FCD} + \frac{2EI}{l}\left[2\theta_C + \theta_D + \frac{3\delta}{l}\right]$

$$= 0 + \frac{2EI}{4}\left[2\theta_C + \theta_D - \frac{3\delta}{4}\right]$$

$$= \frac{EI}{2}\left[2\theta_C + \theta_D - \frac{3\delta}{4}\right] \text{ - - - - - - - } ⑤$$

$$M_{DC} = M_{FDC} + \frac{2EI}{l}\left[2\theta_D + \theta_C + \frac{3\delta}{l}\right]$$

$$= 0 + \frac{2EI}{4}\left[2\theta_D + \theta_C - \frac{3\delta}{4}\right]$$

$$M_{DC} = \frac{EI}{2}\left[2\theta_D + \theta_C - \frac{3\delta}{4}\right] \text{ - - - - - - } ⑥$$

c) Joint equilibrium equations:

At Joint B, $M_{BA} + M_{BC} = 0.$

From equ. ② and ③,

$$\frac{2EI}{5}\left(2\theta_B - \frac{3\delta}{5}\right) + (-25) + \frac{EI}{2}(2\theta_B + \theta_C) = 0.$$

$$EI\left(\frac{4}{5}\theta_B - \frac{6}{25}\delta + \theta_B + \frac{\theta_C}{2}\right) - 25 = 0$$

$$EI\left(\frac{9}{5}\theta_B - \frac{6}{25}\delta + \frac{\theta_C}{2}\right) = 25$$

$$EI(1.8\theta_B + 0.5\theta_C - 0.24\delta) = 25. \text{ - - - - } ⑦$$

At Joint C, $M_{CB} + M_{CD} = 0.$ from equ. ④ and ⑤

$$25 + \frac{EI}{2}(2\theta_C + \theta_B) + \frac{EI}{2}\left(2\theta_C + \theta_D - \frac{3\delta}{4}\right) = 0.$$

$$\frac{EI}{2}\left(2\theta_C + \theta_B + 2\theta_C + \theta_D - \frac{3\delta}{4}\right) = -25$$

$$EI\left(4\theta_C + \theta_B + \theta_D - \frac{3\delta}{4}\right) = -50 \text{ - - - - - - } ⑧$$

At joint D, $M_{DC} = 0$, $\frac{EI}{2}\left(2\theta_D + \theta_C - \frac{3\delta}{4}\right) = 0$

$$2\theta_D + \theta_C - \frac{3\delta}{4} = 0 \ ; \ \theta_D = \frac{3\delta}{8} - \frac{\theta_C}{2} \text{ - - - - - - } ⑨$$

Sub. ⑨ in ⑧,

$$4\theta_C + \theta_B + \frac{3\delta}{8} - \frac{\theta_C}{2} - \frac{3\delta}{4} = -\frac{50}{EI}$$

$$\theta_B + 3.5\,\theta_C - 0.375\delta = -\frac{50}{EI} \quad \ldots\ldots ⑩$$

d) Shear equation:

$$\frac{M_{AB}+M_{BA}}{l_{AB}} + \frac{M_{CD}}{l_{CD}} = 0. \qquad (\because M_{DC}=0)$$

$$\frac{\frac{2EI}{5}\left(\theta_B - \frac{3\delta}{5}\right) + \frac{2EI}{5}\left(2\theta_B - \frac{3\delta}{5}\right)}{5} + \frac{\frac{EI}{2}\left(2\theta_C + \theta_D - \frac{3\delta}{4}\right)}{4} = 0$$

$$\frac{2EI}{25}\left(3\theta_B - \frac{6\delta}{5}\right) + \frac{EI}{8}\left(2\theta_C + \frac{3\delta}{8} - \frac{\theta_C}{2}\right) = 0$$

$$EI\left(\frac{6}{25}\theta_B - \frac{12\delta}{125} + \frac{\theta_C}{4} + \frac{3\delta}{64} - \frac{\theta_C}{16}\right) = 0.$$

$$0.24\theta_B + 0.1875\theta_C - 0.143\delta = 0. \quad \ldots\ldots ⑪$$

Solving equations, ⑦, ⑩ and ⑪, we get,

$$\theta_C = \frac{-19.14}{EI}\ ;\ \delta = \frac{9.21}{EI}\ ;\ \theta_B = \frac{20.43}{EI}\ ;\ \theta_D = \frac{13.02}{EI}.$$

e) Final moments:

$M_{AB} = 5.962$ KNm ; $M_{BC} = -14.135$ KNm ; $M_{CD} = -16.08$ KNm.

$M_{BA} = 14.134$ KNm ; $M_{CB} = +16.08$ KNm. ; $M_{DC} = 0$.

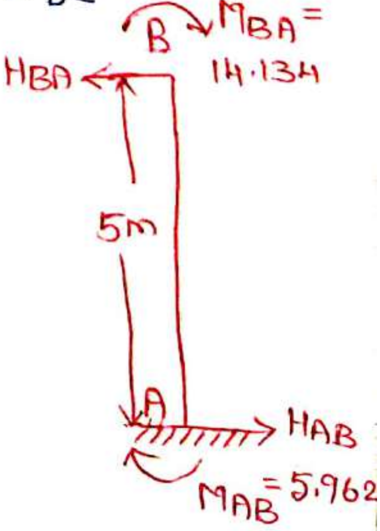
f) To draw SFD:

Span AB: Taking moment about A,

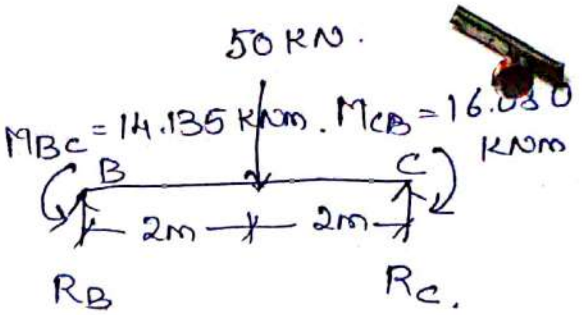
$$M_{AB} + M_{BA} - H_{BA}(5) = 0.$$

$$5.962 + 14.134 - H_{BA}(5) = 0$$

$$\boxed{H_{BA} = 4.0192\ \text{KN} = H_{AB}}$$



Span BC:



Taking moments about B,

$M_{CB} + (50 \times 2) - M_{Bc} - R_c(4) = 0.$

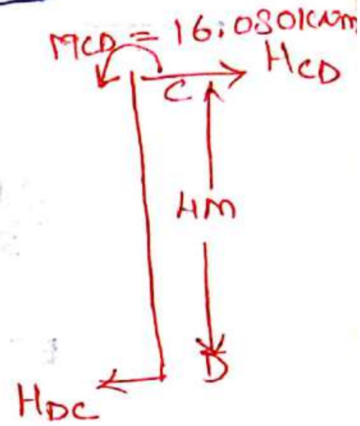
$16.080 + 100 - 14.135 - R_c(4) = 0$

$\boxed{R_c = 25.4865 \text{ KN.}}$

$R_B =$ Total load $- R_c = 50 - 25.4865$

$\boxed{R_B = 24.5135 \text{ KN}}$

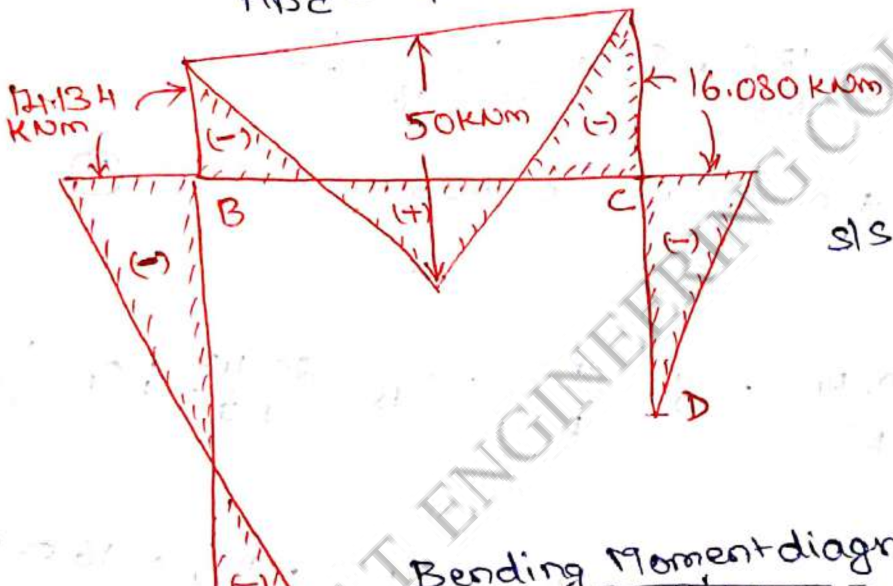
Span CD:



Taking moment about C,

$-M_{CD} + H_{DC}(4) = 0$

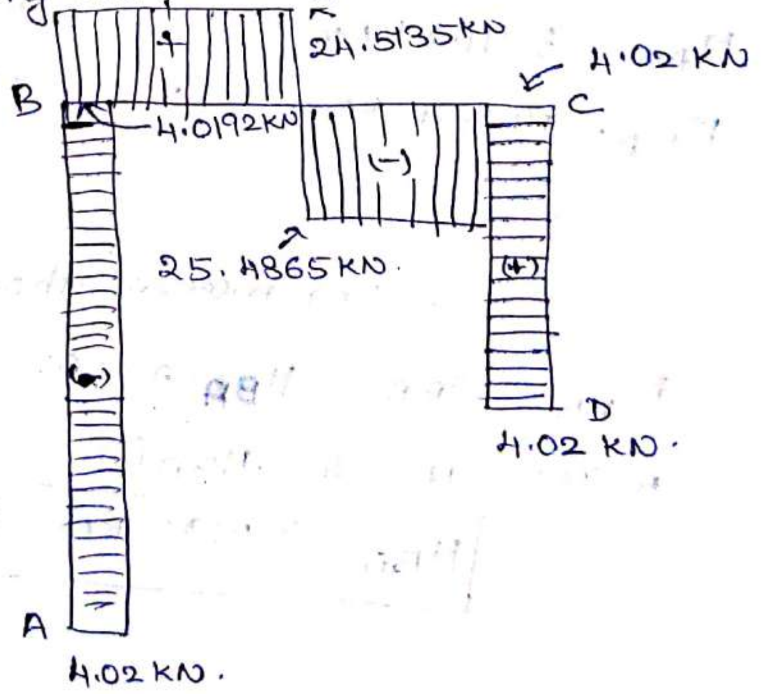
$H_{DC} = 4.02$ KN $= H_{CD}$.



s/s BM $= \frac{Wab}{l}$

$= 50 \times \frac{4}{4} = 50$ KNm.

Bending Moment diagram.



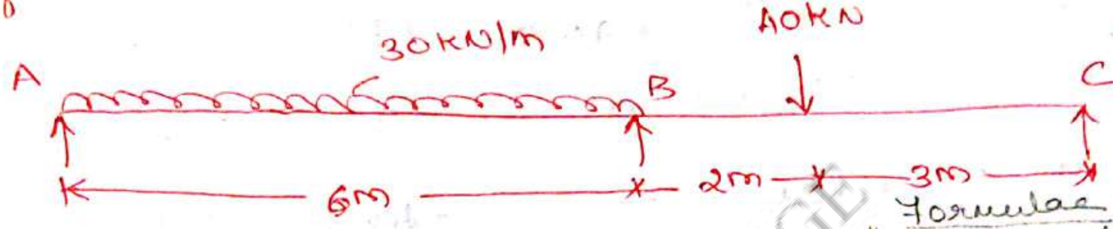
Shear Force Diagram.

UNIT III

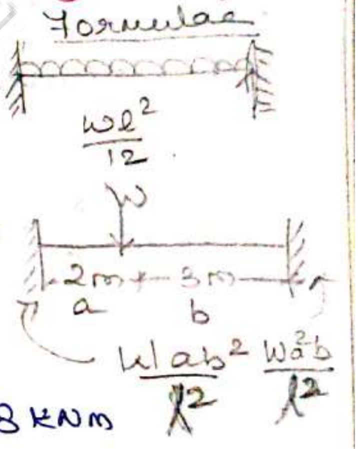
Moment Distribution method.

Distribution and carryover of moments - stiffness and carryover factors - Analysis of continuous beams - Plane rigid frames with and without sway - Neylor's simplification

① Analyse the continuous beam shown in fig. by the method of moment distribution.



I uniform.



a) Fixed end moments:

$$M_{FAB} = -\frac{wl^2}{12} = -\frac{30\times6^2}{12} = -90\text{ KNm}.$$

$$M_{FBA} = +\frac{wl^2}{12} = \frac{30\times6^2}{12} = +90\text{ KNm}.$$

$$M_{FBC} = -\frac{Wab^2}{l^2} = -\frac{40\times2\times3^2}{5^2} = -28.8\text{ KNm}$$

$$M_{FCB} = +\frac{Wa^2b}{l^2} = +\frac{40\times2^2\times3}{5^2} = +19.2\text{ KNm}.$$

b) Distribution factor: Moments has to be distributed at jt B where two members meet. (BA and BC) Since their ends are hinged, their stiffness is $\frac{3EI}{l}$ ~~Hinged ends} = $\frac{3EI}{l}$.~~

Their stiffness are in the ratio $K_{BA} : K_{BC}$ ~~Fixed = $\frac{4EI}{l}$~~

∴ Hinged $\frac{3}{4}\frac{I}{l}$ ✓

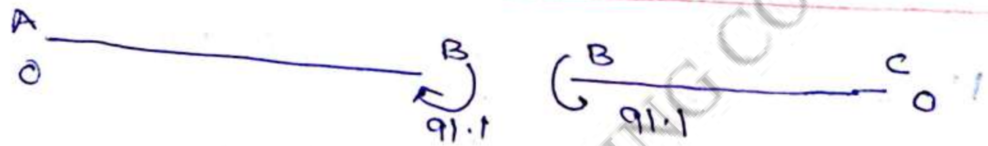
$= \frac{3I}{4l} : \frac{3}{4}\frac{I}{l}$

Fixed $\frac{I}{l}$ ✓ & continuous.

joint	Member	Relative stiffness	Sum	Distribution factor.
B	BA	$\frac{3}{4}\times\frac{I}{6} = \frac{3I}{24}$ 0.5EI.	$\frac{3I}{24} + \frac{3I}{20}$ $= \frac{33I}{120}$ 0.5+0.6 = 1.1EI	$\left(\frac{3I}{24}\right) / \left(\frac{33I}{120}\right) = \frac{5}{11}$ $\frac{0.5}{1.1}$ = 0.45
	BC	$\frac{3}{4}\times\frac{I}{5} = \frac{3I}{20}$ 0.6 EI		$\left(\frac{3I}{20}\right) / \left(\frac{33I}{120}\right) = \frac{6}{11}$ $\frac{0.6}{1.1}$ = 0.55

c) Moment distribution:

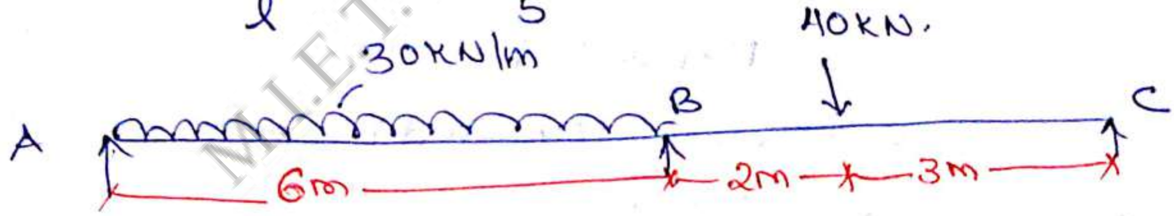
Line	Joint	A	B		C
1	Member	AB	BA	BC	CB
2	D.F	—	5/11	6/11	—
3	FEM	−90	+90	−28.8	19.2
4	Release A and C and c/o	$+90 \times \frac{1}{2} \rightarrow$	+45	$-9.6 \leftarrow \frac{1}{2} \times$	−19.2
5	Initial Moments	0	135	−38.4	0
6	Balance B	—	96.6 × 5/11 change sign −43.9	diff 96.6 × 6/11 −52.7	—
7	Final moments	0	91.1	−91.1	0

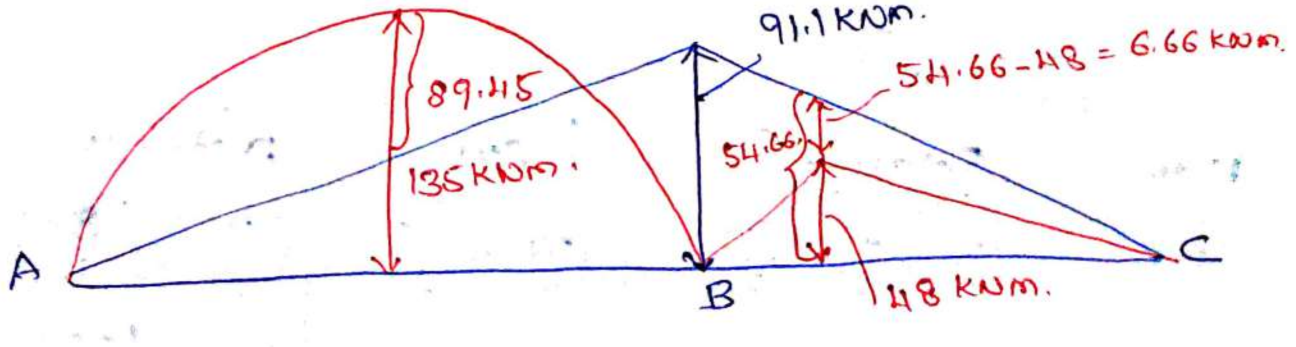


Simply supported moments:

In AB: $\frac{wl^2}{8} = \frac{30 \times 6^2}{8} = 135$ kNm.

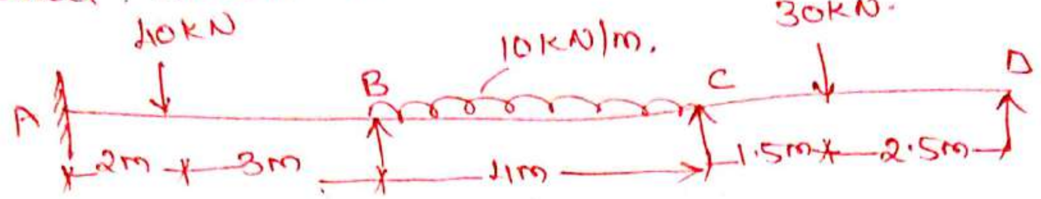
In BC: $\frac{Wab}{l} = \frac{40 \times 2 \times 3}{5} = 48$ kNm.





BMD.

② Analyse the continuous beam by moment distribution method. Sketch BMD and SFD.



a) Fixed end moments:

Span AB : $M_{FAB} = -\frac{Wab^2}{l^2} = \frac{40 \times 2 \times 3^2}{5^2} = -28.8\,kNm.$

$M_{FBA} = \frac{Wa^2b}{l^2} = \frac{40 \times 2^2 \times 3}{5^2} = 19.2\,kNm.$

Span BC : $M_{FBC} = \frac{-wl^2}{12} = -\frac{10 \times 4^2}{12} = -13.33\,kNm$

$M_{FCB} = +\frac{wl^2}{12} = +\frac{10 \times 4^2}{12} = +13.33\,kNm$

Span CD : $M_{FCD} = -\frac{Wab^2}{l^2} = -\frac{30 \times 1.5 \times (2.5)^2}{4^2} = -17.58\,kNm.$

$M_{FDC} = +\frac{Wa^2b}{l^2} = +\frac{30 \times 1.5^2 \times 2.5}{4^2} = +10.55\,kNm.$

b) Distribution factor:

continuous & fixed $\frac{I}{l}$

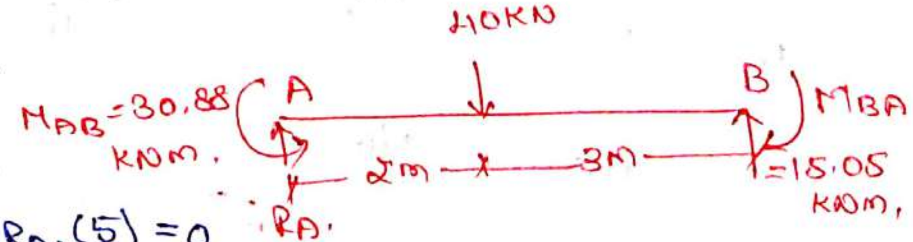
s/s $\frac{3}{4}\cdot\frac{I}{l}$

joints.	Members.	Relative stiffness (R.S).	Total stiffness (T.S).	distribution factor (RS/T.S).
B	BA	$I/5$	$\frac{I}{5} + \frac{I}{4} = \frac{9I}{20}$	$\frac{I/5}{9I/20} = \frac{4}{9}$
	BC	$I/4$		$\frac{I/4}{9I/20} = \frac{5}{9}$
C	CB	$I/4$	$\frac{I}{4} + \frac{3I}{16} = \frac{7I}{16}$	$\frac{I/4}{7I/16} = \frac{4}{7}$
	CD	$\frac{3}{4} \times \frac{I}{4} = \frac{3I}{16}$		$\frac{3I/16}{7I/16} = \frac{3}{7}$

c) Moment distribution:

Joint.	A	B		C		D
Member.	AB	BA	BC	CB	CD	DC
Distribution factor	—	4/9	5/9	$\frac{4}{7}$	$\frac{3}{7}$	—
Fixed end moments Release & carry over to C.	−28.8	+19.2	−13.33	+13.33	−17.58 −5.28	+10.55 −10.55
Initial moments.	−28.8	+19.2	−13.33	+13.33	−22.86	0
Balancing		4/9 × 5.87 −2.61	× 5/9 −3.26	$\frac{4}{7}$ × −9.53 5.45	× 3/7 4.08	
Carry over	−1.31		2.73	−1.63		
Balancing × ratio opposite		−1.21	−1.52	0.93	0.70.	
CO ½	−0.61		0.47	−0.76		
Balancing		−0.21	−0.26	0.43	0.33	
CO	−0.11		0.22	−0.13		
Bal.		−0.10	−0.12	0.07	0.06	
CO	−0.05		0.04	−0.06		
Bal.		−0.02	−0.02	0.03	0.03	
Final total moments.	−30.88	15.05	−15.05	17.66.	−17.66	0

d) To draw SFD:

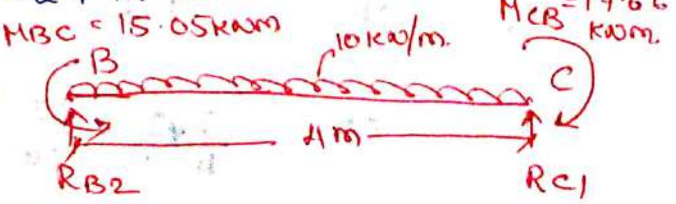


Span AB: clockwise +ve

Taking moment about A,

$150{,}5 + 40(2) - 30.88 - R_{B1}(5) = 0$

$R_{B1} = 12.83$ KN, $R_A = 40 - R_{B1}$, $R_A = 27.17$ KN.

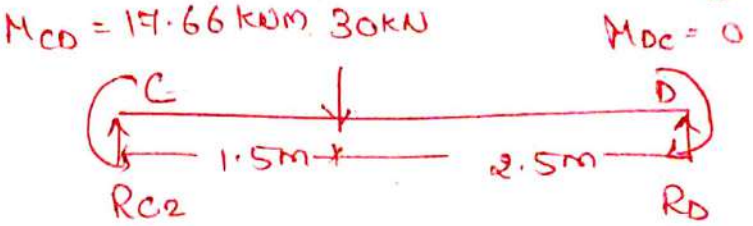


Span BC: Taking moment about B,

$17.66 + \frac{10 \times 4^2}{2} - 15.05 - R_{C1}(4) = 0$

$R_{C1} = 20.65$ KN, $R_{B2} = 10 \times 4 - 20.65 = 19.32$ KN.

Span CD



Taking moment about C,

$0 + 30(1.5) - 17.66 - R_D(4) = 0.$

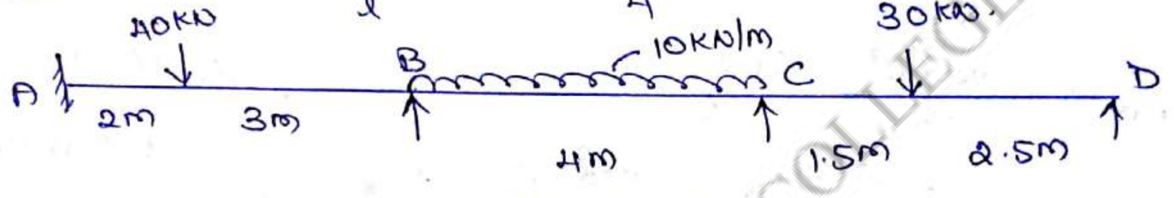
$R_D = 6.84$ kN. , $R_{c2} = 30 - 6.84$, $R_{c2} = 23.16$ kN.

Simply supported bending moments:

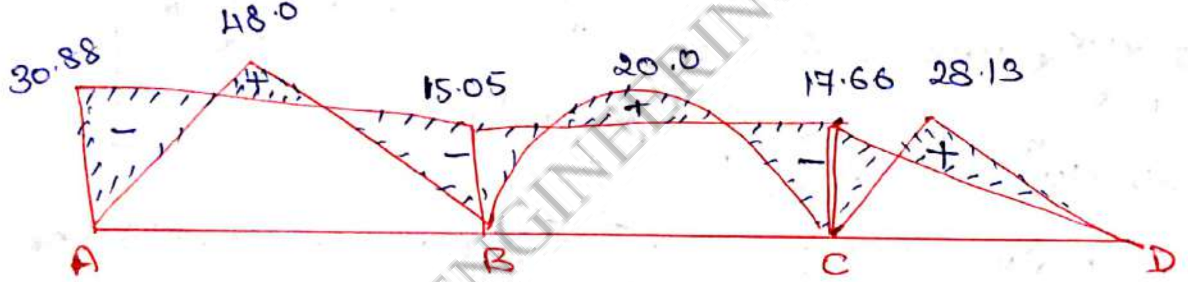
Span AB: $M = \frac{Wab}{l} = \frac{40 \times 2 \times 3}{5} = 48$ kNm.

span BC: $M = \frac{wl^2}{8} = \frac{10 \times 4^2}{8} = 20$ kNm.

span CD : $M = \frac{Wab}{l} = \frac{30 \times 1.5 \times 2.5}{4} = 28.13$ kNm.

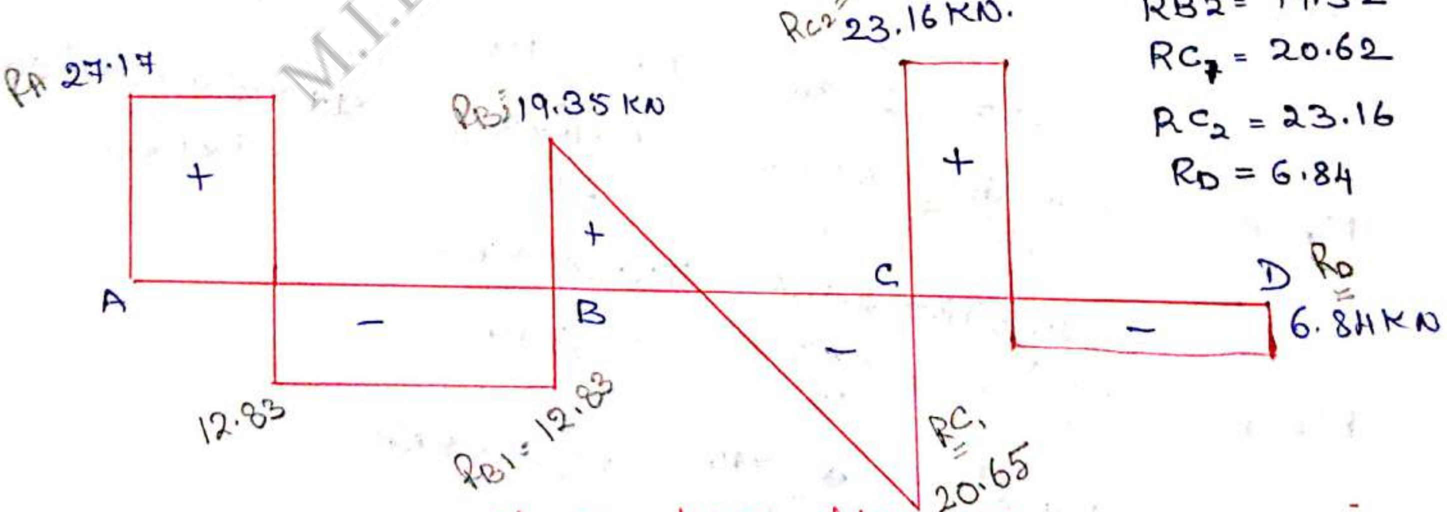


EI constant



Bending Moment diagram.

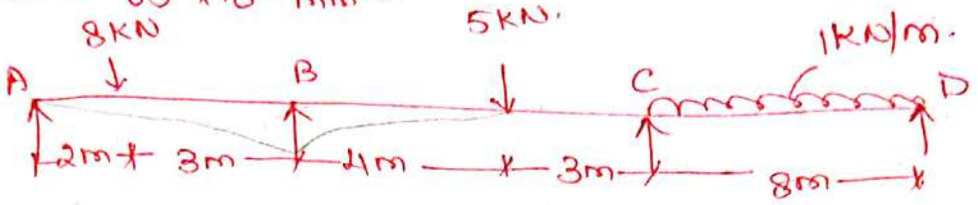
RA1 = 27.17
RB1 = 12.83
RB2 = 19.32
RC_7 = 20.62
RC_2 = 23.16
R_D = 6.84



Shear force diagram.

Sinking of supports:

③ A continuous beam ABCD, 20m long is s/s at its ends is propped at the same level at B and C as shown. If support B sinks by 10mm, analyse the beam by moment distribution method and sketch BMD. Take $E = 2.1\times10^5\ N/mm^2$ and $I = 85\times10^5\ mm^4$.



R.H.S sinking −ve

L.H.S. sinking +ve

Fixed end moments:

$$M_{FAB} = \frac{-Wab^2}{l^2} - \frac{6EI\delta}{l^2} = -5.76 - 4.284$$

$$= -\frac{8\times2\times3^2}{5^2} - \frac{6\times2.1\times10^8\times85\times10^{-7}}{5^2} * \frac{10}{1000} = -10.044\ KNm$$

$$\left[E = \frac{2.1\times10^5}{1000} * 1000^2 = 2.1\times10^8\ KN/m^2 \ ; \ \delta = \frac{10}{1000} m \right.$$

$$\left. I = 85\times10^5\ mm^4 = \frac{85\times10^5}{(1000)^4} = 85\times10^{-7}\ m^4. \right].$$

$$M_{FBA} = \frac{Wa^2b}{l^2} - \frac{6EI\delta}{l^2} = \frac{8\times2^2\times3}{5^2} - 4.284 = -0.444\ KNm.$$

$$M_{FBC} = \frac{-Wab^2}{l^2} + \frac{6EI\delta}{l^2} = -\frac{5\times4\times3^2}{7^2} + \frac{6\times2.1\times10^8\times85\times10^{-7}}{7^2\times1000}\times10$$

$$= -3.673 + 2.186 = -1.487\ KNm.$$

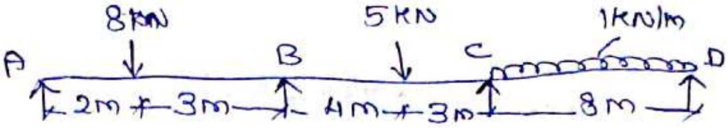
$$M_{FCB} = +\frac{Wa^2b}{l^2} + \frac{6EI\delta}{l^2} = \frac{5\times4^2\times3}{7^2} + \frac{6\times2.1\times10^8\times85\times10^{-7}\times10}{7^2\times1000}$$

$$= 4.898 + 2.186 = 7.084\ KNm$$

$$M_{FCD} = -\frac{wl^2}{12} = -\frac{1\times8^2}{12} = -5.33\ KNm.$$

$$M_{FDC} = +\frac{wl^2}{12} = +\frac{1\times8^2}{12} = +5.33\ KNm.$$

Distribution factor:



Joint	Member	Relative Stiffness	Sum	Distribution factor
B continuous	BA	$\frac{3}{4}\times\frac{I}{l}=\frac{3}{4}\times\frac{I}{5}$ $=3I/20$	$\frac{3I}{20}+\frac{I}{7}=\frac{41I}{140}$	$\left(\frac{3I}{20}\right)/\left(\frac{41I}{140}\right)=0.512$
	BC	$\frac{I}{l}=\frac{I}{7}$.		$\left(\frac{I}{7}\right)/\left(\frac{41I}{140}\right)=0.488$
continuous C	CB	$I/7$	$\frac{I}{7}+\frac{3I}{32}$ $=\frac{53I}{224}$	$\left(\frac{I}{7}\right)/\left(\frac{53I}{224}\right)=0.604$
	CD	$\frac{3}{4}\times\frac{I}{8}=\frac{3I}{32}$		$(3I/32)/\left(\frac{53I}{224}\right)$ $=0.396.$

Moment distribution:

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
Distribution factor	—	0.512	0.488	0.604	0.396	—
FEM	−10.044	−0.444	−1.487	7.084	−5.33	+5.33
Release A, D & carry over	+10.044	$\xrightarrow{1/2}$ 5.022			−2.665 $\xleftarrow{1/2}$	−5.33
Initial moments	0	+4.578	−1.487	+7.084	−7.995	0
Balance	—	−1.583	−1.508	+0.550	−0.911 +0.361	—
Carry over	—		0.275 $\overset{1/2}{\times}$	−0.754	—	—
Balance	—	−0.141	−0.134	0.455	0.299	—
Carry over	—	—	0.228	−0.067	—	—
Balance	—	−0.117	−0.111	0.040	0.027	—
Final moments.	0	+2.737	−2.737	+7.308	−7.308	0

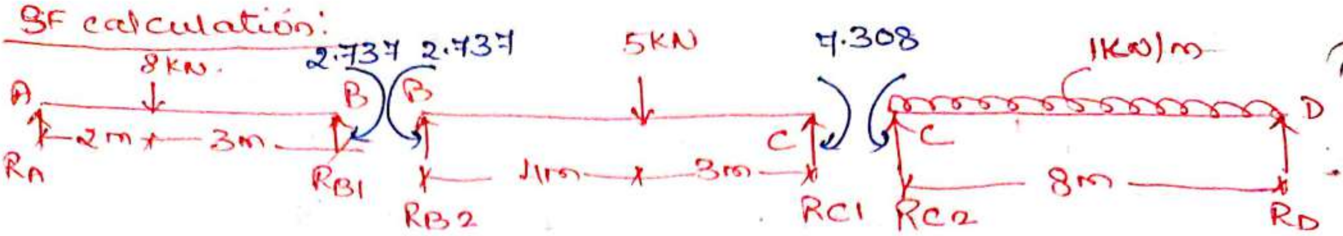
Simply supported bending moments:

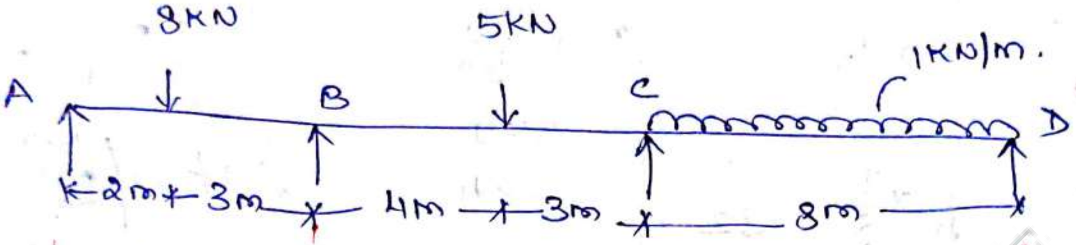
$$M_{AB}=\frac{Wab}{l}=\frac{8\times2\times3}{5}=9.6\text{ KNm.}$$

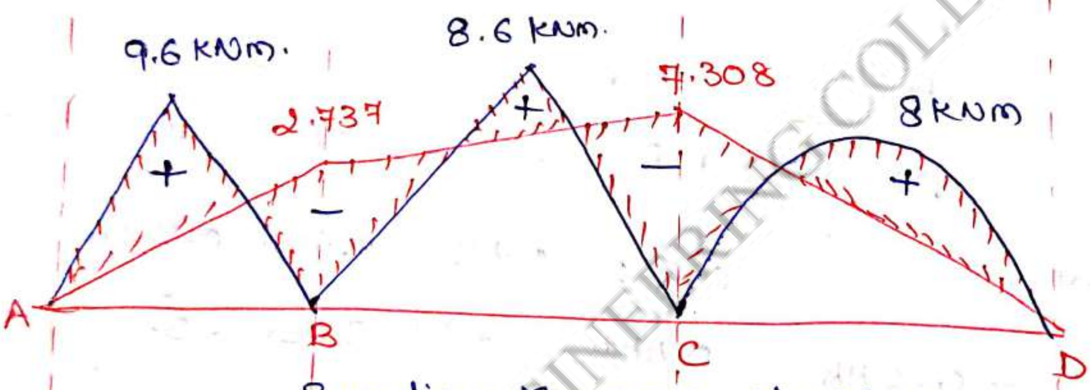
$$M_{BC}=\frac{Wab}{l}=\frac{5\times4\times3}{7}=8.6\text{ KNm.}$$

$$M_{CD}=\frac{wl^2}{8}=\frac{1\times8^2}{8}=8\text{ KNm}$$

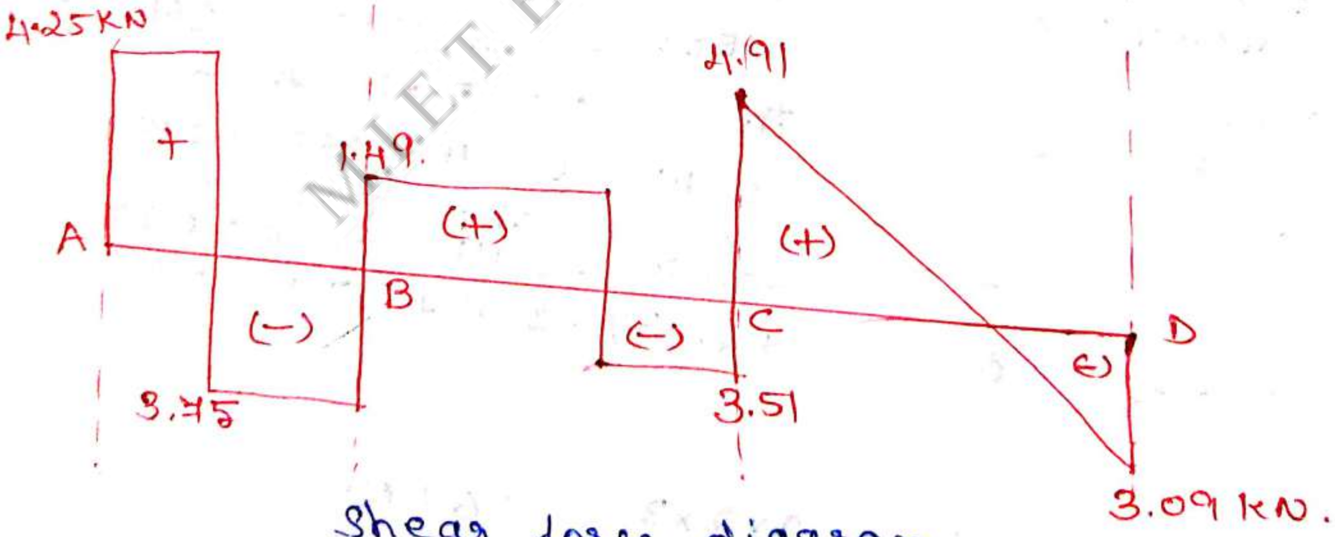
SF calculation:







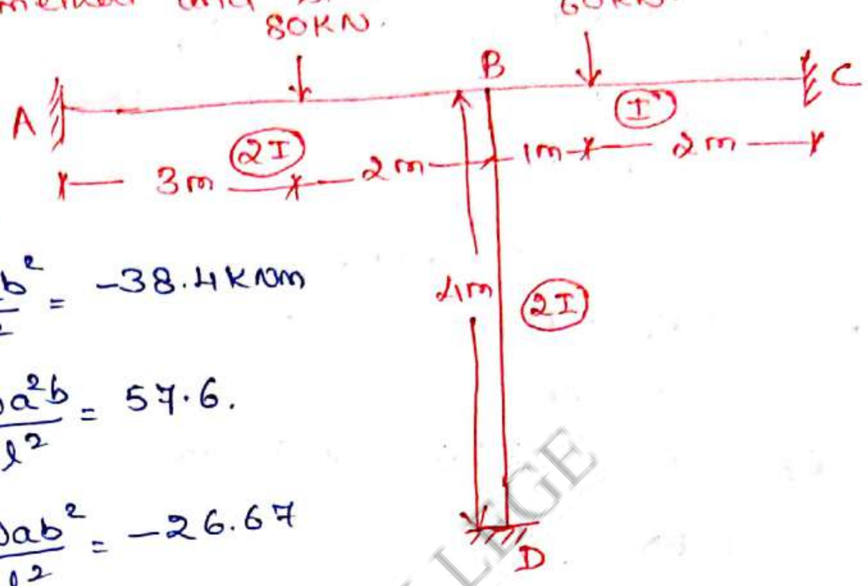
Bending Moment Diagram.



Shear force diagram.

Bending Moment Diagrams of structural frames.

Analyse the structure loaded as shown in fig. by moment distribution method and sketch BMD and SFD.



a) Fixed end moments:

Span AB: $M_{FAB} = -\frac{Wab^2}{l^2} = -38.4\,\text{KNm}$

$M_{FBA} = +\frac{Wa^2b}{l^2} = 57.6.$

Span BC: $M_{FBC} = -\frac{Wab^2}{l^2} = -26.67$

$M_{FCB} = +13.33\,\text{KNm},$

Span BD: $M_{FBD} = 0 = M_{FDB}$.

b) Distribution factors:

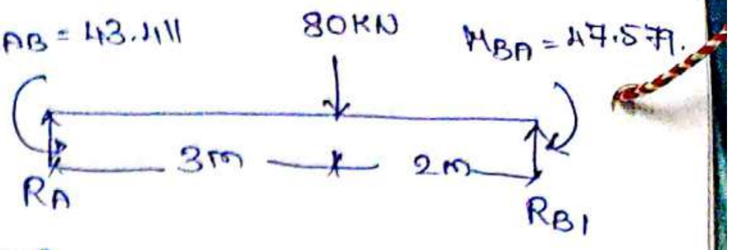
Joint	Members	Relative Stiffness R.S.	Total Stiffness T.S	Distribution factor (R.S/T.S)
B.	BA	$2I/5$ $\cancel{\frac{I}{6}}$	$\frac{2I}{5}+\frac{I}{2}+\frac{I}{3}$ $= \frac{37I}{30}$	0.324
	BD	$\frac{2I}{4} = \frac{I}{2}$		0.406
	BC	$\frac{I}{3}$.		0.270.

c) Moment distribution:

Joint	A	B			C
Members	AB	BA	BD	BC	CB
Distribution factors.	—	0.324	0.406	0.270	—
Fixed End moments.	−38.4	+57.6	0	−26.67	+13.33
Balancing		−10.021	−12.558	−8.351	
carry over.	−5.011				−4.176
Final moments.	−43.411	47.579	−12.558	−35.021	+9.154

$M_{DB} = \frac{1}{2}(-12.558) = -6.279\,\text{KNm}.$

d) To draw SFD:



Span AB:

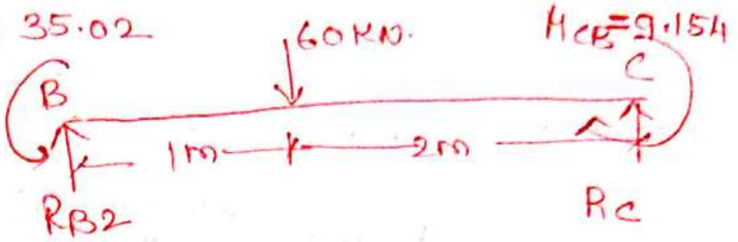
Taking moment about A,

$R_{B1}(5) + M_{AB} - W(3) - M_{BA} = 0.$

$R_{B1} = 48.834$ KNm, R_A = total load $- R_{B1}$

$= 80 - 48.834 = 31.166$ KN.

Span BC:



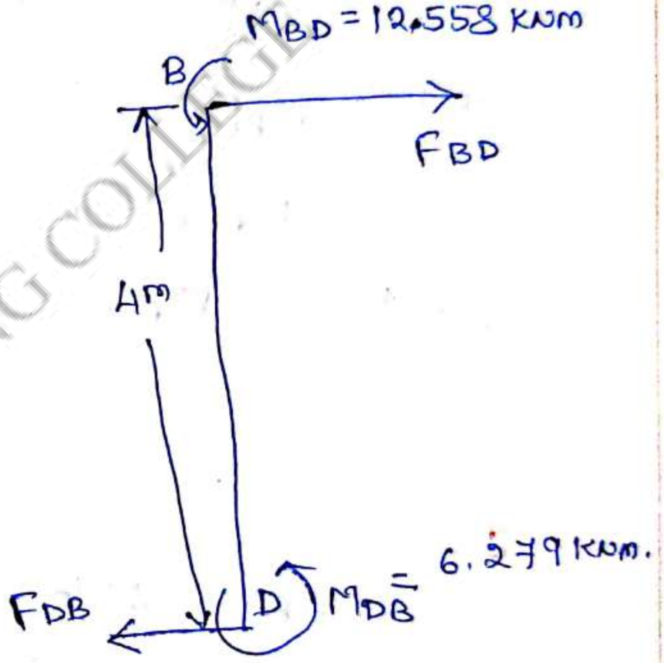
Taking moments about B,

$R_C(3) + M_{BC} - M_{CB} - 60(1) = 0$

$R_C = 11.378$ KNm.

$R_{B2} = 60 - 11.378 = 48.622$ KN.

Span BD:



Taking moments about D,

$F_{BD}(4) - M_{BD} - M_{DB} = 0$

$F_{DB}(4) - 12.558 - 6.279 = 0.$

clockwise +ve
anti " -ve

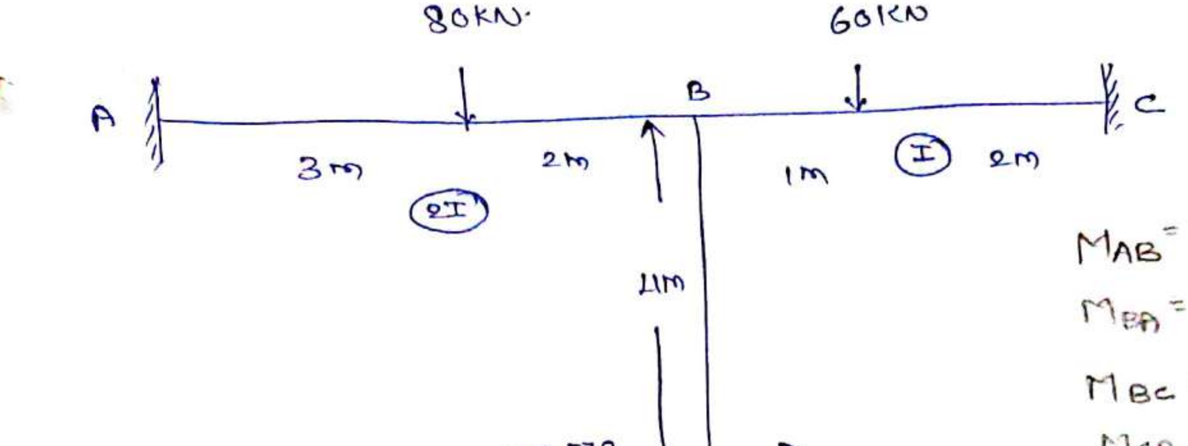
$F_{BD} = 4.709$ KN $(\rightarrow)$

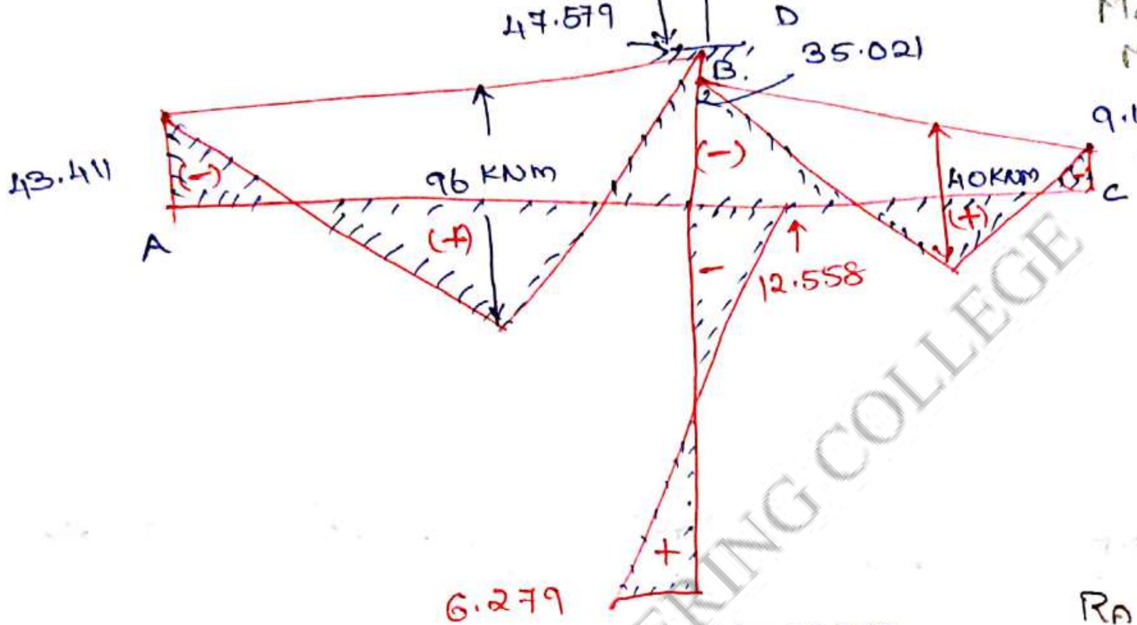
$F_{DB} = 4.709$ KN $(\leftarrow)$.

Simply supported bending moments.

Span AB $= \frac{Wab}{l} = \frac{80 \times 3 \times 2}{5} = 96$ KNm.

Span BC $= \frac{Wab}{l} = \frac{60 \times 1 \times 2}{3} = 40$ KNm.





Bending moment diagram.

M_{AB} = 48.834 kNm

M_{BA} = 47.571

M_{BC} = 35.02

M_{CB} = 9.154

M_{BD} = -12.558

M_{PB} = 6.279

R_A = 31.166

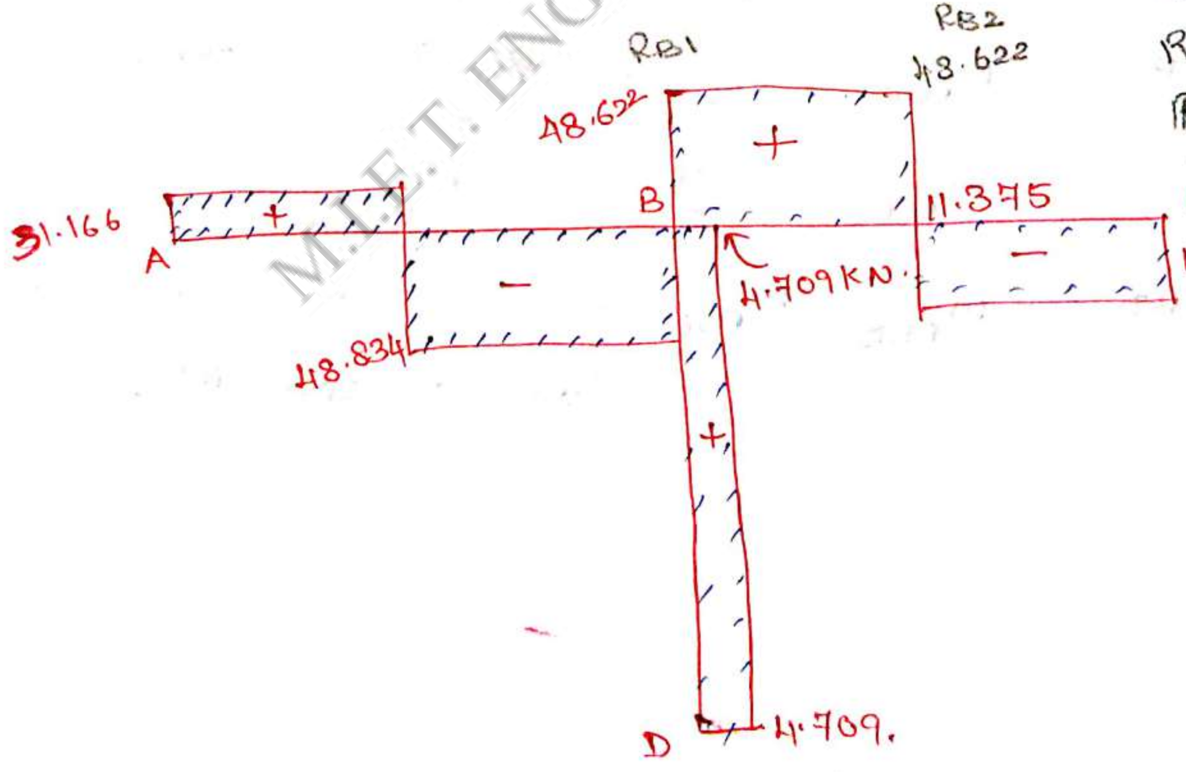
R_{B1} = 48.884

R_{B2} = 48.622

R_C = 11.378.

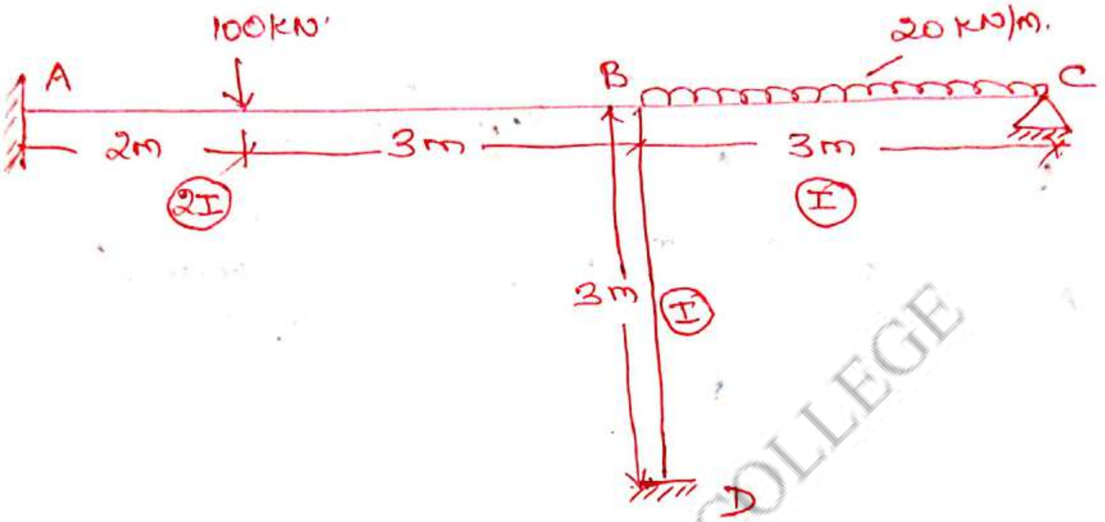
F_{BD} = 4.709 →

F_{DB} = 4.709 ←



Shear force diagram.

A continuous beam ABC is supported on an elastic column BD and is loaded as shown. Treating jt B as rigid, analyse the frame and the BMD and sketch the deflected shape of the structure



a) Fixed end moments:

Span AB: $M_{FAB} = -\frac{Wab^2}{l^2} = \frac{-100\times2\times3^2}{5^2} = -72$ KNm.

$M_{FBA} = +\frac{Wa^2b}{l^2} = \frac{+100\times2^2\times3}{5^2} = +48$ KNm.

Span BC: $M_{FBC} = -\frac{wl^2}{12} = -\frac{20\times3^2}{12} = -15$ KNm.

$M_{FCB} = +\frac{wl^2}{12} = +15$ KNm.

Span BD: $M_{FBD} = M_{FDB} = 0$. (as there is no lateral load on span BD)

b) Distribution factors:

Joint	Members	Relative Stiffness (R.S)	Total Stiffness or Sum (T.S)	Distribution factor R.S/T.S
B	BA $\frac{I}{l}$	$\frac{2I}{5}$ Fixed	$\frac{2I}{5}+\frac{I}{3}+\frac{I}{4}$ $=\frac{59I}{60}$	$\frac{2I/5}{59I/60}=0.407$
	BD $\frac{I}{l}$	$\frac{I}{3}$ Fixed		$\frac{I/3}{59I/60}=0.339$
	BC	$\frac{3}{4}\times\frac{I}{3}=\frac{I}{4}$. s/s.		$\frac{I/4}{59I/60}=0.254$

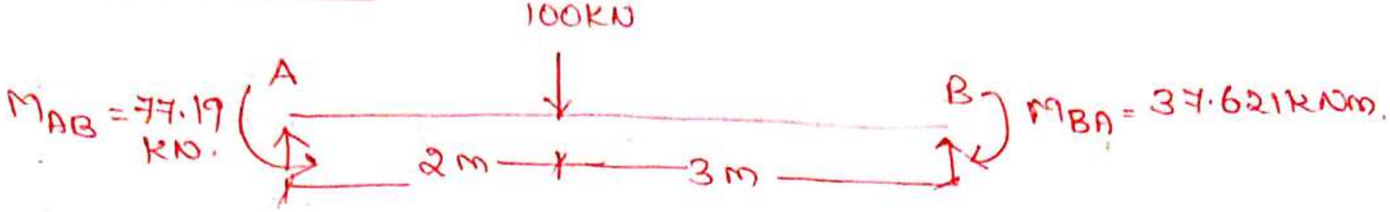
c) Moment distribution:

Joint	A	B			C
Member	AB	BA	BD	BC	CB
Distribution factor.	–	0.407	0.339	0.254	–
Fixed end moments	-72	48	0	-15	15.
Release C and carry over to B.			0	-7.5 $\xleftarrow{1/2}$	-15
Initial Moments.	-72	48	– ~~-8.645~~	-22.5 ~~-6.477~~	0
Balancing.		-10.379	-8.645	-6.477	
Carry over	-5.19				
Final moments.	-77.19	37.621	-8.645	-28.977	0

$$M_{DB}=\frac{1}{2}M_{BD}=\frac{1}{2}(-8.645)=-4.323\ \text{KNm}.$$

d) To draw SFD:

Span AB :



$\Sigma M_B = 0$. (assuming clockwise +ve).

$R_A(5) + M_{BA} - 77.19 - 100(3) = 0$.

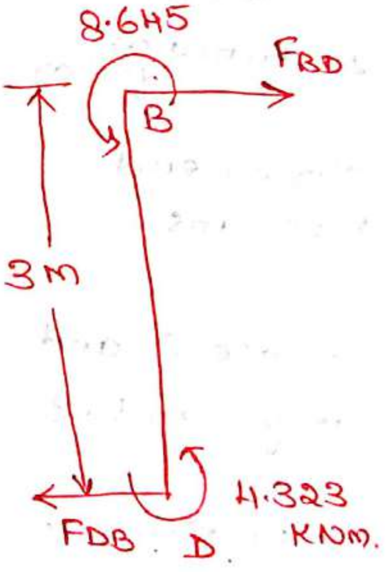
$RA(5) + 37.621 - 77.19 - 300 = 0$

$\boxed{R_A = 67.91 \text{ KN}}$ $R_{B1} = \text{total load} - 67.91$

$= 100 - 67.91$

$\boxed{R_{B1} = 32.086 \text{ KN.}}$

Span BD :



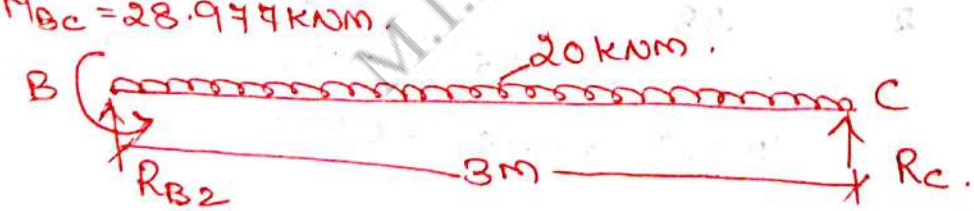
$\Sigma M_D = 0$.

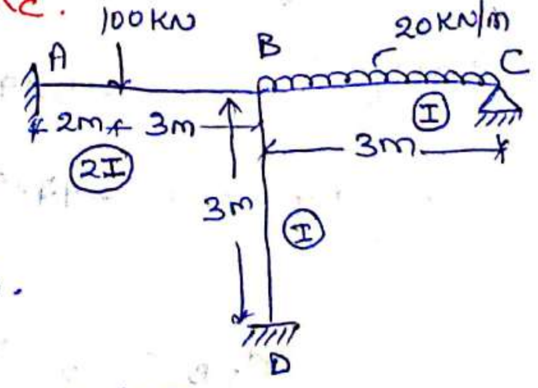
$8.645 + 4.323 - F_{BD}(3) = 0$.

$F_{BD} = 4.323$ KN (→)

$F_{DB} = 4.323$ KN (←).

Span BC : $\Sigma M_c = 0$.





Taking moments about C.

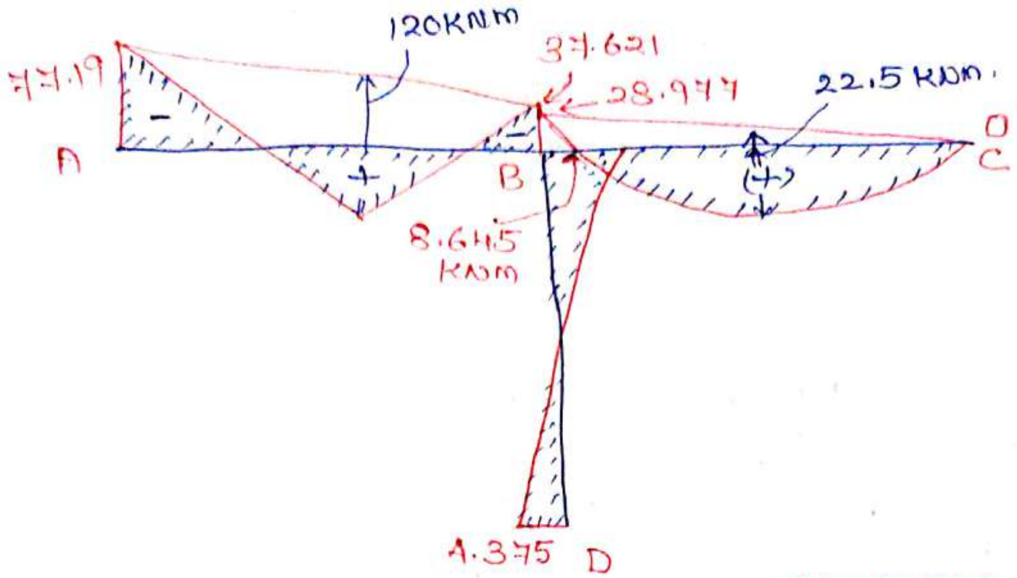
$R_{B2}(3) - M_{BC} - \frac{wl^2}{2} = 0$.

$R_{B2}(3) - 28977 - \frac{20 \times 3^2}{2} = 0$.

$R_{B2} = 39.659$ KN.

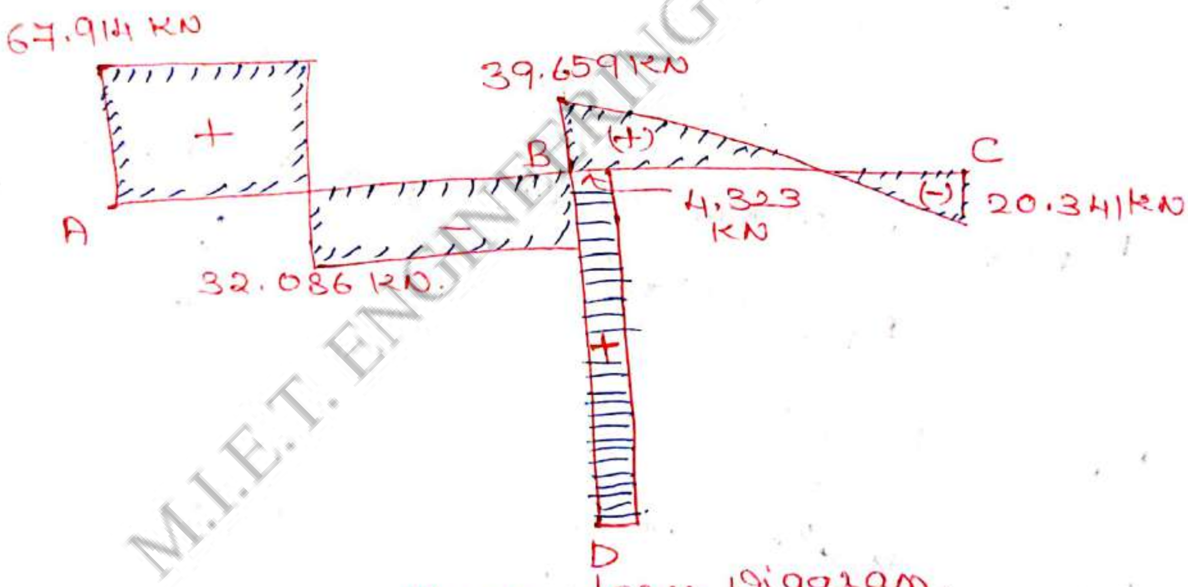
$R_c = \text{Total load} - R_{B2} = 20 \times 3 - 39.659 = 20.341$ KN.

$M_{AB} = -77.19$, $M_{BA} = 37.621$, $M_{BD} = -8.645$, $M_{Bc} = -28.977$, $M_{CB}=0$.

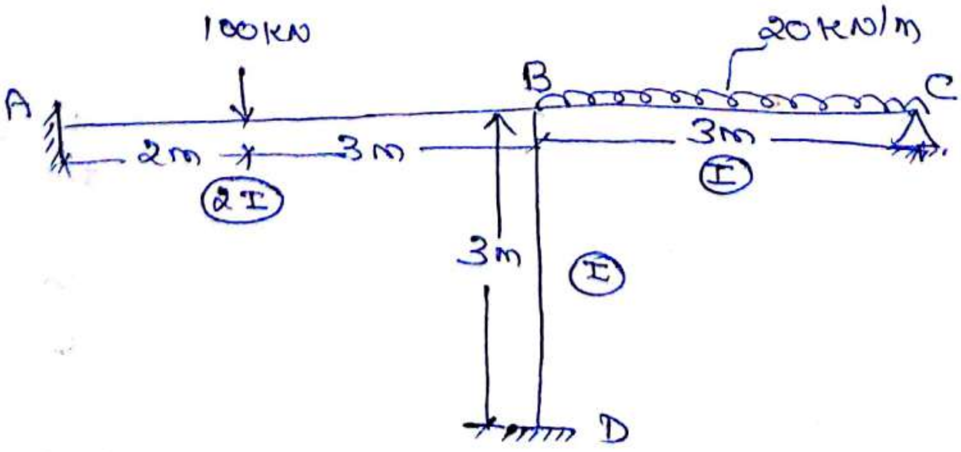


Bending moment diagram.

Simply supported bending moments:

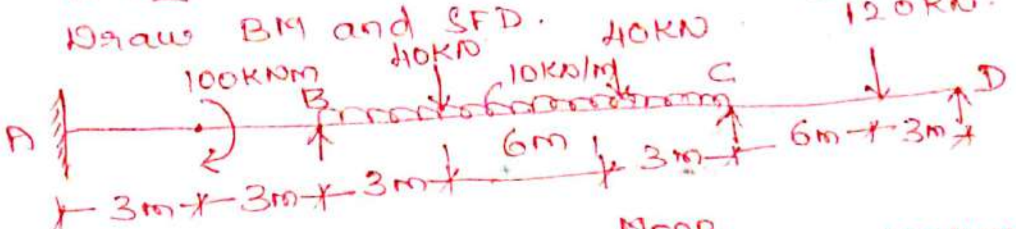


Shear force Diagram.

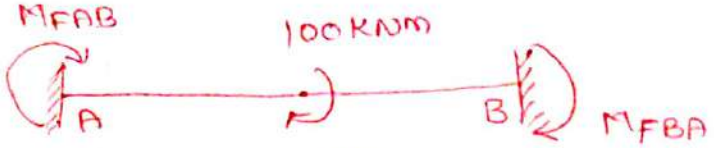


Analyse the continuous beam by moment distribution

Draw BM and SFD.



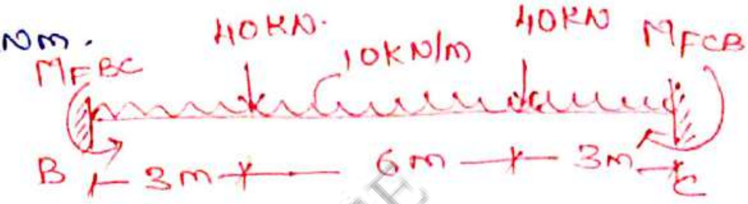
a) **Fixed end moments:**



Span AB:

$$M_{FAB} = +\frac{M_0}{4} \text{ where } M_0 = +100\text{KNm}.$$

$$M_{FAB} = \frac{+100}{4} = 25 \text{ KNm}.$$



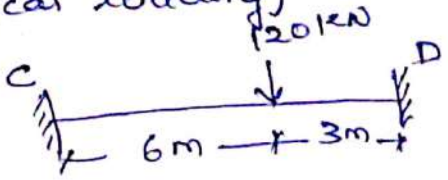
Span BC:

$$M_{FBC} = -\frac{wl^2}{12} - \frac{W_1a_1b_1^2}{l^2} - \frac{W_2a_2b_2^2}{l^2}$$

$$= -\frac{10\times12^2}{12} - \frac{40\times3\times9^2}{12^2} - \frac{40\times9\times3^2}{12^2}$$

$$= -120 - 67.5 - 22.5 = -210 \text{ kNm}.$$

$$M_{FCB} = +210\text{KNm}. \text{ (since symmetrical loading)}$$



Span CD: $M_{FCD} = -\frac{Wab^2}{l^2}$

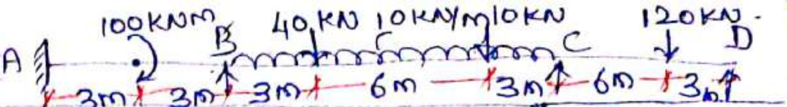
$$M_{FCD} = -\frac{120\times6\times3^2}{9^2} = -80\text{kNm}$$

$$M_{FDC} = \frac{Wa^2b}{l^2} = \frac{120\times6^2\times3}{9^2} = 160 \text{ kNm}$$

b) **Distribution factors:**

Joint	Member	Relative Stiffness (R.S)	Total Stiffness (T.S)	Distribution factor = R.S./T.S
B	BA	$I/6$,	$\frac{I}{6}+\frac{I}{12} = \frac{I}{4}$	$\frac{I/6}{I/4} = \frac{2}{3} = 0.667$
	BC	$I/12$.		$\frac{I/12}{I/4} = \frac{1}{3} = 0.333$.
C	CB	$I/12$	$\frac{I}{12}+\frac{I}{12} = \frac{I}{6}$.	$\frac{I/12}{I/6} = \frac{1}{2} = 0.5$
	CD	$\frac{3}{4}\times\frac{I}{9} = \frac{I}{12}$		$\frac{I/12}{I/6} = \frac{1}{2} = 0.5$

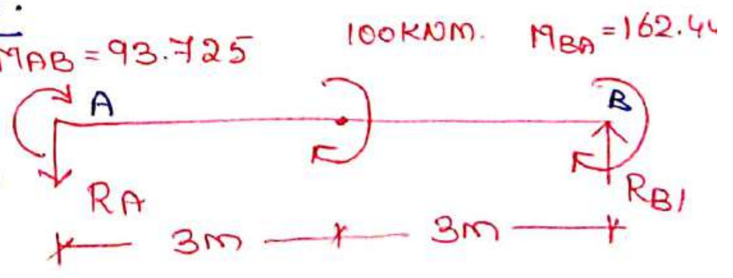
c) Moment Distribution:



Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
Distribution factor	–	0.667	0.333	0.5	0.5	–
Fixed end moments	25	25	-210	+210	-80	160
Release D & C.O to C					-80	-160.
Initial moments	25	25	-210	+210	-160	0
Balance		123.395	61.605	-25	-25	
Carry over	61.698		-12.5	30.803	0	0
Balance		8.338	4.163	-15.402	-15.402	–
Carry over	4.169		-7.701	2.082	–	–
Balance	–	5.137	2.564	-1.041	-1.041	
carry over	2.569		-0.521	1.282		
Balance.		0.348	0.173	-0.6410	-0.641	
carry over	0.174		-0.321	0.086		
Balance		0.214	0.107	-0.044	-0.044	
carry over	0.107		-0.022	0.054		
Balance		0.015	0.007	-0.027	-0.027	
C.O to ends.	0.008					
Final moments.	93.725	162.447	-162.446	202.153	-202.155	0.

To draw shear force diagram:

Span AB:



Taking moment about A,

$(R_{B1} \times 6) - 100 - 162.447 - 93.725 = 0$

$\boxed{R_{B1} = 59.362 \text{ KN} (\uparrow)}$

$R_A = -R_{B1} = -59.362 \text{ KN}.$

$\boxed{R_A = 59.362 \text{ KN} (\downarrow)}$

CE 8502 STRUCTURAL ANALYSIS II

UNIT IV: FLEXIBILITY METHOD.

Equilibrium and compatability – Determinate Vs indeterminate structures – Indeterminacy – Primary structure – compatibility conditions – Analysis of indeterminate pin-jointed plane frames, continuous beams, rigid jointed plane frames (with redundancy restricted to two).

Matrix methods

Flexibility method. or force method or compatability method. (includes moments, stresses, reactions etc.).

Stiffness method. or displacement method or equilibrium method. (includes rotations, strains, twists etc).

$\boxed{P = K \cdot \Delta}$ M =

force, stiffness, disp.

The behaviour of a structure is largely defined by defining force-displacement relationship in the form of a matrix.

Matrix method.

Stiffness matrix

$\boxed{P = K \cdot \Delta}$ deflection.

force, stiffness

If $\Delta = 1$, $\boxed{P = K}$

∴ Force reqd. to produce unit deflection = Stiffness

Flexibility matrix.

∴ Flexibility (f) → is opposite to Stiffness.

$P = K \cdot \Delta \Rightarrow P = \frac{1}{f} \cdot \Delta$

$\Delta = f \cdot P$ if $P = 1$.

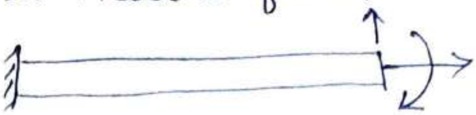
$\boxed{\Delta = f}$

∴ flexibility = displacement produced for unit force.

$\Delta = f \cdot P$.

writing in Matrix form,

$m \times n$
$n \times P$.



$$\begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{Bmatrix}_{3\times1} = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}_{3\times3} \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix}_{3\times1}$$

$\underset{\text{displacement} \quad \text{force}}{f_{11}}$ => displacement occur in direction ① due to unit force at ①.

$\underset{\text{force} \quad \text{displacement}}{K_{11}}$ => force required in direction ① to produce unit displacement at ①.

$P = K \cdot \delta$.

=> $\delta = 1$, $\boxed{P = K}$

$$\text{flexibility matrix} = \begin{bmatrix} \frac{L^2}{3EI} & \frac{L^2}{2EI} \\ \frac{L^2}{2EI} & \frac{L}{EI} \end{bmatrix}.$$

same.

Make flexibility matrix?

(Q.1)

DOF = 2., order of flexibility matrix = 2×2.

f_{11} = displacement at the direction ① due to unit force at ①.
deflection ~~rotation~~ due to load = 1
$\Rightarrow \frac{PL^3}{3EI} = \frac{L^3}{3EI}$

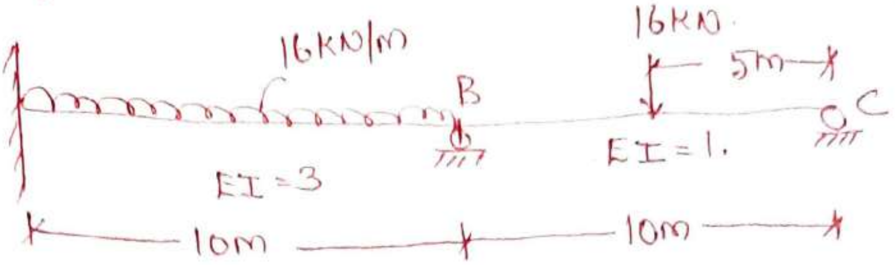
f_{12} = displacement in the direction ① due to unit force at ②
= deflection due to moment = 1 $\Rightarrow \frac{ML^2}{2EI} = \frac{L^2}{2EI}$

f_{21} = displacement in the direction ② due to unit force at ①
rotation due to load at ①. $\Rightarrow \frac{PL^2}{2EI} = \frac{L^2}{2EI}$.

f_{22} = displacement in the direction ② due to unit force at ②.
- rotation due to moment $\Rightarrow \frac{ML}{EI} = \frac{L}{EI}$

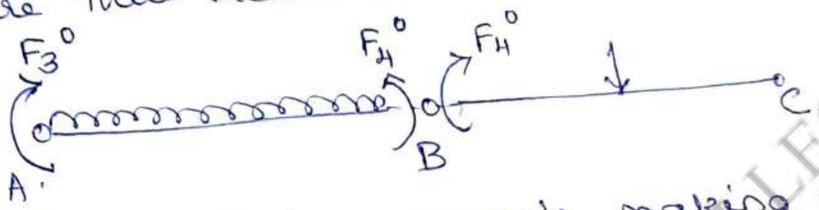
Stiffness matrix = opp. of flexibility
$[K] = [f]^{-1} = \frac{adj[f]}{|f|}$

① Analyse the continuous beam by flexibility method.



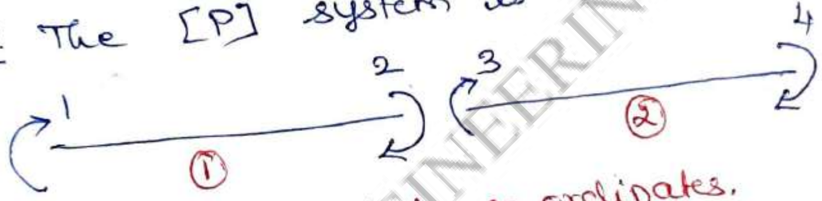
Degree of redundancy = 4 − 2 = 2.

Step 1: Introduce hinges at A and B. The moments at A and B are thus redundants.



F_4^0 is a pair of moments making a BM (sagging) BM in this case.

Step 2: The [P] system is as under.



Element co-ordinates.

Step 3: The [F] system is in 2 parts. $[F]^*$ and $[F]^0$.

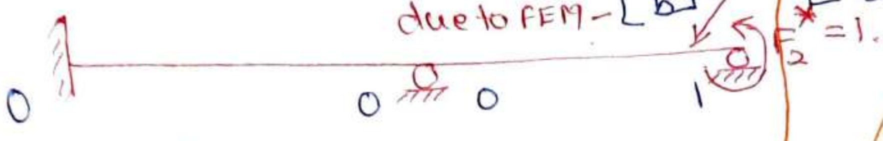
F^* is primarily made up of the effects of lateral loads on AB and BC. These are opposite to fixed end moments

Span	FEM. Fixed end moment (kNm)	Name of moment.	Equivalent joint loads.
AB	$-\frac{wl^2}{12} = -133.3$	M_{AB}	133.33
	$+\frac{wl^2}{12} = +133.33$	M_{BA}	−133.33 } −113.3 = F_1^*
BC	$-\frac{Wl}{8} = -20.0$	M_{BC}	+20.0
	$+\frac{Wl}{8} = +20.0$	M_{CB}	−20.0 $F_2^* = -20$
	∴ $F_1^* = -113.3$, $F_2^* = -20.0$.	M_{AB} can be	ignored as it is fixed.

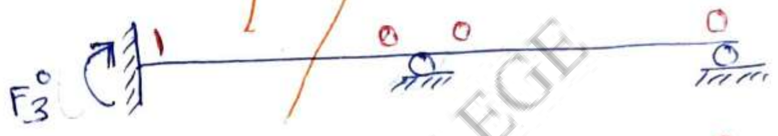
The [b] matrix is generated by introducing $F_1^* = 1$, $F_2^* = 1$, $F_3^0 = 1$ and $F_4^0 = 1$ in 4 steps and finding P_1, P_2, P_3 and P_4.

$$\text{Hence } [b] = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

due to FEM – $[b]^*$ $[b]^0$ Redundants (consider sign)



$$F^* = \begin{Bmatrix} -113 \\ -20 \end{Bmatrix}$$

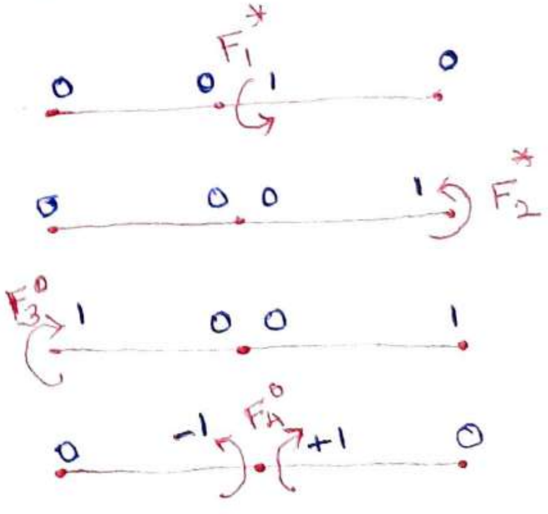


The assembled element flexibility matrix $[\alpha]$ is :

$$[\alpha] = \begin{bmatrix} [\alpha_1]_1 & [0] \\ [0] & [\alpha_2]_2 \end{bmatrix} = \begin{bmatrix} \frac{l}{6EI}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \frac{l}{6EI}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{10}{18}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \frac{10}{6}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \end{bmatrix}$$

$$= \frac{10}{18}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix}$$



$$a = [b]^T [\alpha] [b]$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} b^{*T} \\ b^{0T} \end{bmatrix} [\alpha] [b^* \quad b^0]$$

$$a_{22} = a_{00} = b^{0T} [\alpha] b^0$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \frac{10}{18} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$[a_{00}] = \frac{10}{18} \begin{bmatrix} 2 & 1 \\ 1 & 8 \end{bmatrix}$$

$$\frac{1}{(10/18)(16-1)} \begin{bmatrix} 8 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= 0.12$$

$$[a_{00}]^{-1} = 0.12 \begin{bmatrix} 8 & -1 \\ -1 & 2 \end{bmatrix}$$

$$[a_{0*}] = [b_0]^T [\alpha] [b_*]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \frac{10}{18} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{10}{18} \begin{bmatrix} 0 & 0 \\ +6 & -3 \end{bmatrix}$$

where $\{F\}^* = \begin{Bmatrix} -113.3 \\ -20 \end{Bmatrix}$

$$\{F\}^0 = -[a_{00}]^{-1} [a_{0*}] \{F\}^*$$

$$= -0.12 \begin{bmatrix} 8 & -1 \\ -1 & 2 \end{bmatrix} \frac{10}{18} \begin{bmatrix} 0 & 0 \\ 6 & -3 \end{bmatrix} \begin{Bmatrix} -113.3 \\ -20.0 \end{Bmatrix}$$

$$\{F\}^0 = \begin{Bmatrix} -41.3 \\ 82.7 \end{Bmatrix} \Rightarrow \{F\} = \begin{Bmatrix} F^* \\ F^0 \end{Bmatrix} = \begin{Bmatrix} -113.3 \\ -20 \\ -41.3 \\ 82.7 \end{Bmatrix}$$

$$\{F\}^T = [-113.3 \quad -20 \quad -41.3 \quad 82.7]$$

The element forces [P] are given by $[P] = [b]\{F\}$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} -113.3 \\ -20.0 \\ -41.3 \\ 82.7 \end{Bmatrix} = \begin{Bmatrix} -41.3 \\ -82.7 \\ -30.6 \\ -20.0 \end{Bmatrix}$$

The final forces $[P]^f = [P] - \{P\}^e$

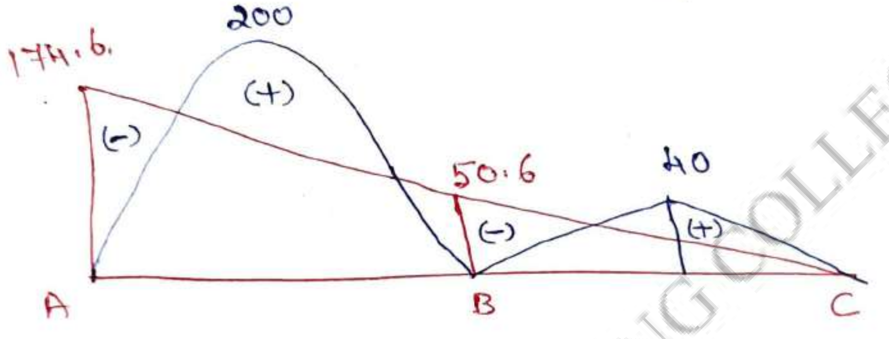
where $\{P\}^e$ is the vector of equivalent joint forces.

$$\{P\}^f = \begin{Bmatrix} -41.3 \\ -82.7 \\ -30.6 \\ -20.0 \end{Bmatrix} - \begin{Bmatrix} 133.3 \\ -133.3 \\ 20 \\ -20 \end{Bmatrix} = \begin{Bmatrix} -174.6 \\ 50.6 \\ -50.6 \\ 0 \end{Bmatrix}$$

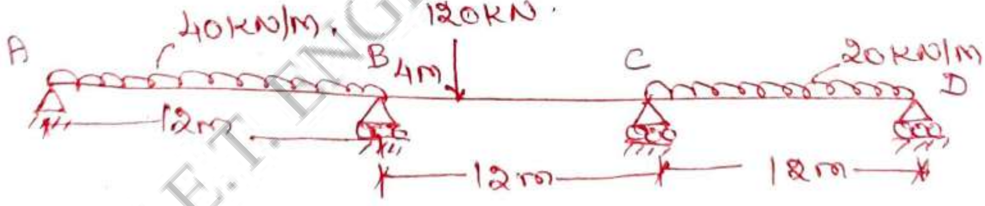
Free BM:

M_{max} in span AB $= \frac{wl^2}{8} = \frac{16\times100}{8} = 200$ KNm.

M_{max} in span BC $= \frac{Wl}{4} = \frac{16\times10}{4} = 40$ KNm.



② Analyse the continuous beam by flexibility method. Assume EI is constant. Draw BMD.



Degree of redundancy = 2

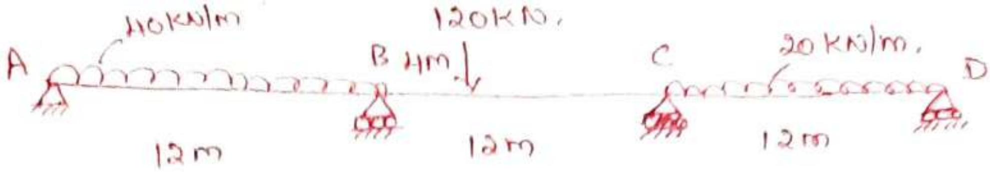
Treating M_B and M_C as redundants $\{F^0\}$

The $\{F^*\}$ forces are made up of equivalent joint forces arising out of transverse loads on each span.

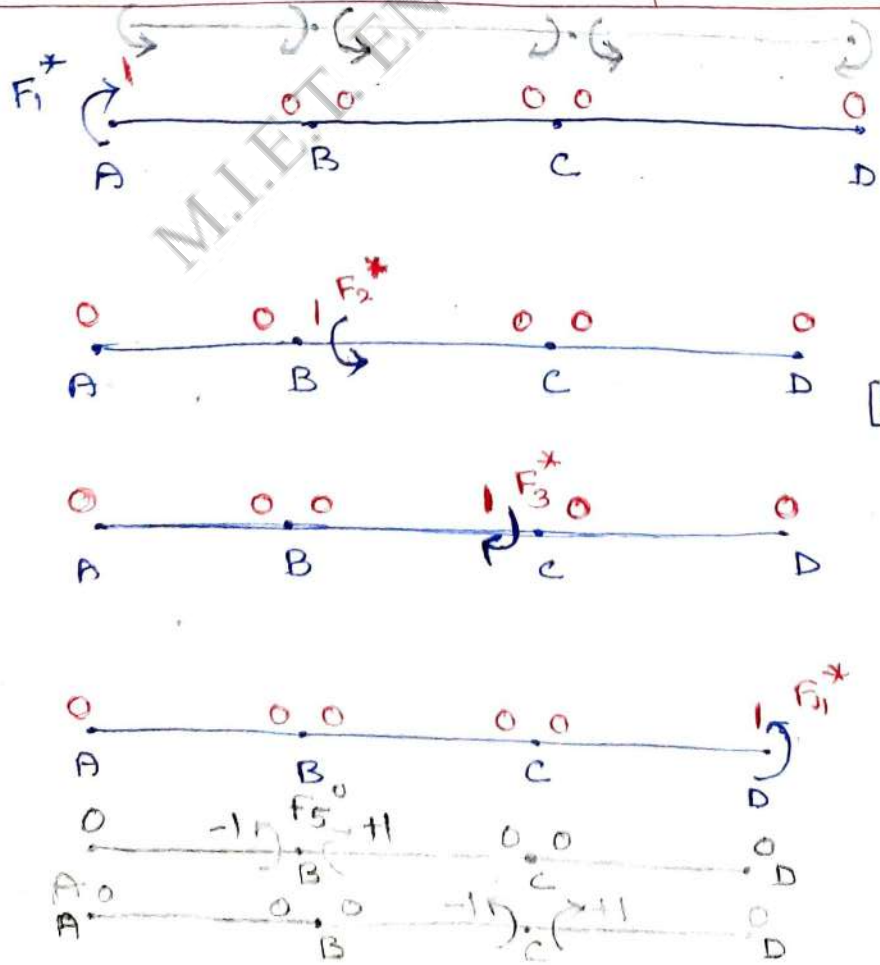
Redundant forces $\{F^0\} = -[a_{00}]^{-1}[a_{0*}]\{F^*\}$.

$[a_{00}] = [b^0]^T[\alpha][b^0]$

④



Span	Fixed end moments (KNm)	Name of moment	Equivalent joint force KNm $\{P\}^c$	F^*
AB	$-\frac{wl^2}{12} = -\frac{40\times12^2}{8} = -480$	M_{AB}	$+480$	$480 = F_1^*$ A
	$+\frac{wl^2}{12} = \frac{+40\times12^2}{12} = 480.$	M_{BA}	-480	$-266.67 = F_2^*$ B
BC	$-\frac{Wab^2}{l^2} = \frac{-120\times4\times8^2}{12^2} = -213.33$	M_{BC}	$+213.33$	
	$\frac{Wab^2}{l^2} = \frac{120\times4^2\times8}{\cancel{213.33}\;12^2} = 106.67$	M_{CB}	-106.67	133.33 F_3^* C
CD	$-\frac{wl^2}{12} = -\frac{20\times12^2}{12} = -240$	M_{CD}	$+240$	
	$\frac{wl^2}{12} = +240$	M_{DC}	$-240.$	-240 F_4^* D



$$[b] = \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

Assembled element flexibility matrix $[\alpha]$:

$$[\alpha] = \begin{bmatrix} \alpha_1 & & \\ & \alpha_2 & \\ & & \alpha_3 \end{bmatrix} = \frac{l}{6EI} \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 & -1 & 2 \end{bmatrix}$$

$$\frac{12}{6EI} = \frac{2}{EI}$$

Redundant forces $\{F^0\}$.

$$\{F^0\} = -[a_{oo}]^{-1}[a_o^*]\{F^*\}.$$

$$[a_{oo}] = [b^0]^T[\alpha][b^0].$$

$$= \underbrace{\begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix}}_{2\times6} \frac{2}{EI} \underbrace{\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 & -1 & 2 \end{bmatrix}}_{6\times6} \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{2}{EI} \underbrace{\begin{bmatrix} 1 & -2 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 2 & -1 \end{bmatrix}}_{2\times6} \underbrace{\begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{6\times2}$$

$$= \frac{2}{EI} \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

$$[a_{oo}]^{-1} = \frac{1}{2/EI} \times \frac{1}{(16-1)} \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} = \frac{EI}{30} \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}.$$

$$[a_{o*}] = [b_o]^T[\alpha][b_*]$$

$$= \frac{2}{EI} \underbrace{\begin{bmatrix} 1 & -2 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 2 & -1 \end{bmatrix}}_{2\times6} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{6\times4}$$

$$= \frac{2}{EI} \underbrace{\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & -1 \end{bmatrix}}_{2\times4}.$$

Redundant forces: $\{F^o\}$:

$$\{F_0\} = -[a_{00}]^{-1}[a_{0*}]\{F_*\}$$

$$\{F_0\} = -\frac{EI}{30}\begin{bmatrix}4 & -1\\ -1 & 4\end{bmatrix}\frac{2}{EI}\begin{bmatrix}1 & 2 & -1 & 0\\ 0 & 1 & -2 & -1\end{bmatrix}\begin{bmatrix}480\\ -266.67\\ 133.33\\ -240\end{bmatrix}$$

$$= -\frac{1}{15}\begin{bmatrix}4 & 7 & -2 & 1\\ -1 & 2 & -7 & -4\end{bmatrix}\begin{Bmatrix}480\\ -266.67\\ 133.37\\ -240\end{Bmatrix}$$

$$= \begin{Bmatrix}30.22\\ 65.78\end{Bmatrix} \text{ KN.}$$

Element forces $\{P\}$:

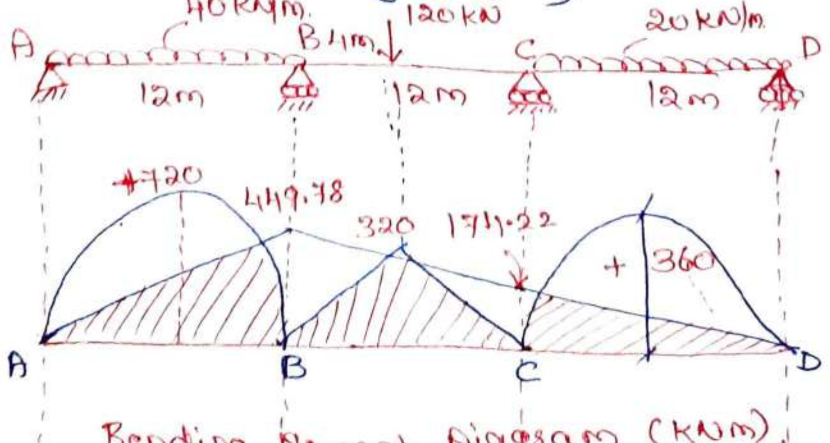
$$\{P\} = [b]\{F\}.$$

$$= \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1 & 0 & 0\end{bmatrix}\begin{Bmatrix}480\\ -266.67\\ 133.33\\ -240.0\\ 30.22\\ 65.78\end{Bmatrix}$$

$$= \begin{Bmatrix}480\\ -30.22\\ -236.45\\ 67.55\\ 65.78\\ -240.0\end{Bmatrix}$$

Final forces: $\{P\}^f = \{P\} - \{P\}^e$

$$\begin{Bmatrix}M_{AB}\\ M_{BA}\\ M_{BC}\\ M_{CB}\\ M_{CD}\\ M_{DA}\end{Bmatrix} = \begin{Bmatrix}480\\ -30.22\\ -236.45\\ 67.55\\ 65.78\\ -240.0\end{Bmatrix} - \begin{Bmatrix}480\\ -480\\ 213.33\\ -106.67\\ 240.0\\ -240.0\end{Bmatrix} = \begin{Bmatrix}0\\ 449.78\\ -449.78\\ +174.22\\ -174.22\\ 0\end{Bmatrix}$$



Bending Moment Diagram (KNm).

Step by step procedure

Flexibility matrix method.

1. Decide on the primary structure. Indicate redundants $\{F\}$

2. Find fixed end moments.

3. Generate $[b]$ matrix such that $\{P\} = [b]\{F\}$

$$[b] = \left[[b^*] \;\vdots\; [b^0] \right]$$

4. Assemble element flexibility matrix $[\alpha]$.

5. $a = [b]^T [\alpha][b]$

$$a_{00} = [b_0]^T [\alpha][b_0]$$

$$[a_{0*}] = [b_0]^T [\alpha][b_*]$$

$$\{F\}^0 = -[a_{0a}]^{-1}[a_{0*}]\{F\}^*$$

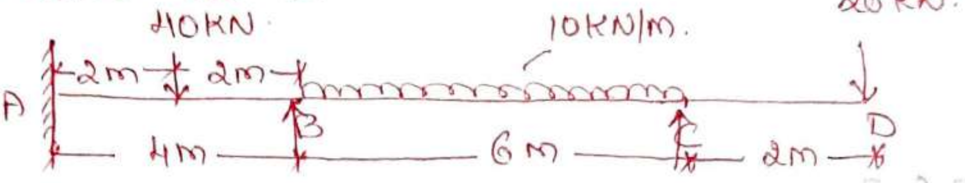
$$\{F\}^T = \begin{bmatrix} \{F\}^* \\ \{F\}^0 \end{bmatrix}^T \Rightarrow \{F\} = \begin{bmatrix} \{F\}^* \\ \{F\}^0 \end{bmatrix}$$

$$[P] = [b]\{F\}.$$

6. Final forces $\{P\}^f = \{P\} - \{P\}^e$.

where $\{P\}^e$ = equivalent joint forces.

③ Analyse the continuous beam ABCD by flexibility method. Support condition: A - fixed. B & C continuous support and D - free. Span AB = 4m, BC = 6m, CD = 2m. Load: 40 KN at 2m from A. 10 KN/m over the span BC and 20KN at D.



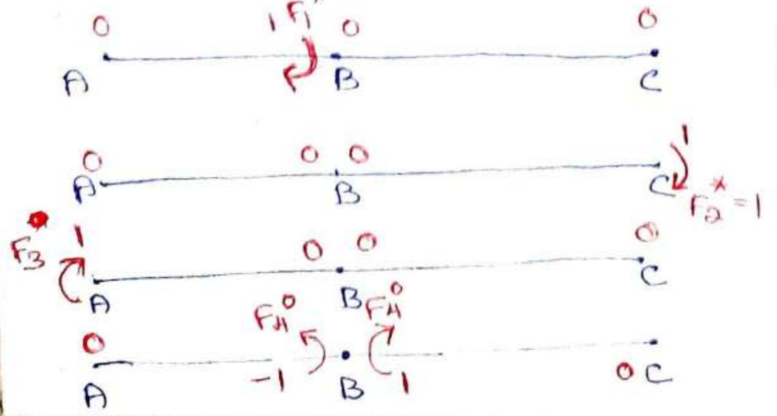
Degree of redundancy = (3+2+2) - 9 = 2.

Treat~~ing~~ the support moments at A and B as redundants.

The {F}* forces are made up of equivalent joint forces arising out of transverse loads on spans AB, BC & CD.

Span.	Fixed end moments (KNm)	Name of moment	Equivalent joint force $\{P\}^e$	F^*
AB	$-\frac{Wl}{8} = \frac{-40\times4}{8} = -20$	M_{FAB}.	20	Ignored.
	$\frac{Wl}{8} = \frac{40\times4}{8} = 20$	M_{FBA}	-20	$10F_1^*$
BC	$-\frac{wl^2}{12} = \frac{-10\times6^2}{12} = -30$	M_{FBC}	30	
	$\frac{wl^2}{12} = \frac{10\times6^2}{12} = 30$	M_{FCB}	-30	$10F_2^*$
CD	$-20\times2 = -40$.	M_{FCD}.	40	

$F_1^* = 10$ kNm ; $F_2^* = 10$ kNm.



$$[b] = \begin{array}{cc|cc} b^* & & b^0 & \\ \hline 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}$$

$$[\alpha] = \begin{bmatrix} \alpha_1 & \\ & \alpha_2 \end{bmatrix}$$

$$[\alpha_1] = \frac{l_1}{6EI_1}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \frac{\overset{2}{\cancel{4}}}{3\,\cancel{6}EI}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$ (for span AB).

$$[\alpha_2] = \frac{l_2}{6EI_2}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \frac{\overset{2\times3}{\cancel{6}}}{\underset{3\times2}{\cancel{6}}EI}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$ (for span BC).

$$= \frac{2}{3EI}\begin{bmatrix} 3 & -1.5 \\ -1.5 & 3 \end{bmatrix}$$

Assembled element flexibility matrix:

$$[\alpha] = \begin{bmatrix} \alpha_1 & \\ & \alpha_2 \end{bmatrix} = \frac{2}{3EI}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 3 & -1.5 \\ 0 & 0 & -1.5 & 3 \end{bmatrix}$$

Redundant forces $\{F^0\}$.

$$\{F^0\} = -[a_{00}]^{-1}[a_{0*}]\{F^*\}.$$

$$[a_{00}]^{-1} = [b_0]^T[\alpha][b_0]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}\frac{2}{3EI}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 3 & -1.5 \\ 0 & 0 & -1.5 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{2}{3EI}\begin{bmatrix} 2 & -1 & 0 & 0 \\ 1 & -2 & 3 & -1.5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$[a_{00}] = \frac{2}{3EI}\begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}$$

$$[a_{00}]^{-1} = \frac{3EI}{2(10-1)}\begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix} = \frac{EI}{6}\begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix}$$

$$[a_{0*}] = [b_0]^T[\alpha][b_*].$$

$$= \frac{2}{3EI}\begin{bmatrix} 2 & -1 & 0 & 0 \\ 1 & -2 & 3 & -1.5 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \frac{2}{3EI}\begin{bmatrix} -1 & 0 \\ -2 & -1.5 \end{bmatrix}.$$

(Q)
(F)

$$\{F_0\} = -\frac{EI}{6}\begin{bmatrix}5 & -1\\ -1 & 2\end{bmatrix}\frac{2}{3EI}\begin{bmatrix}-1 & 0\\ -2 & -1.5\end{bmatrix}\begin{Bmatrix}10\\ 10\end{Bmatrix}$$

$$= -\frac{1}{9}\begin{bmatrix}-3 & 1.5\\ -3 & -3\end{bmatrix}\begin{Bmatrix}10\\ 10\end{Bmatrix} = \begin{Bmatrix}1.667\\ 6.667\end{Bmatrix}.$$

Element forces $\{P\} = [b]\{F\}$.

$$= \begin{bmatrix}0 & 0 & 1 & 0\\ 1 & 0 & 0 & -1\\ 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\end{bmatrix}\begin{Bmatrix}10\\ 10\\ 1.667\\ 6.667\end{Bmatrix} = \begin{Bmatrix}1.667\\ 3.333\\ 6.667\\ 10\end{Bmatrix}.$$

Final forces: $\{P\}^f = \{P\} - \{P^e\}$.

$$= \begin{Bmatrix}1.667\\ 3.333\\ 6.667\\ 10\end{Bmatrix} - \begin{Bmatrix}20\\ -20\\ 30\\ -30\end{Bmatrix} = \begin{Bmatrix}-18.333\\ 23.333\\ -23.333\\ 40.\end{Bmatrix}$$

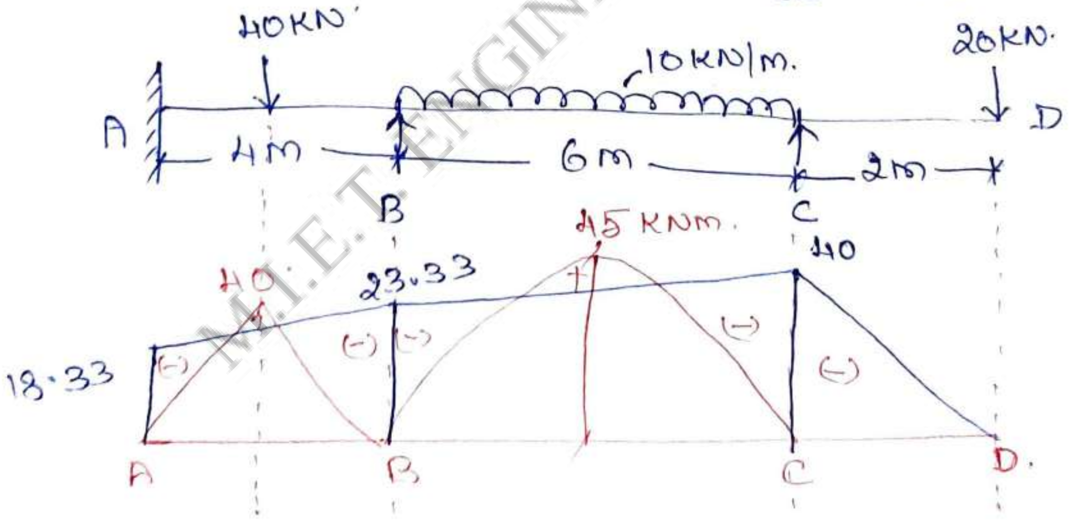
$M_{AB} = -18.333$ KNm.

$M_{BA} = 23.33$ KNm

$M_{BC} = -23.33$ KNm

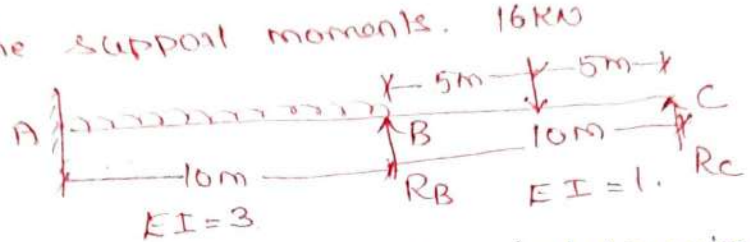
$M_{CB} = 40$ KNm.

$M_{CD} = -40$ KNm (overhanging moment).



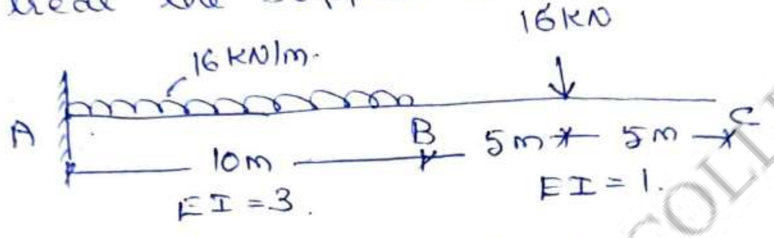
Bending Moment Diagram (KNm).

④ Analyse the beam in the previous example by treating the reactions at B and C as redundants. obtain the support moments.



The structure is statically indeterminate by 2 degrees.

Let us treat the support reactions RB and RC as redundants.



Primary structure.

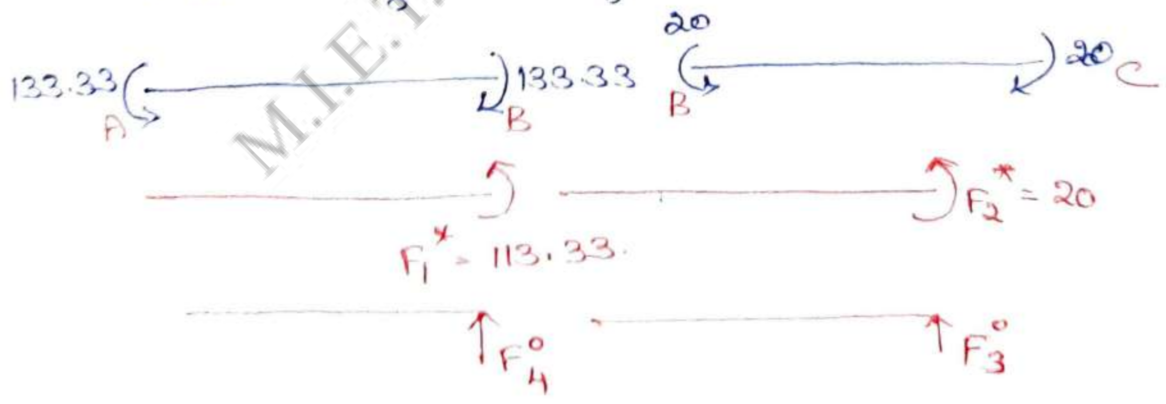
Equivalent jt force

$$M_{FAB} = \frac{-wl^2}{12} = -\frac{16\times10^2}{12} = -133.33 \text{ KNm.}$$

$$M_{FBA} = +\frac{wl^2}{12} = +\frac{16\times10^2}{12} = 133.33 \text{ KNm.}$$ 113.33 −113.33.

$$M_{FBC} = -\frac{Wl}{8} = \frac{-16\times10}{8} = -20 \text{ KNm.}$$ −20.

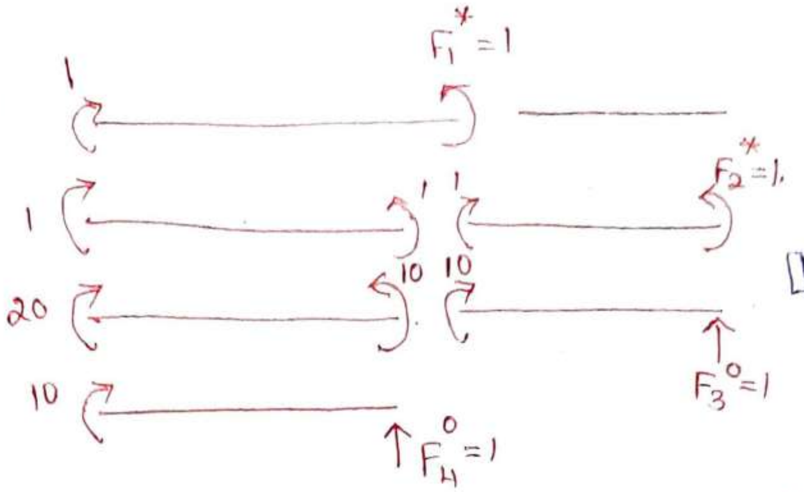
$$M_{FCB} = +\frac{Wl}{8} = \frac{16\times10}{8} = 20 \text{ KNm.}$$



Applying F_1^*, F_2^*, F_3^0 and F_4^0 = unity, one at a time, [b] matrix is formed

$$\{F\} = \begin{Bmatrix} -113.33 \\ -20 \end{Bmatrix} \qquad \{F^*\} = \begin{Bmatrix} 113.33 \\ 20 \end{Bmatrix}$$

when sign is included in b^* matrix donot put sign for F^* matrix.



$$[b] = \begin{array}{cc|cc} [b^*] & & [b^\circ] & \\ 1 & 1 & 20 & 10 \\ -1 & -1 & -10 & 0 \\ 0 & 1 & 10 & 0 \\ 0 & 1 & 0 & 0 \end{array}$$

$$[a] = [b]^T [\alpha][b]$$

$$[a_{00}] = [b^\circ]^T [\alpha][b^\circ]$$

$$[a_{0*}] = [b^\circ][\alpha][b^*]$$

Element flexibility matrix:

$$[\alpha_i] = \frac{l}{6EI}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \frac{10}{6\times 3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$[\alpha_1] = \frac{10}{18}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

$$[\alpha_2] = \frac{10}{6\times 1}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \frac{10}{6}\times\frac{3}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \frac{10}{18}\begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}.$$

$$[\alpha] = \begin{bmatrix} [\alpha_1] & 0 \\ 0 & [\alpha_2] \end{bmatrix} = \frac{10}{18}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix}$$

$$[a_{00}] = [b^\circ]^T [\alpha][b^\circ]$$

$$= \begin{bmatrix} 20 & -10 & 10 & 0 \\ 10 & 0 & 0 & 0 \end{bmatrix} \times \frac{10}{18}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix}\begin{bmatrix} 20 & 10 \\ -10 & 0 \\ 10 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{10}{18}\begin{bmatrix} 20 & -10 & 10 & 0 \\ 10 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 50 & 20 \\ -40 & -10 \\ 60 & 0 \\ -30 & 0 \end{bmatrix} = \frac{10}{18}\begin{bmatrix} 2000 & 500 \\ 500 & 200 \end{bmatrix}$$

$$[a_{00}]^{-1} = \frac{18}{10}\,\frac{1}{(2000\times 200 - 500\times 500)}\begin{bmatrix} 200 & -500 \\ -500 & 2000 \end{bmatrix}$$

$$= \frac{18\times 100}{10\times 1{,}50{,}000}\begin{bmatrix} 2 & -5 \\ -5 & 20 \end{bmatrix}$$

$$[a_{00}]^{-1} = \frac{18}{1{,}50{,}000}\begin{bmatrix} 2 & -5 \\ -5 & 20 \end{bmatrix}$$

$$[a_{0*}] = [b^0]^T [\alpha][b^*]$$

$$= \begin{bmatrix} 20 & -10 & 10 & 0 \\ 10 & 0 & 0 & 0 \end{bmatrix} \times \frac{10}{18} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$= \frac{10}{18} \begin{bmatrix} 20 & -10 & 10 & 0 \\ 10 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ -3 & -3 \\ 0 & 9 \\ 0 & -9 \end{bmatrix} = \frac{10}{18} \begin{bmatrix} 90 & 180 \\ 30 & 30 \end{bmatrix}$$

Redundant forces are given by $\{F^0\} = -[a_{00}]^{-1}[a_{0*}]\{F^*\}$.

$$\{F^0\} = \frac{-18}{15000} \begin{bmatrix} 2 & -5 \\ -5 & 20 \end{bmatrix} \times \frac{10}{18} \begin{bmatrix} 90 & 180 \\ 30 & 30 \end{bmatrix} \begin{Bmatrix} 113.33 \\ 20 \end{Bmatrix}.$$

$$= \frac{-1}{1500} \begin{bmatrix} 30 & 210 \\ 150 & -300 \end{bmatrix} \begin{Bmatrix} 113.33 \\ 20 \end{Bmatrix} = \frac{-1}{1500} \begin{Bmatrix} 7599.9 \\ 10999.5 \end{Bmatrix}$$

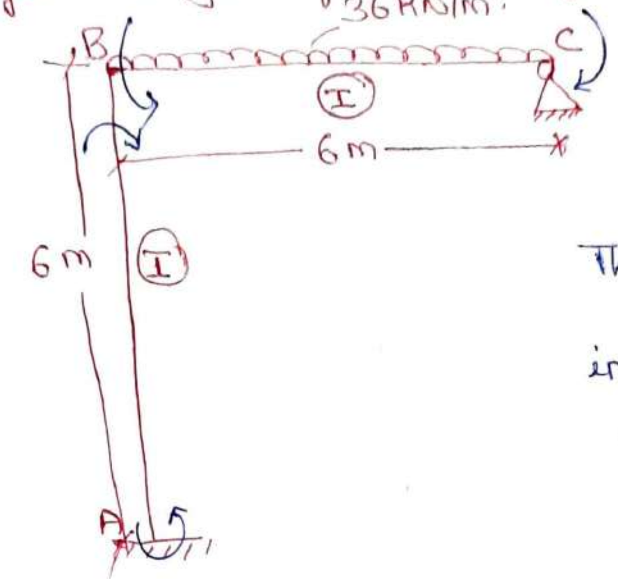
$$\{F^0\} = \begin{Bmatrix} -5.067 \\ -7.333 \end{Bmatrix}.$$

$$\{P\} = [b]\{F\}$$

$$\{P\} = \begin{bmatrix} 1 & 1 & 20 & 10 \\ -1 & -1 & -10 & 0 \\ 0 & 1 & 10 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 113.33 \\ 20 \\ -5.067 \\ -7.333 \end{Bmatrix} = \begin{Bmatrix} -41.34 \\ -82.66 \\ -30.67 \\ -20.00 \end{Bmatrix}$$

Final force $\{P^f\} = \begin{Bmatrix} -133.33 \\ 133.33 \\ -20.00 \\ 20.00 \end{Bmatrix} + \begin{Bmatrix} -41.34 \\ -82.66 \\ -30.67 \\ -20.00 \end{Bmatrix} = \begin{Bmatrix} -174.67 \\ 50.67 \\ -50.67 \\ 0 \end{Bmatrix}$

5) Analyse the given frame using matrix flexibility method.



The degree of static indeterminacy $= (3+2)-3 = 2$.

Let us make the given frame to a primary structure by removing V_c and H_c treating them as redundants.

Fixed end moments

$M_{FAB} = M_{FBA} = 0$

$M_{FBC} = \frac{-wl^2}{12} = -\frac{36\times 6^2}{12} = -108$ kNm.

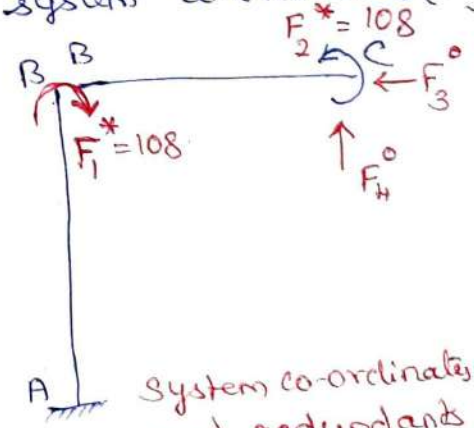
$M_{FCB} = +\frac{wl^2}{12} = 108$ kNm.

Equivalent joint loads.

0	A
+108 kNm.	B
−108	C

$$\{F\} = \begin{bmatrix} F_1^* \\ F_2^* \\ F_3^0 \\ F_4^0 \end{bmatrix}$$

The primary structure with system co-ordinates $\{F^*$ and redundants $\{F^0\}$.



System co-ordinates and redundants

Element flexibility matrix:

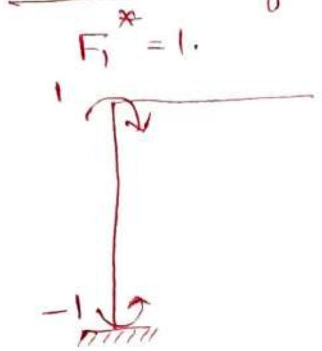
$$[\alpha_1] = [\alpha_2] = \frac{l}{6EI}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{6}{6EI}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

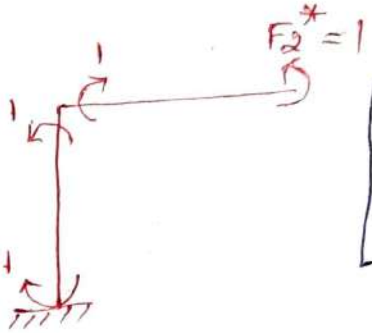
Assembled element flexibility matrix:

$$[\alpha] = \frac{1}{EI}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

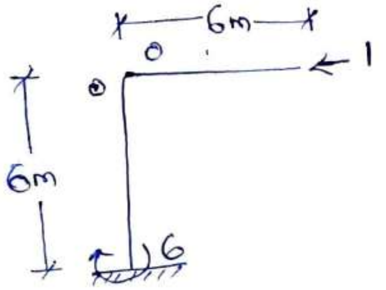
Formation of [b] matrix:



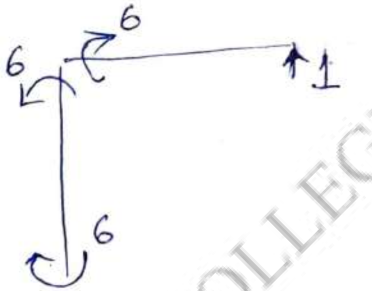
$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$



$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$



$$\begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



$$\begin{bmatrix} +6 \\ -6 \\ 6 \\ 0 \end{bmatrix}$$

∴ [b] matrix is given by

$$[b] = \left[\begin{array}{cc:cc} -1 & 1 & 6 & 6 \\ 1 & -1 & 0 & -6 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\qquad b^* \qquad\qquad b^0$$

Redundant forces: $\{F^0\} = -[a_{22}]^{-1}[a_{21}]\{F^*\}$

$$[a_{00}] = [b_0]^T [\alpha][b_0].$$

$$[a_{00}] = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 6 & -6 & 6 & 0 \end{bmatrix} \frac{1}{EI} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 6 & 6 \\ 0 & -6 \\ 0 & 6 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{1}{EI} \begin{bmatrix} 12 & -6 & 0 & 0. \\ 18 & -18 & +12 & -6 \end{bmatrix} \begin{bmatrix} 6 & 6 \\ 0 & -6 \\ 0 & 6 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{1}{EI} \begin{bmatrix} 72 & 108 \\ 108 & 288 \end{bmatrix}$$

$$[a_{0*}] = [b_0]^T [\alpha][b_*]$$

$$= \begin{bmatrix} 6 & 0 & 0 & 0 \\ 6 & -6 & 6 & 0 \end{bmatrix} \frac{1}{EI} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} = \frac{1}{EI} \begin{bmatrix} -18 & 18 \\ -36 & 54 \end{bmatrix}$$

∴ Redundant forces:

$$\{F^o\} = -[a_{oo}]^{-1}[a_{ox}]\{F^*\}$$

$$= -EI\begin{bmatrix}72 & 108\\108 & 288\end{bmatrix}^{-1}\frac{1}{EI}\begin{bmatrix}-18 & 18\\-36 & 54\end{bmatrix}\begin{bmatrix}108\\108\end{bmatrix}$$

$$= -\frac{1}{9072}\begin{bmatrix}288 & -108\\-108 & 72\end{bmatrix}\begin{bmatrix}0\\1944\end{bmatrix}$$

$$= \begin{Bmatrix}23.14\\-15.43\end{Bmatrix}$$

Element forces: $\{P\}=[b]\{F\} = [b^*|b^o]\begin{Bmatrix}F^*\\F^o\end{Bmatrix}$

$$\{P\} = \begin{bmatrix}-1 & 1 & 6 & 6\\1 & -1 & 0 & -6\\0 & 1 & 0 & 6\\0 & -1 & 0 & 0\end{bmatrix}\begin{bmatrix}108\\108\\23.14\\-15.43\end{bmatrix} = \begin{bmatrix}46.26\\92.58\\15.42\\-108\end{bmatrix}$$

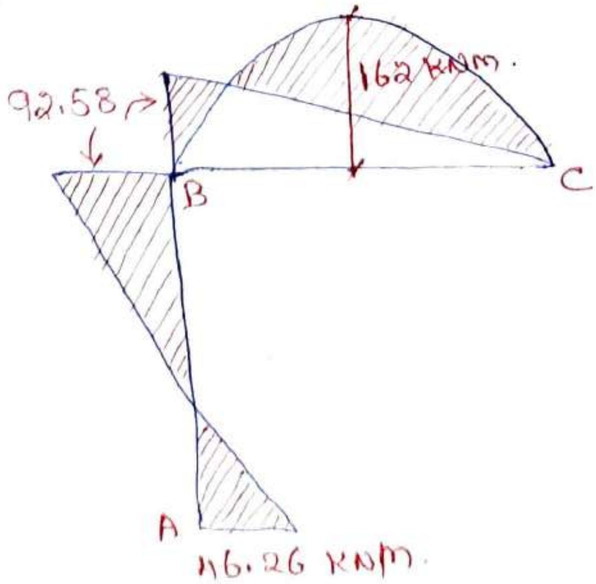
Final forces: $\{P\}^f = \{P\} + \{\text{Fixed end moments}\}$

$$= \begin{Bmatrix}46.26\\92.58\\15.42\\-108\end{Bmatrix} + \begin{Bmatrix}0\\0\\-108\\108\end{Bmatrix}$$

$$\begin{Bmatrix}M_{AB}\\M_{BA}\\M_{BC}\\M_{CB}\end{Bmatrix} = \begin{Bmatrix}46.26\\92.58\\-92.58\\0\end{Bmatrix}$$

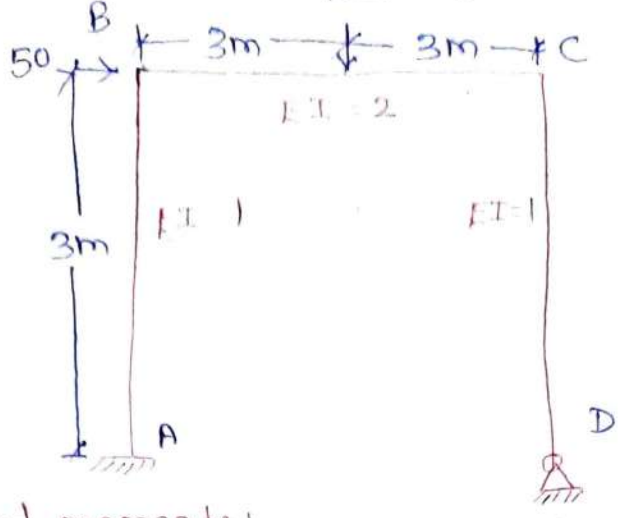
Simply supported moments:

Span BC = $\frac{wl^2}{8} = \frac{36\times 6^2}{8} = 162$ KNm



Bending moment diagram.(BMD).

⑥ Analyse the frame using flexibility matrix method.



SI = 2

consider V_D and H_D as redundants.

Fixed end moments:

$$M_{FBC} = \frac{-100\times 6}{8} = -75 \text{ KNm.}$$

$$M_{FCB} = \frac{100\times 6}{8} = +75 \text{ KNm.}$$

Equivalent jt forces

+75

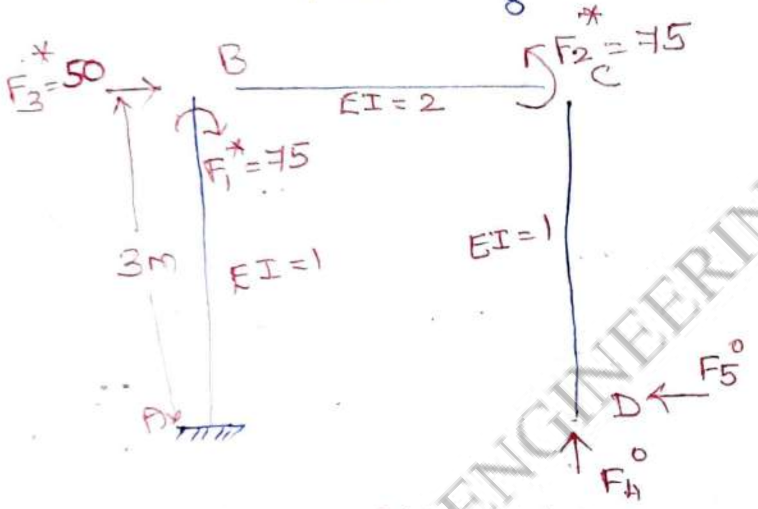
−75

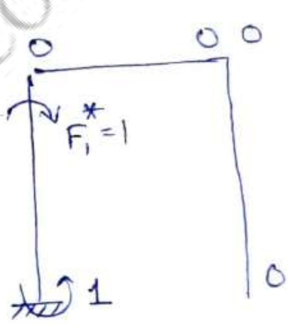
F^*

$F_1^* = 75$

$F_2^* = -75$

$F_3^* = 50$

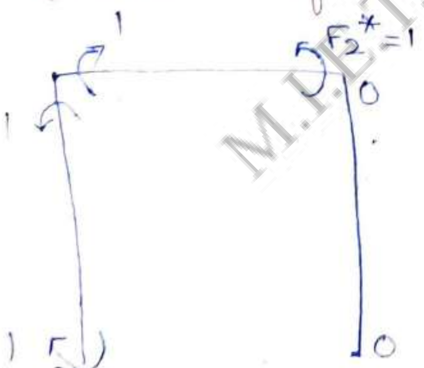




b^*

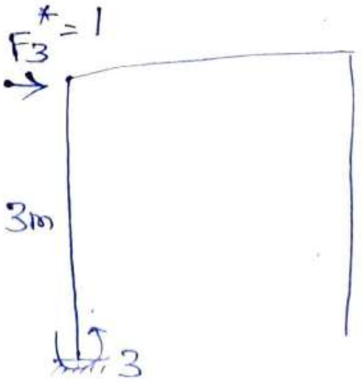
$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Derivation of [b] matrix.



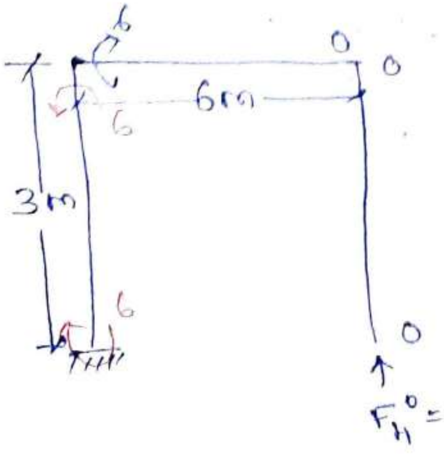
b^*

$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

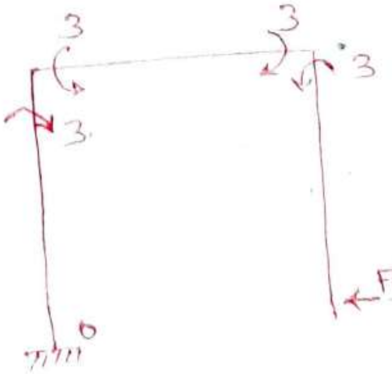


$$\begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

b^*



$$\begin{bmatrix} 6 \\ -6 \\ 6 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



$$\begin{bmatrix} 0 \\ 3 \\ -3 \\ 3 \\ -3 \\ 0 \end{bmatrix}$$

Element flexibility matrix $[\alpha]$:

$$[\alpha_i] = \frac{l}{6EI}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \; ; \; [\alpha_1] = \frac{3}{6\times 1}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$[\alpha_2] = \frac{6}{6\times 2}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \; ; \; [\alpha_3] = \frac{3}{6\times 1}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

Assembled element flexibility matrix:

$$[\alpha] = \frac{1}{2}\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

To find redundant forces:

$$\{F^o\} = -[a_{oo}]^{-1}[a_{o*}]\{F^*\}$$

$$[a_{oo}] = [b_o]^T[\alpha][b_o]$$

$$= \begin{bmatrix} 6 & -6 & 6 & 0 & 0 & 0 \\ 0 & 3 & -3 & 3 & -3 & 0 \end{bmatrix} \times \frac{1}{2}\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 6 & 0 \\ -6 & 3 \\ 6 & -3 \\ 0 & 3 \\ 0 & -3 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 18 & -18 & 12 & -6 & 0 & 0 \\ -3 & 6 & -9 & 9 & -6 & 3 \end{bmatrix}\begin{bmatrix} 6 & 0 \\ -6 & 3 \\ 6 & -3 \\ 0 & 3 \\ 0 & -3 \\ 0 & 0 \end{bmatrix}$$

$$[a_{oo}] = \frac{1}{2}\begin{bmatrix} 288 & -108 \\ -108 & 90 \end{bmatrix} = \begin{bmatrix} 144 & -54 \\ -54 & 45 \end{bmatrix}$$

$$[a_{oo}]^{-1} = \frac{1}{(144\times 45 - 54\times 54)}\begin{bmatrix} 45 & 54 \\ 54 & 144 \end{bmatrix} = \frac{1}{3564}\begin{bmatrix} 45 & 54 \\ 54 & 144 \end{bmatrix}$$

$$[a_{0*}] = [b_0]^T [\alpha][b_*]$$

$$= \begin{bmatrix} 6 & -6 & 6 & 0 & 0 & 0 \\ 0 & 3 & -3 & 3 & -3 & 0 \end{bmatrix} \times \frac{1}{2} \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 & -3 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 18 & -18 & 12 & -6 & 0 & 0 \\ -3 & 6 & -9 & 9 & -6 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 & -3 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[a_{0*}] = \frac{1}{2} \begin{bmatrix} -36 & 54 & -54 \\ 9 & -27 & 9 \end{bmatrix}$$

$$\{F_0^*\} = -[a_{00}]^{-1} [a_{0*}] \{F^*\}$$

$$= -\frac{1}{3564} \begin{bmatrix} 45 & 54 \\ 54 & 144 \end{bmatrix} \times \frac{1}{2} \begin{bmatrix} -36 & 54 & -54 \\ 9 & -27 & 9 \end{bmatrix} \begin{Bmatrix} 75 \\ 75 \\ 50 \end{Bmatrix}$$

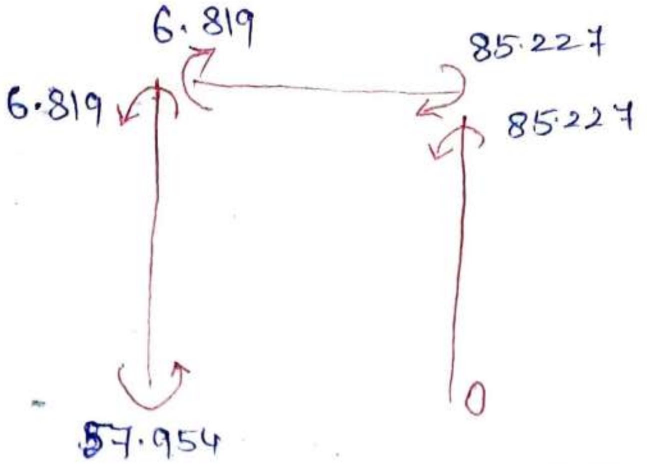
$$= -\frac{1}{3564 \times 2} \begin{bmatrix} 45 & 54 \\ 54 & 144 \end{bmatrix} \begin{bmatrix} -1350 \\ -900 \end{bmatrix} = \frac{-1}{3564 \times 2} \begin{Bmatrix} -109350 \\ -202500 \end{Bmatrix}$$

$$\{F_0\} = \begin{Bmatrix} 15.341 \\ 28.409 \end{Bmatrix}$$

Element forces: $\{P\} = [b]\{F\}$ where $\{F^*\} = \begin{Bmatrix} F^* \\ F^0 \end{Bmatrix}$ (no sign for F^* as it is included in b^*)

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \begin{bmatrix} -1 & 1 & -3 & 6 & 0 \\ 1 & -1 & 0 & -6 & 3 \\ 0 & 1 & 0 & 6 & -3 \\ 0 & -1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 75 \\ 75 \\ 50 \\ 15.341 \\ 28.409 \end{Bmatrix} = \begin{Bmatrix} -57.954 \\ -6.819 \\ 81.819 \\ 10.227 \\ -85.227 \\ 0 \end{Bmatrix}$$

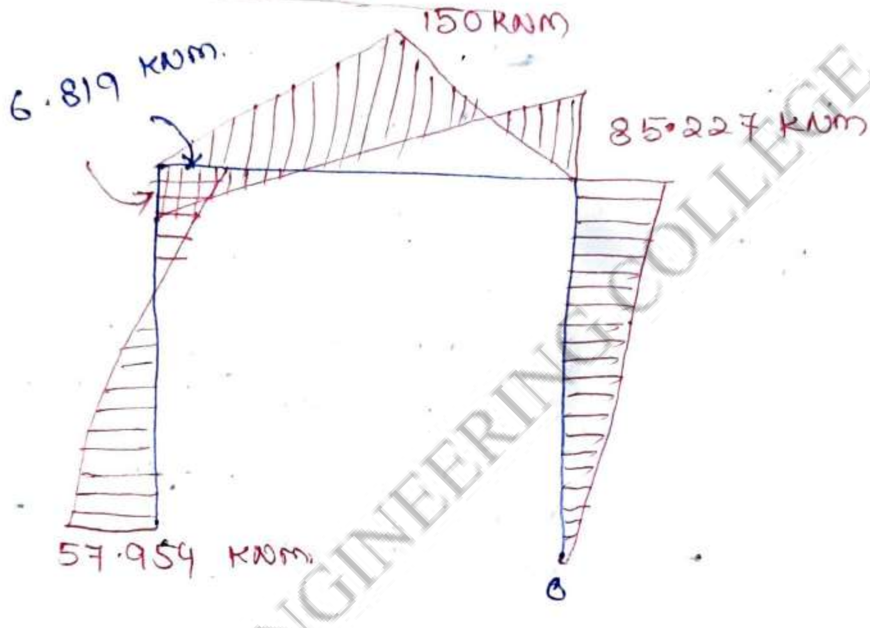
$$\{P^f\} = \{P\} + \{FEM\} = \begin{Bmatrix} -57.954 \\ -6.819 \\ 81.819 \\ 10.227 \\ -85.227 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ -75 \\ 75 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -57.954 \\ -6.819 \\ 6.819 \\ 85.227 \\ -85.227 \\ 0 \end{Bmatrix}$$



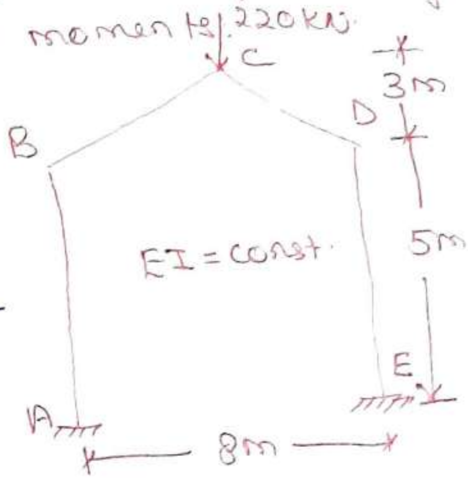
-57.954
-6.819
6.819
85.227
-85.227
0

Free BM = $\frac{Wl}{4} = \frac{100 \times 6}{4} = 150$ KNm.

Bending moment diagram:



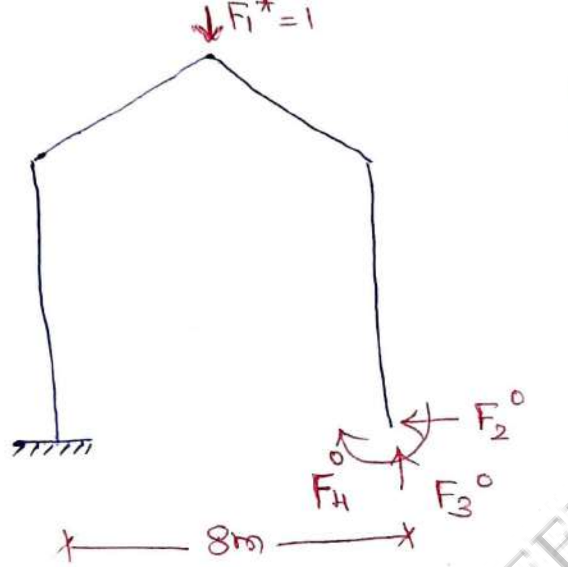
Analyse the frame shown by matrix flexibility method and obtain the support moments.



SI = 3.

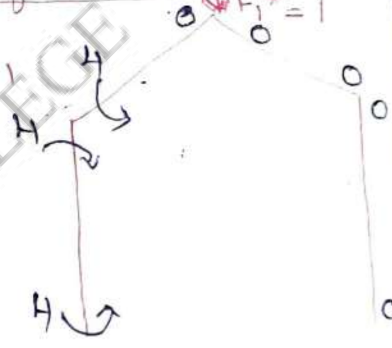
consider H_E, V_E and M_E as redundants.

Load is acting @ node, hence there will be no equivalent jt. load.

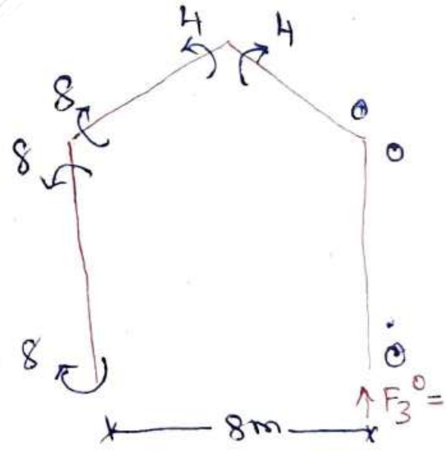


Formation of [b] matrix.

Applying $F_1^* = 1$

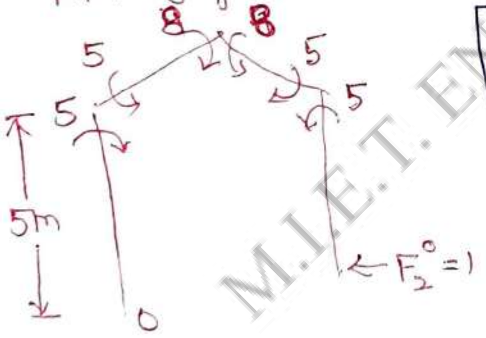


$$\begin{bmatrix} -4 \\ 4 \\ -4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

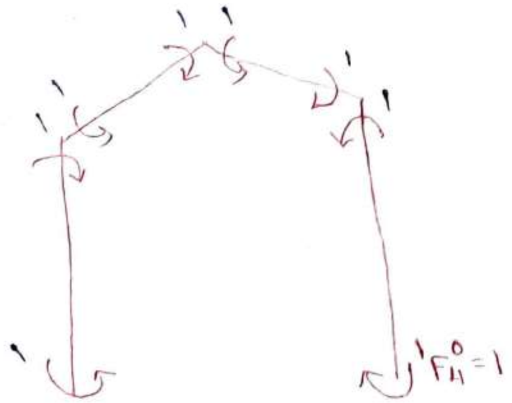


$$\begin{bmatrix} 8 \\ -8 \\ 8 \\ -4 \\ 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $F_2^0 = 1$



$$\begin{bmatrix} 0 \\ 5 \\ -5 \\ 8 \\ -8 \\ 5 \\ -5 \\ 0 \end{bmatrix}$$



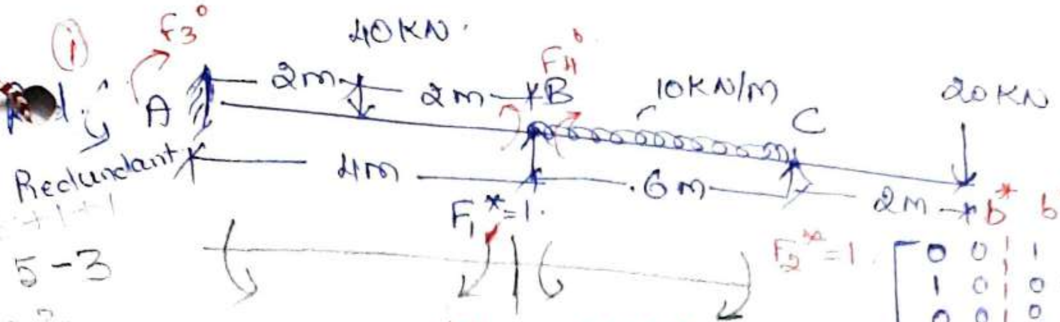
$$\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$[b] = \left[\begin{array}{c:ccc} -4 & 0 & 8 & -1 \\ 4 & 5 & -8 & 1 \\ -4 & -5 & 8 & -1 \\ 0 & 8 & -4 & 1 \\ 0 & -8 & 4 & -1 \\ 0 & 5 & 0 & 1 \\ 0 & -5 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Element flexibility matrix: ∵ length of all elements and EI are const., 'α' is same for all elements.

$$[\alpha_1] = [\alpha_2] = [\alpha_3] = [\alpha_4] = \frac{5}{6EI}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}. \qquad [F^*] = [220].$$

①



5 - 3 = 2

MA = 0.

$F_1^* = +10$

b^* b° $F_2^* = +10$.

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

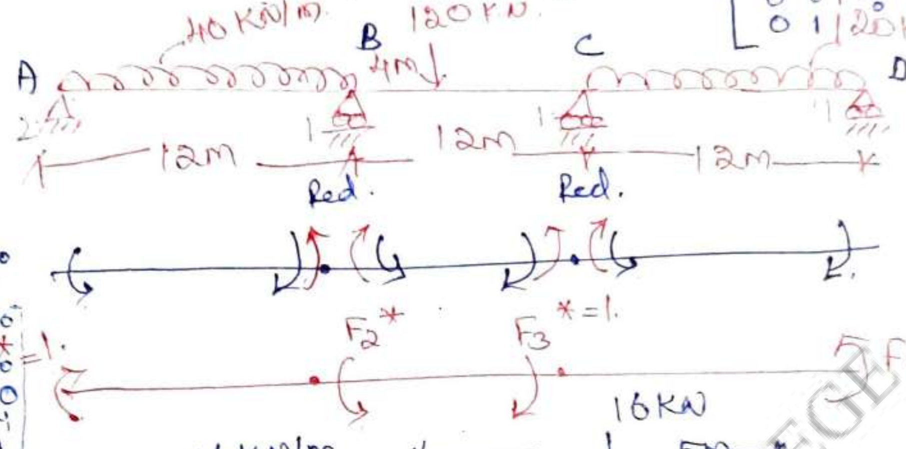
$F_1^* = 48$.

$F_2^* = -266$

$F_3^* = 133.33$.

$F_4^* = -240$.

②



2+1+1+1
5-3 = 2

actual. b°

b^*

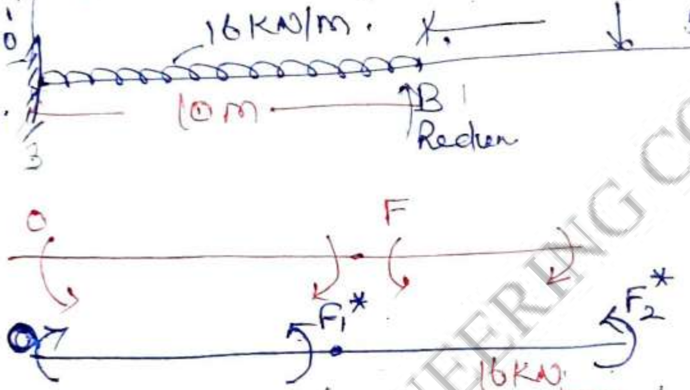
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$F_1^* = 1.$

b^*

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$$

③



5-3 = 2

F_1^*

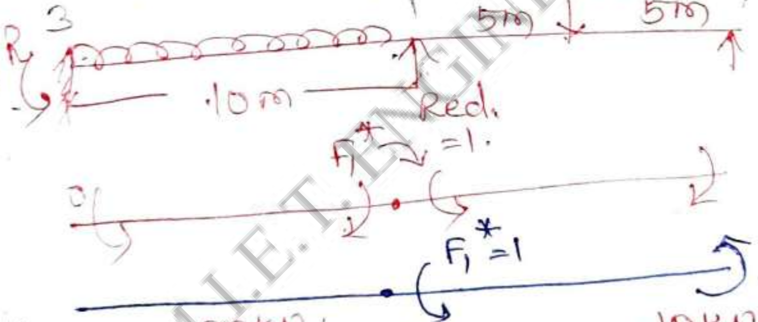
EJF

$M_{FAB} = -133$ +133 0

$M_B = +113$. $-113 = F_1^*$

$M_C = +20$ $-20 = F_2^*$

④



3+1+1
5
-3
2

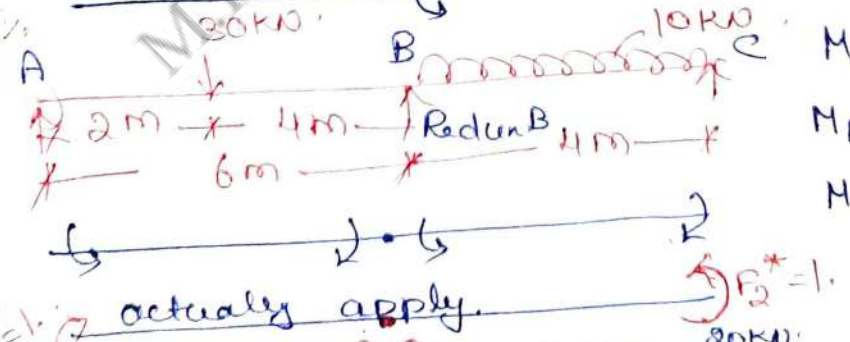
MFAB. 0

MB -113 F_1^*

Mc 20 F_2^*

$F_1^* = +1$

⑤



4-3
1

$M_{AB} = -26.67$.

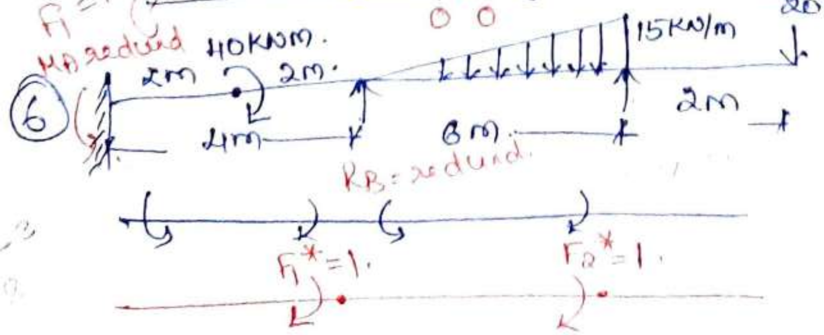
$M_B = 0$

$M_C = 13.33$. $F_2^* = -1$

1 0
0 0
0 0
0 1

EJL 0

⑥



5-3

MAB = -10.

MB = -8 +8

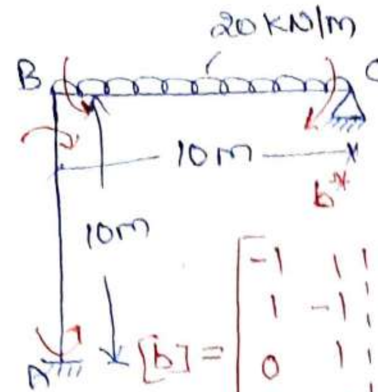
Mc = -13 +13

$F_1^* = +1$

$F_2^* = +1$.

Equivalent joint force KNm. $\{P_i\}^e$

3m + r − 3j = 6 + 5 − 9 = 2



b°

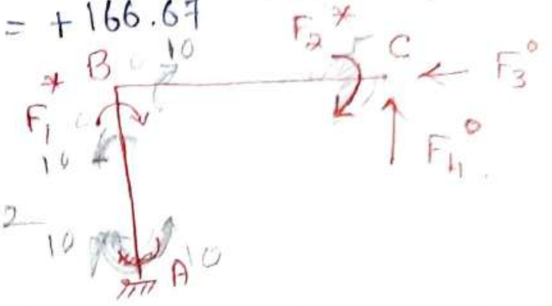
$M_{FAB} = 0$

$M_{FBA} = 0$

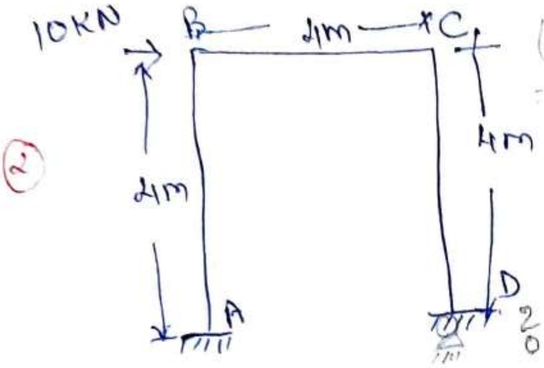
$M_{FBC} = -166.67$ } $166.67 F_1^*$

$M_{FCB} = +166.67$ $-166.67 F_2^*$

$$[b] = \left[\begin{array}{cc|cc} -1 & 1 & 10 & 10 \\ 1 & -1 & 0 & -10 \\ 0 & 1 & 0 & 10 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

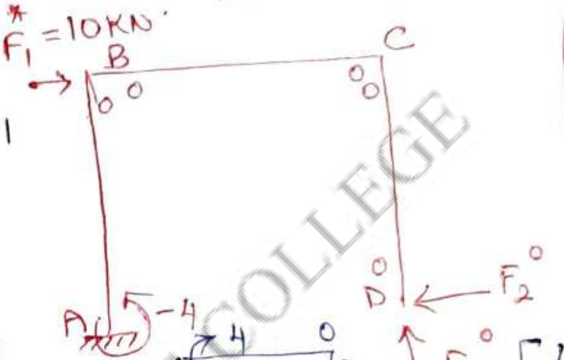


②

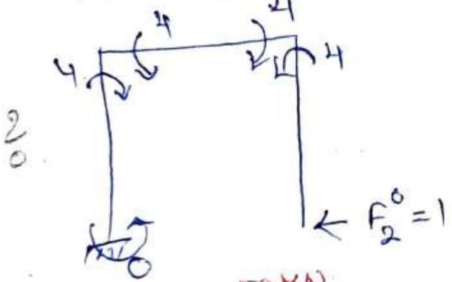


(3×3) + 6 − 12 = 15 − 12 = 3

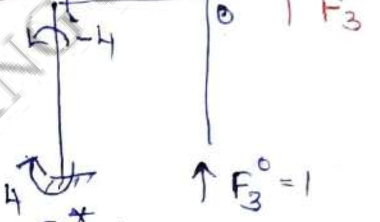
Redundant = 2,



$$\begin{bmatrix} -4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

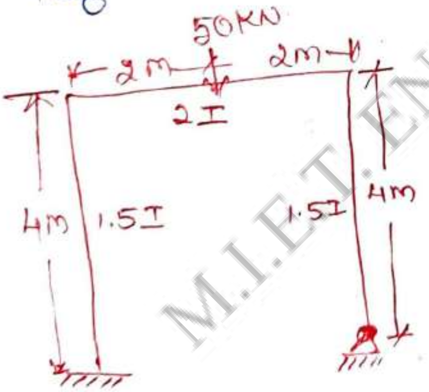


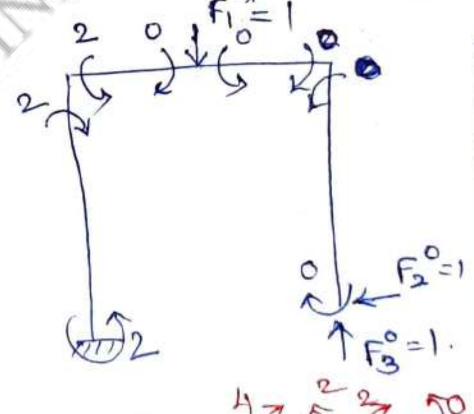
$$\begin{bmatrix} 0 \\ +4 \\ -4 \\ +4 \\ -4 \\ 0 \end{bmatrix}$$



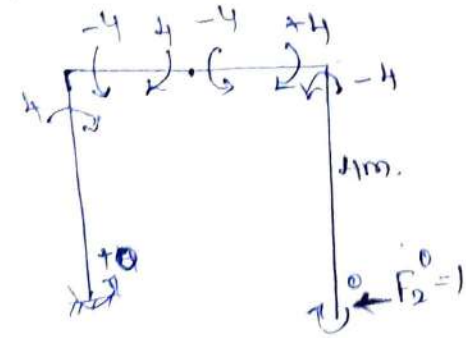
$$\begin{bmatrix} 4 \\ -4 \\ 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

③

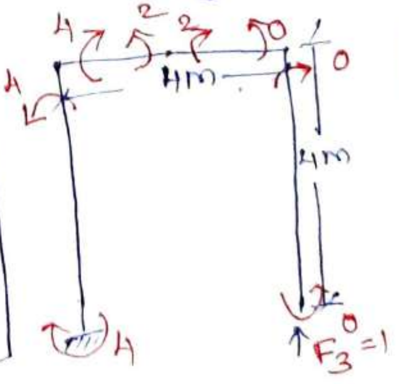




$$\begin{bmatrix} -2 \\ 2 \\ -2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



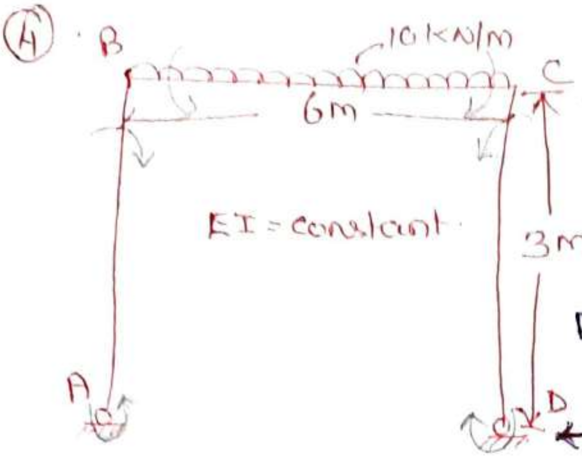
$$\begin{bmatrix} +0 \\ 4 \\ -4 \\ 4 \\ -4 \\ 4 \\ -4 \\ 0 \end{bmatrix}$$



$$\begin{bmatrix} 4 \\ -4 \\ 4 \\ -2 \\ +2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

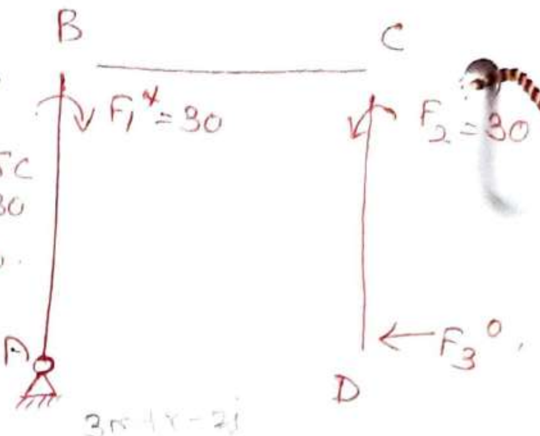
$$[b] = \left[\begin{array}{c|cc} -2 & 0 & 4 \\ 2 & 4 & -4 \\ -2 & -4 & 4 \\ 0 & 4 & -2 \\ 0 & -4 & 2 \\ 0 & 4 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

④



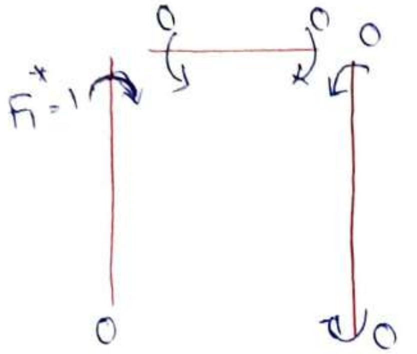
$M_{FAB} = M_{FBA} = M_{FCD}$
$= M_{FDC} = 0$ EJC
$M_{FBC} = -30$ +30
$M_{FCB} = 30$ -30

$\leftarrow H_D$(Redundant).

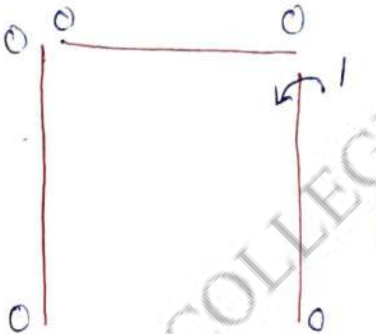


3m + r - 3j
9 + 4 - 12 = 0

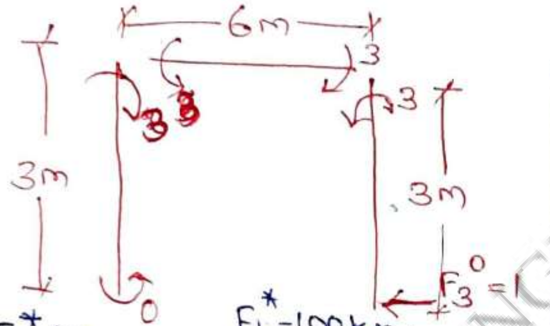
Redundant = 1.



$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

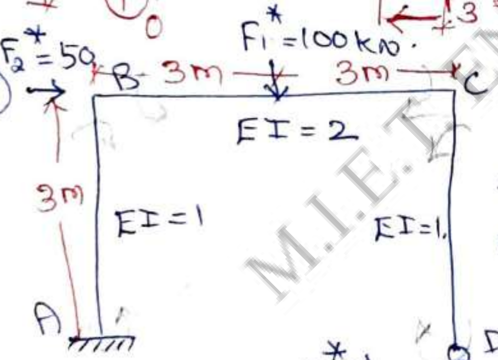


$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$



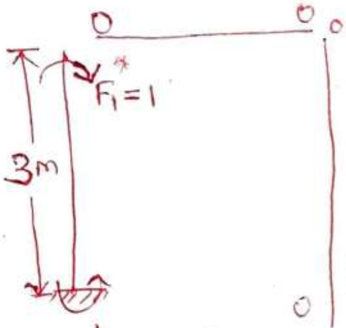
$$\begin{bmatrix} 0 \\ +3 \\ -3 \\ 3 \\ -3 \\ 0 \end{bmatrix}$$

⑤

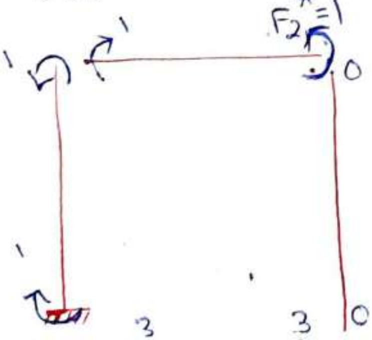


3m + r - 3j
9 + 5 - 12
= 2

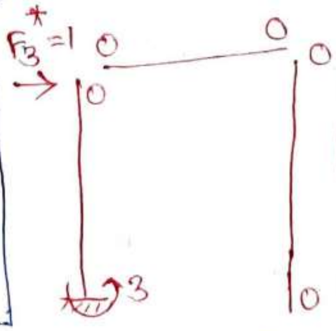
$M_{F_{AB}} = M_{FBC}$
$= M_{FCD} = M_{FDC} = 0$ EJF
$M_{FBC} = -75$ kNm +75
$M_{FCB} = 75$ kNm. -75



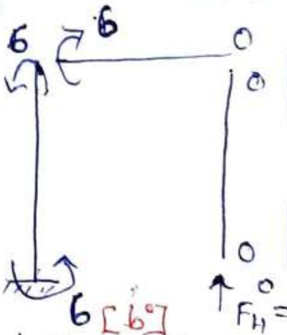
$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



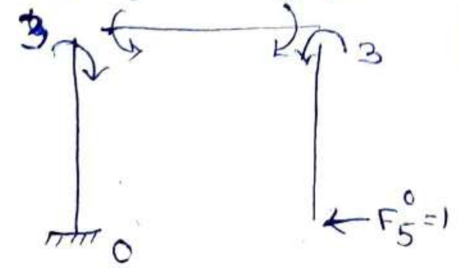
$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$



$$\begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



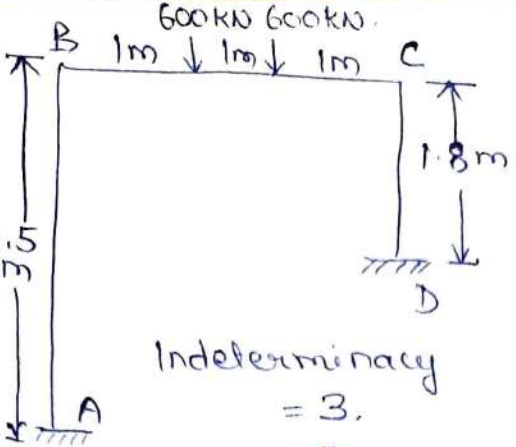
$$\begin{bmatrix} -6 \\ 6 \\ -6 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



$$\begin{bmatrix} 0 \\ 3 \\ -3 \\ 3 \\ -3 \\ 0 \end{bmatrix}$$

$$[b] = \begin{array}{c} \begin{array}{cc} [b^*] & 6\,[b^0] \end{array} \\ \left[\begin{array}{ccc|cc} -1 & 1 & -3 & -6 & 0 \\ 1 & -1 & 0 & 6 & 3 \\ 0 & 1 & 0 & -6 & -3 \\ 0 & -1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

⑥



Indeterminacy = 3.

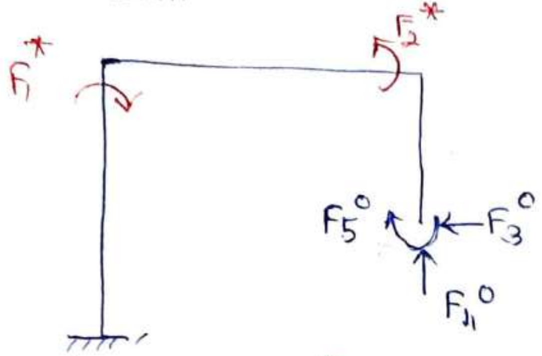
$M_{FAB} = M_{FBA} = M_{FCD} = M_{FDC} = 0.$

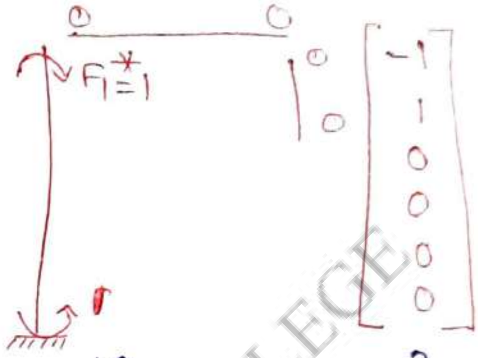
$M_{FBC} = -\frac{600\times1\times2^2}{3^2} - \frac{600\times2\times1^2}{3^2} = -400$ kNm.

+400

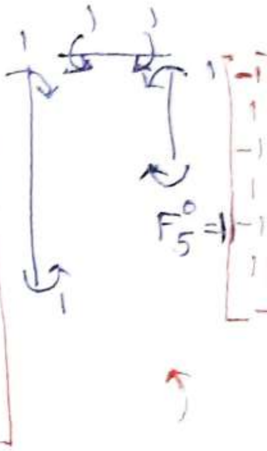
$M_{FCB} = 400$ kNm.

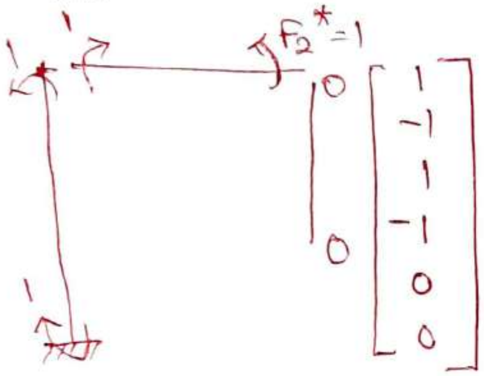
-400



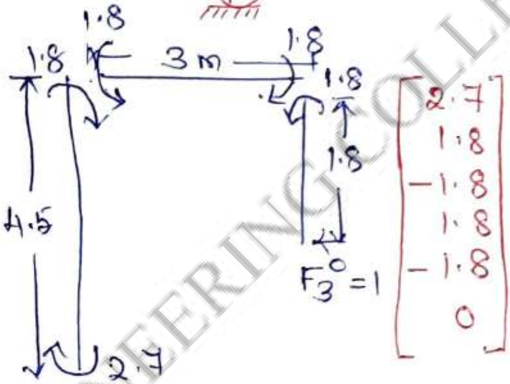


$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

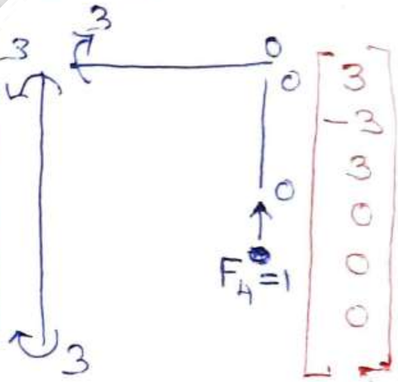




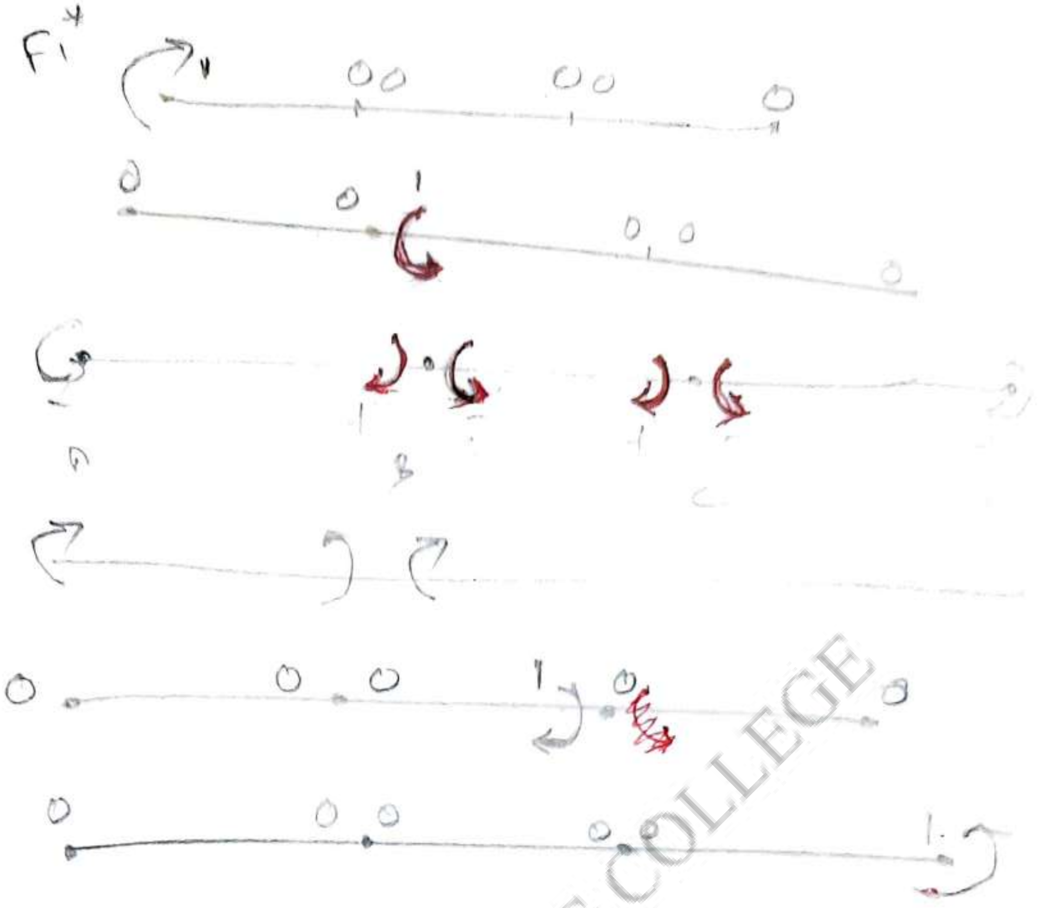
$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$



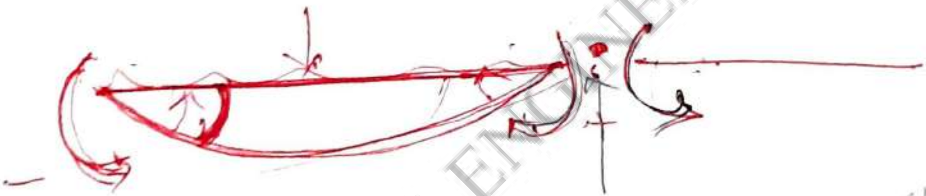
$$\begin{bmatrix} 2.7 \\ 1.8 \\ -1.8 \\ 1.8 \\ -1.8 \\ 0 \end{bmatrix}$$



$$\begin{bmatrix} 3 \\ -3 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



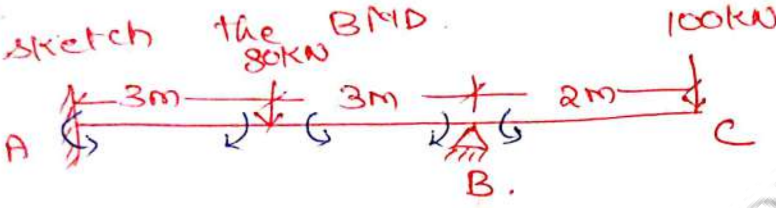
anticlock



UNIT V : STIFFNESS MATRIX METHOD.

Element and global stiffness matrices - Analysis of continuous beams - co-ordinates transformations - Rotation matrix - Transformations of stiffness matrices, load vectors and displacements vectors - Analysis of pin-jointed plane frames and rigid frames (with redundancy limited to two).

1. Analyse the beam loaded as shown by matrix stiffness and sketch the BMD.



Steps: [P°]

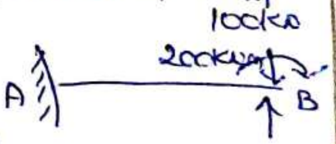
1. Find [B].
2. Find [F°]
3. Find [Ff]
4. Find [k]
5. Find [K]
6. Find {u}.
7. {S}.
8. {P'}.
9. (Pf).
10. Draw BMD.

Fixed end moments:

$$M_{FAB} = -\frac{Wl}{8} = \frac{-80\times6}{8} = -60 \text{ KNm.}$$

$$M_{FBA} = \frac{Wl}{8} = \frac{80\times6}{8} = +60 \text{ KNm.}$$

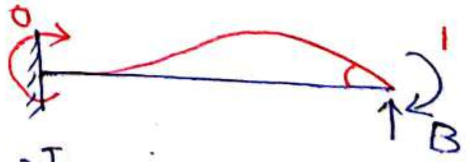
Load in BC will be equivalent to 100 kN download and a moment at B equal to $M_B = +Wl = 100\times2 = 200\text{ kN}$.



Forces in the element co-ordinates at fixed state (FEM)

$$\{P^o\} = \begin{bmatrix} -60 \\ 60 \end{bmatrix}$$

Transformation matrix $[B] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



Fixed co-ordinate forces $\{F^o\} = \{B\}^T * \{P^o\}$

$$= [0 \quad 1] \begin{Bmatrix} -60 \\ 60 \end{Bmatrix}$$

$$= 60.$$

Force applied at the system co-ordinates $[F^F] = \{200\}$

$$[F] = [F^F] - [F^o]$$

$$= [200] - [60] = [140]$$

Element stiffness matrix $[k] = \frac{EI}{l}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} = \frac{EI}{6}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

System stiffness matrix $[K] = [\beta]^T [k][\beta]$.

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{EI}{6}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{EI}{6}\begin{bmatrix} 2 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{2EI}{3}.$$

$$[K]^{-1} = \frac{3}{2EI}$$

System displacement $\{u\} = [K]^{-1}\{F\}$

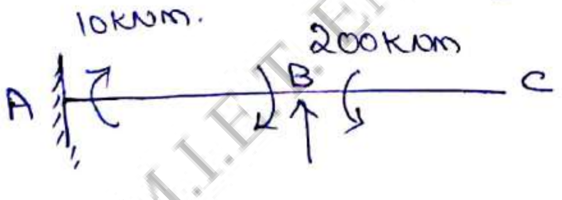
$$= \left[\frac{3}{2EI}\right][140] = \left[\frac{210}{EI}\right]$$

Element forces $[P'] = [k][\beta][u]$

$$= \frac{EI}{6}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\left[\frac{210}{EI}\right] = \begin{bmatrix} 70 \\ 140 \end{bmatrix}.$$

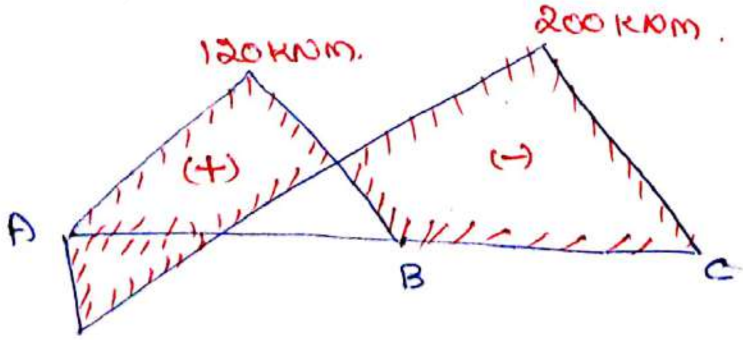
Final forces, $[P]^F = [P]^0 + [P']$

$$= \begin{bmatrix} -60 \\ 60 \end{bmatrix} + \begin{bmatrix} 70 \\ 140 \end{bmatrix} = \begin{bmatrix} 10 \\ 200 \end{bmatrix}.$$



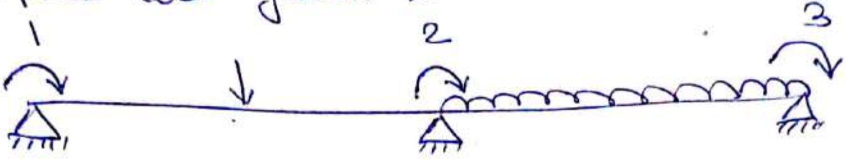
Free BM, $M'_{AB} = \frac{Wab}{l}$

$$= \frac{80\times3\times3}{6} = 120\ kNm.$$

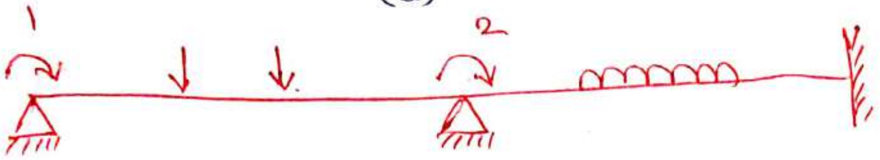


Bending Moment diagram.

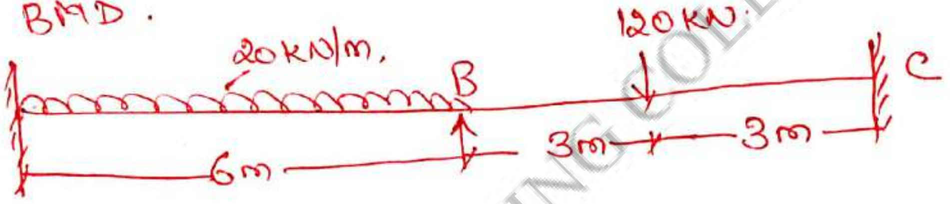
Assign system co-ordinate number to every nodal degree of freedom - rotational or translation. Few examples are given below.



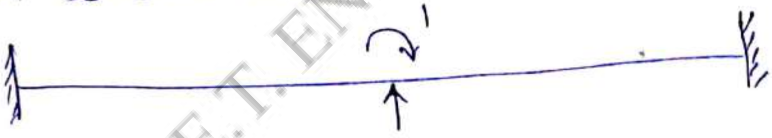
(a)



② Analyse the continuous beam by stiffness method. Draw BMD.



The structure is kinematically indeterminate to first degree ie θ_B is an unknown independent displacement (system co-ordinate).



Fixed end moments:

$$M_{FAB} = -\frac{wl^2}{12} = \frac{-20\times 6^2}{12} = -60 \text{ KNm.}$$

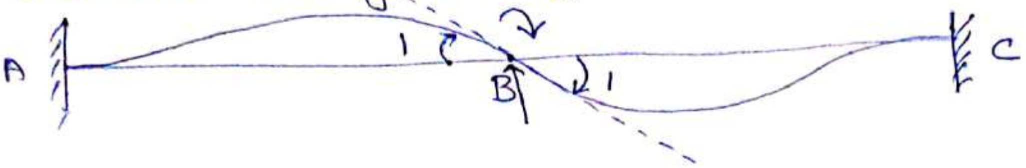
$$M_{FBA} = +\frac{wl^2}{12} = \frac{20\times 6^2}{12} = +60 \text{ KNm.}$$

$$M_{FBC} = -\frac{Wl}{8} = \frac{-120\times 6}{8} = -90 \text{ KNm.}$$

$$M_{FCB} = \frac{Wl}{8} = \frac{120\times 6}{8} = 90 \text{ KNm.}$$

$$[P^0] = \begin{bmatrix} -60 \\ 60 \\ -90 \\ 90 \end{bmatrix}$$

To form [β]: Applying unit ~~displa~~ rotation at the system co-ordinate, the resultant displacements at element co-ordinates are given by [β].



$$[\beta] = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$[F^0] = [\beta]^T [P]^0$$

$$= \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -60 \\ 60 \\ -90 \\ 90 \end{bmatrix} = [-30]$$

Forces applied at system co-ordinate $[F^f] = 0$

(∵ there is no external force at system co-ordinate).

Element stiffness matrix $[K_n]$:

$$[k_1] = \frac{EI}{l}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} = \frac{EI}{6}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} = EI\begin{bmatrix} 0.67 & 0.33 \\ 0.33 & 0.67 \end{bmatrix}$$

$$[k_1] = [k_2] = EI\begin{bmatrix} 0.67 & 0.33 \\ 0.33 & 0.67 \end{bmatrix}$$

Assembled element stiffness matrix [k]:

$$[k] = EI\begin{bmatrix} 0.67 & 0.33 & 0 & 0 \\ 0.33 & 0.67 & 0 & 0 \\ 0 & 0 & 0.67 & 0.33 \\ 0 & 0 & 0.33 & 0.67 \end{bmatrix}$$

System stiffness matrix [K]: $[\beta]^T [k] [\beta]$

$$[K] = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix} EI \begin{bmatrix} 0.67 & 0.33 & 0 & 0 \\ 0.33 & 0.67 & 0 & 0 \\ 0 & 0 & 0.67 & 0.33 \\ 0 & 0 & 0.33 & 0.67 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$[K] = EI\begin{bmatrix} 0.33 & 0.67 & 0.67 & 0.33 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = EI\,[0.67 + 0.67]$$

$$[K] = 1.34\,EI$$

$$[K]^{-1} = \frac{1}{1.34EI}$$

System displacement $\{u\} = [K]^{-1}\{[F^f]-[F^0]\}$

$$\{u\} = \frac{1}{1.34EI}\left[0-[-30]\right]$$

$$\{u\} = \frac{30}{1.34EI} = \frac{22.39}{EI} .$$

Element displacement $[\delta]$: $= [\beta]\{u\}$

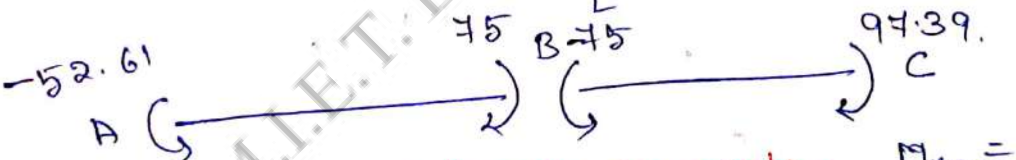
$$= \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \times \frac{22.39}{EI} = \frac{1}{EI}\begin{bmatrix}0\\22.39\\22.39\\0\end{bmatrix}$$

$$[P'] = [k][\delta] = EI\begin{bmatrix}0.67 & 0.33 & 0 & 0\\0.33 & 0.67 & 0 & 0\\0 & 0 & 0.67 & 0.33\\0 & 0 & 0.33 & 0.67\end{bmatrix}\frac{1}{EI}\begin{bmatrix}0\\22.39\\22.39\\0\end{bmatrix}$$

$$= \begin{bmatrix}7.39\\15.00\\15.00\\7.39\end{bmatrix}$$

Final forces. $[P^f] = [P^0] + [P']$

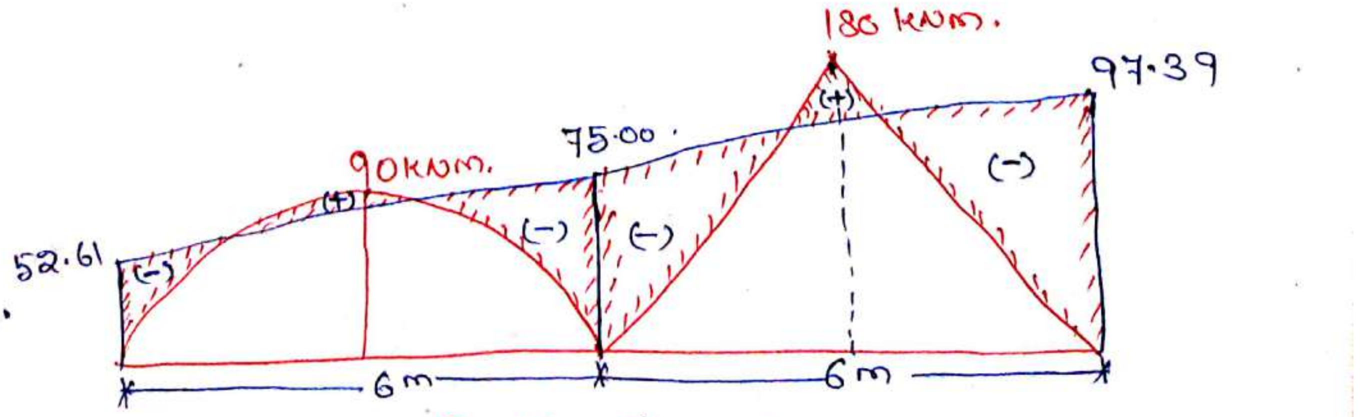
$$= \begin{bmatrix}-60\\60\\-90\\90\end{bmatrix} + \begin{bmatrix}7.39\\15.00\\15.00\\7.39\end{bmatrix} = \begin{bmatrix}-52.61\\75.00\\-75.00\\97.39\end{bmatrix} \text{KNm.}$$



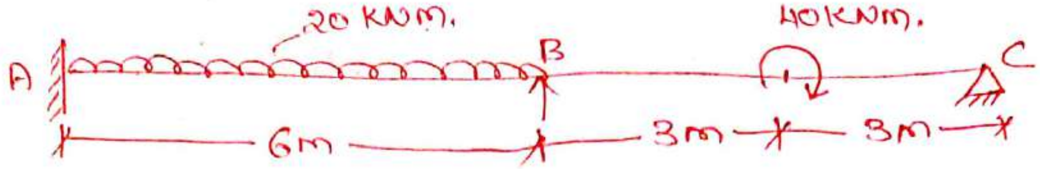
Simply supported beam moments :

$$M_{AB} = \frac{wl^2}{8} = \frac{20\times6^2}{8} = 90 \text{ KNm.}$$

$$M_{BA} = \frac{Wl}{4} = \frac{120\times6}{4} = 180 \text{ KNm}$$



Bending Moment

③


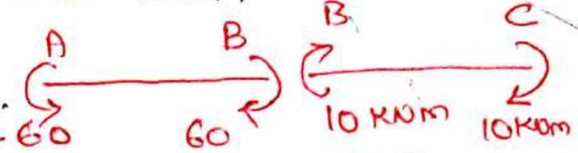
θ_B and θ_C are unknown displacements – KI = 2

∴ Stiffness matrix K will be of order 2×2.

Fixed end moments:

$$M_{FAB} = \frac{-wl^2}{12} = \frac{-20\times 6^2}{12} = -60 \text{ KNm}.$$

$$M_{FBA} = \frac{wl^2}{12} = +60 \text{ KNm}.$$

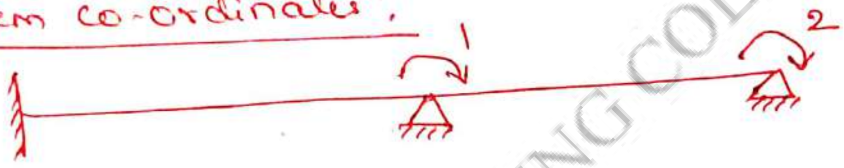


$$M_{FBC} = \frac{M}{4} = \frac{40}{4} = 10 \text{ KNm}.$$

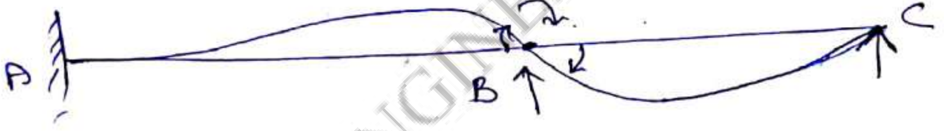
$$M_{FCB} = \frac{M}{4} = \frac{40}{4} = 10 \text{ KNm}.$$

$$[P^0] = \begin{bmatrix} -60 \\ 60 \\ 10 \\ 10 \end{bmatrix}$$

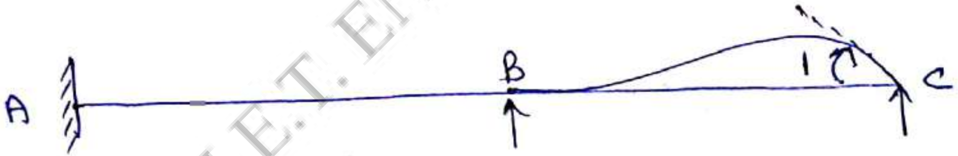
System co-ordinates:



Formation of [B] matrix:



$$\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$



$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Fixed co-ordinate forces $[F^0] = [B]^T [P^0]$

$$= \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -60 \\ 60 \\ 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 70 \\ 10 \end{bmatrix}.$$

Forces applied at system co-ordinates.

$$[F^f] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ (∵ no forces in the given loading at the system co-ordinates)

Element stiffness matrix [ki] :

$$[k_1] = \frac{EI}{l}\begin{bmatrix}4 & 2\\ 2 & 4\end{bmatrix} = \frac{EI}{6}\begin{bmatrix}4 & 2\\ 2 & 4\end{bmatrix} = EI\begin{bmatrix}4/6 & 2/6\\ 2/6 & 4/6\end{bmatrix}$$

$$[k_1] = EI\begin{bmatrix}0.667 & 0.333\\ 0.333 & 0.667\end{bmatrix}$$

$$[k_2] = \frac{EI}{6}\begin{bmatrix}4 & 2\\ 2 & 4\end{bmatrix} = EI\begin{bmatrix}0.667 & 0.333\\ 0.333 & 0.667\end{bmatrix}$$

Assembled element stiffness matrix:

$$[k] = EI\begin{bmatrix}0.667 & 0.333 & 0 & 0\\ 0.333 & 0.667 & 0 & 0\\ 0 & 0 & 0.667 & 0.333\\ 0 & 0 & 0.333 & 0.667\end{bmatrix}$$

System stiffness matrix : $[K] = [B]^T[k][B]$

$$[K] = \begin{bmatrix}0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0.667 & 0.333 & 0 & 0\\ 0.333 & 0.667 & 0 & 0\\ 0 & 0 & 0.667 & 0.333\\ 0 & 0 & 0.333 & 0.667\end{bmatrix}\begin{bmatrix}0 & 0\\ 1 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix}$$

$$[K] = EI\begin{bmatrix}1.333 & 0.333\\ 0.333 & 0.664\end{bmatrix} ; \quad [K]^{-1} = \frac{1}{EI}\begin{bmatrix}0.857 & -0.429\\ -0.429 & 1.714\end{bmatrix}$$

$$(0.889 - 0.11) = 0.778$$

System displacement $\{u\} = [K]^{-1}\{[F^f] - [F^0]\}$

$$= \frac{1}{EI}\begin{bmatrix}0.857 & -0.429\\ -0.429 & 1.714\end{bmatrix}\left\{\begin{bmatrix}0\\ 0\end{bmatrix} - \begin{bmatrix}70\\ 10\end{bmatrix}\right\}$$

$$\{u\} = \frac{1}{EI}\begin{bmatrix}-55.71\\ +12.86\end{bmatrix}$$

Element displacement {δ} are $\{\delta\} = [B][u]$

$$\{\delta\} = \begin{bmatrix}0 & 0\\ 1 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix}\frac{1}{EI}\begin{bmatrix}-55.71\\ +12.86\end{bmatrix} = \frac{1}{EI}\begin{bmatrix}0\\ -55.71\\ -55.71\\ 12.86\end{bmatrix}$$

The element forces $\{P'\} = [k]\{\delta\}$

$$\{P'\} = EI \begin{bmatrix} 0.667 & 0.333 & 0 & 0 \\ 0.333 & 0.667 & 0 & 0 \\ 0 & 0 & 0.667 & 0.333 \\ 0 & 0 & 0.333 & 0.667 \end{bmatrix}_{4\times4} \frac{1}{EI} \begin{bmatrix} 0 \\ -55.71 \\ -55.71 \\ 12.86 \end{bmatrix}_{4\times1}$$

$$\{P'\} = \begin{bmatrix} -18.57 \\ -37.14 \\ -32.86 \\ -10 \end{bmatrix}_{4\times1}$$

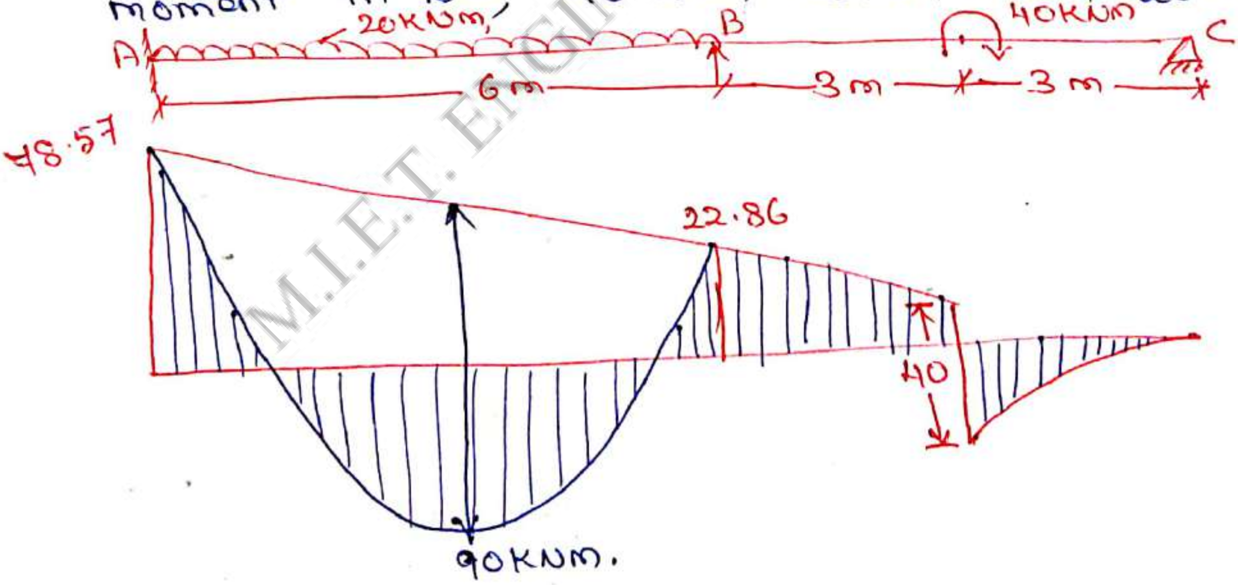
The final forces $\{P^f\} = \{P^o\} + \{P'\}$.

$$= \begin{bmatrix} -60 \\ 60 \\ 10 \\ 10 \end{bmatrix} + \begin{bmatrix} -18.57 \\ -37.4 \\ -32.86 \\ -10 \end{bmatrix} = \begin{bmatrix} -78.57 \\ 22.86 \\ -22.86 \\ 0 \end{bmatrix}$$

simply supported beam moments:

moment in AB $= \frac{wl^2}{8} = \frac{20\times6^2}{8} = 90$ KNm.

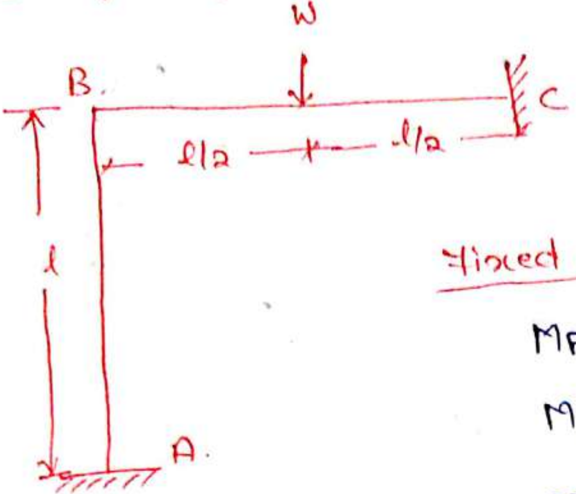
moment in BC, 40 kNm concentrated at mid span.



Bending moment diagram.

Analysis of rigid jointed frames:

④



KI = 1. θ_B = unknown

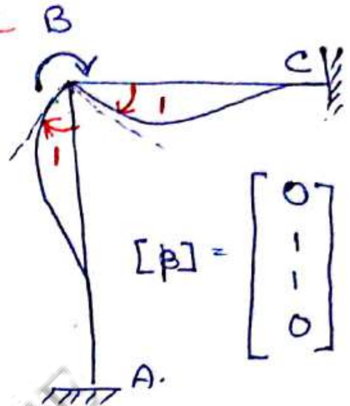
stiffness matrix: 1×1.

Fixed end moments:

$M_{FAB} = M_{FBA} = 0.$

$M_{FBC} = -\frac{Wl}{8}$ #

$M_{FCB} = \frac{Wl}{8}$



$$[\beta] = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Fixed end co-ordinate forces $[F^0] = [\beta]^T [P^0]$

$$[P^0] = \begin{bmatrix} 0 \\ 0 \\ -Wl/8 \\ Wl/8 \end{bmatrix}$$

$$[F^0] = [0 \quad 1 \quad 1 \quad 0] \begin{bmatrix} 0 \\ 0 \\ -Wl/8 \\ Wl/8 \end{bmatrix}$$

$[F^0] = -\frac{Wl}{8}$. Forces applied at system co-ordinates $[F^f] = [0]$.

(∵ there is no external forces at the system co-ordinates).

$[F^f] - [F^0] = [0] - [-\frac{Wl}{8}]$

$= +\frac{Wl}{8}$.

Element stiffness matrix $[k_n]$:

$$[k_1] = \frac{EI}{l}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}; \quad [k_2] = \frac{EI}{l}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}.$$

Assembled element stiffness matrix $[k]$:

$$[k] = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} = \frac{EI}{l}\begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

System stiffness matrix [K] : $[K] = [\beta]^T [k] [\beta]$.

$$[K] = [0\ \ 1\ \ 1\ \ 0] \times \frac{EI}{l}\begin{bmatrix}4&2&0&0\\2&4&0&0\\0&0&4&2\\0&0&2&4\end{bmatrix}\begin{bmatrix}0\\1\\1\\0\end{bmatrix}$$

$$= \frac{EI}{l}[2\ \ 4\ \ 4\ \ 2]\begin{bmatrix}0\\1\\1\\0\end{bmatrix} = \frac{EI}{l}[8].$$

System displacements {u} : $\{u\} = [K]^{-1}\{[F^f] - [F^0]\}$

$$[K]^{-1} = \frac{l}{8EI}$$

$$\{u\} = \frac{l}{8EI} \times \frac{Wl}{8} = \frac{Wl^2}{64EI}.$$

Element displacements {δ} : $\{\delta\} = [\beta]\{u\}$.

$$\{\delta\} = \begin{bmatrix}0\\1\\1\\0\end{bmatrix}\frac{Wl^2}{64EI}[1]$$

$[P'] = [k][\delta]$

$$= \frac{EI}{l}\begin{bmatrix}4&2&0&0\\2&4&0&0\\0&0&4&2\\0&0&2&4\end{bmatrix} \times \frac{Wl^2}{64EI}\begin{bmatrix}0\\1\\1\\0\end{bmatrix}$$

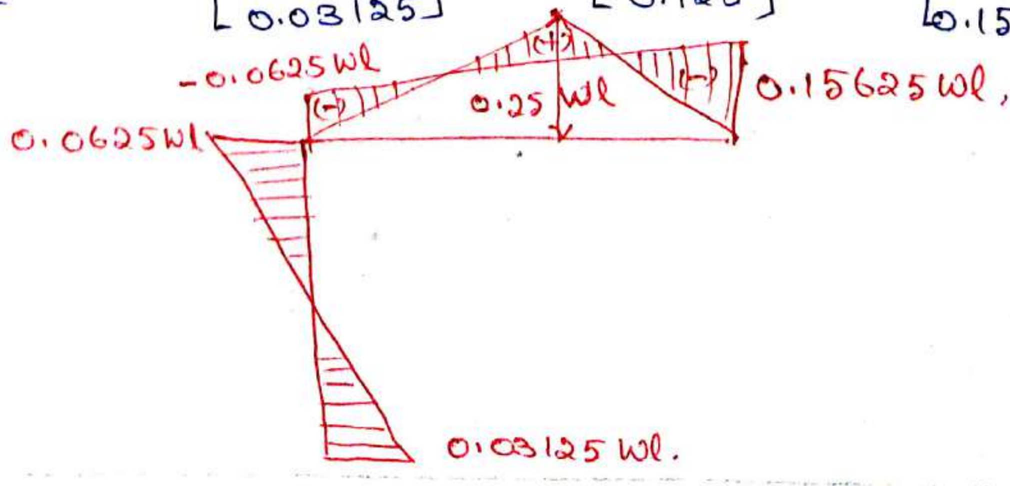
$$[P'] = \frac{Wl}{64}\begin{bmatrix}2\\4\\4\\2\end{bmatrix}$$

Final forces $[P^f] = [P'] + [P^0]$

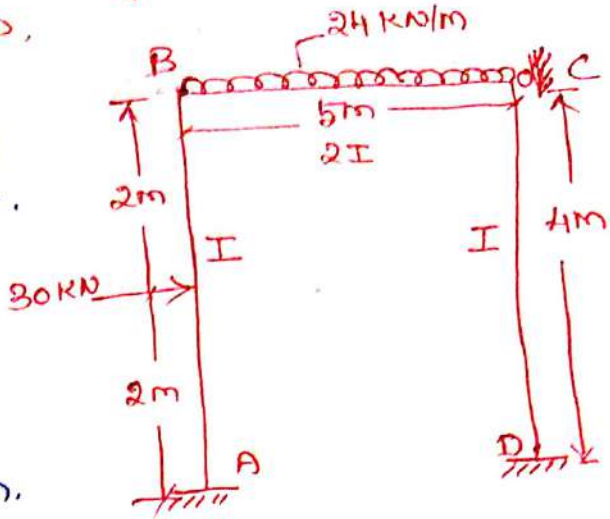
$$[P^f] = \frac{Wl}{64}\begin{bmatrix}2\\4\\4\\2\end{bmatrix} + \begin{bmatrix}0\\0\\-Wl/8\\Wl/8\end{bmatrix}$$

SSBM:
$M_{AB} = 0$
$M_{BC} = \frac{Wl}{4}$

$$[P^f] = Wl\begin{bmatrix}0.03125\\0.0625\\0.0625\\0.03125\end{bmatrix} + Wl\begin{bmatrix}0\\0\\-0.125\\0.125\end{bmatrix} = Wl\begin{bmatrix}0.03125\\0.0625\\-0.0625\\0.15625\end{bmatrix}$$



⑤ Analyse the foll structure by matrix stiffness method and sketch the BMD.



This structure is kinematically indeterminate to second degree. ie rotation at B & C (sway arrested by support C.

Fixed end moments:

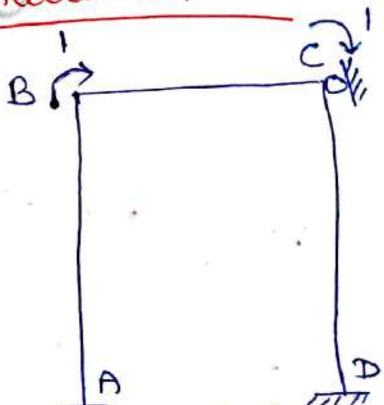
$M_{FAB} = -\frac{Wl}{8} = \frac{-30\times4}{8} = -15$ KNm.

$M_{FBA} = +\frac{Wl}{8} = 15$ KNm,

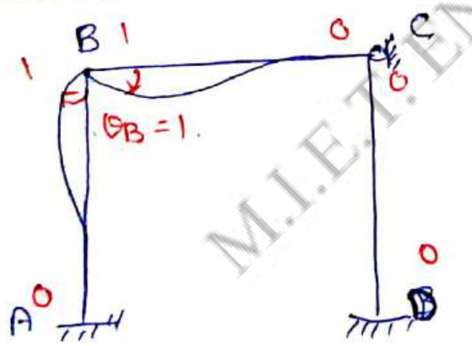
$M_{FBC} = -\frac{wl^2}{12} = \frac{-24\times5^2}{12} = -50$ KNm. ; $M_{FCB} = +50$ KNm.

$$\{P^o\} = \begin{bmatrix} -15 \\ 15 \\ -50 \\ 50 \\ 0 \\ 0 \end{bmatrix}$$

Transformation matrix [B]:

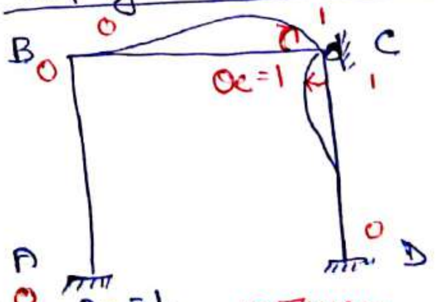


apply unit rotation at B:



$$\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0. \end{bmatrix}$$

apply unit rotation at C:

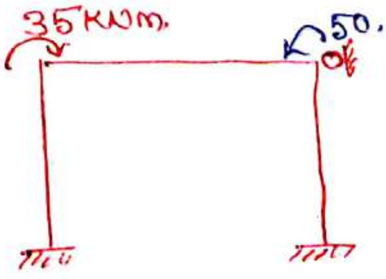


$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Transformation matrix [B]:

$$\begin{array}{cc} \theta_B = 1 & \theta_c = 1. \\ \left[\begin{array}{c} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{array}\right. & \left.\begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{array}\right] \end{array}$$

$$[F^o] = [B]^T [P^o] = \begin{bmatrix} -35 \\ 50 \end{bmatrix}$$



$M_B = M_{FBA} + M_{FBC}$

$= 15 - 50 = -35$ KNm. EJL = 35 KNm ,clockwise.

$M_c = M_{FCB} + M_{FCD} = +50 + 0 = 50$ EJL = −50 anticlockwise.

$$\{F\} = \begin{Bmatrix} 35 \\ -50 \end{Bmatrix} \quad [F^o] = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -15 \\ 15 \\ 50 \\ -50 \\ 0 \\ 0 \end{bmatrix}$$

Element stiffness matrix: $[k] = \frac{EI}{l}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

$[k_1] = \frac{EI}{4}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$; $[k_2] = \frac{E(2I)}{5}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$; $[k_3] = \frac{EI}{4}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

Assembled element stiffness matrix:

$$[k] = \frac{EI}{20}\begin{bmatrix} 20 & 10 & 0 & 0 & 0 & 0 \\ 10 & 20 & 0 & 0 & 0 & 0 \\ 0 & 0 & 32 & 16 & 0 & 0 \\ 0 & 0 & 16 & 32 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20 & 10 \\ 0 & 0 & 0 & 0 & 10 & 20 \end{bmatrix}$$

System stiffness matrix: $[K] = [B]^T[k][B]$.

$$[K] = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \frac{EI}{20}\begin{bmatrix} 20 & 10 & 0 & 0 & 0 & 0 \\ 10 & 20 & 0 & 0 & 0 & 0 \\ 0 & 0 & 32 & 16 & 0 & 0 \\ 0 & 0 & 16 & 32 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20 & 10 \\ 0 & 0 & 0 & 0 & 10 & 20 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$[K] = \frac{EI}{20}\begin{bmatrix} 52 & 16 \\ 16 & 52 \end{bmatrix}$$

$$[K]^{-1} = \frac{20}{EI}\begin{bmatrix} 0.0212 & -0.0065 \\ -0.0065 & 0.0212 \end{bmatrix}.$$

System displacements: $\{u\} = [K]^{-1}\{F\}$.

$$= \frac{20}{EI}\begin{bmatrix} 0.0212 & -0.0065 \\ -0.0065 & 0.0212 \end{bmatrix}\begin{Bmatrix} 35 \\ -50 \end{Bmatrix}.$$

$$= \frac{20}{EI}\begin{Bmatrix} 1.067 \\ -1.2875 \end{Bmatrix}.$$

Element forces: $[P'] = [k][B]\{u\}$

$$[P'] = \frac{EI}{20}\begin{bmatrix} 20 & 10 & 0 & 0 & 0 & 0 \\ 10 & 20 & 0 & 0 & 0 & 0 \\ 0 & 0 & 32 & 16 & 0 & 0 \\ 0 & 0 & 16 & 32 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20 & 10 \\ 0 & 0 & 0 & 0 & 10 & 20 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\frac{20}{EI}\begin{Bmatrix} 1.067 \\ -1.2875 \end{Bmatrix}$$

$$\{P'\} = \begin{bmatrix} 10 & 0 \\ 20 & 0 \\ 32 & 16 \\ 16 & 32 \\ 0 & 20 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} 1.067 \\ -1.287 \end{bmatrix} = \begin{bmatrix} 10.67 \\ 21.34 \\ 13.55 \\ -24.2 \\ -25.8 \\ -12.9 \end{bmatrix}$$

$$\{P^f\} = \{P'\} + \{P^o\}$$

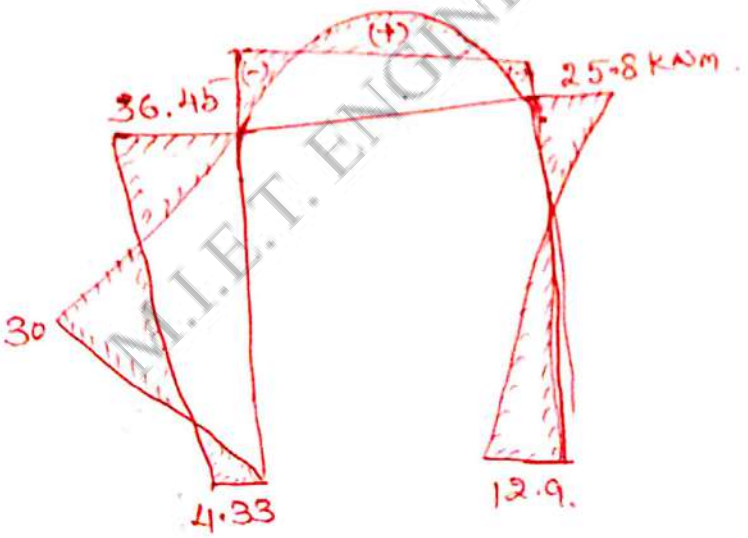
$$= \begin{bmatrix} 10.67 \\ 21.34 \\ 13.55 \\ -24.2 \\ -25.8 \\ -12.9 \end{bmatrix} + \begin{bmatrix} -15 \\ 15 \\ -50 \\ 50 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -4.33 \\ 36.45 \\ -36.45 \\ 25.8 \\ -25.8 \\ -12.9 \end{bmatrix}$$

S/S span Bending moment:

BM in span AB $= \frac{Wl}{4} = \frac{30 \times 4}{4} = 30$ KNm.

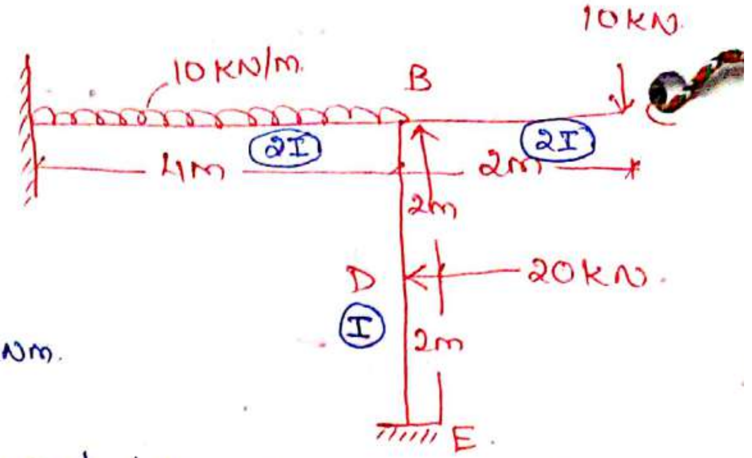
BM in span BC $= \frac{wl^2}{8} = \frac{24 \times 5^2}{8} = 75$ KNm.

BM in span CD = 0 (No external load).



Bending moment diagram (BMD).

⑥ Kinematic Indeterminacy = 1.
i.e rotation at B



Fixed end moments:

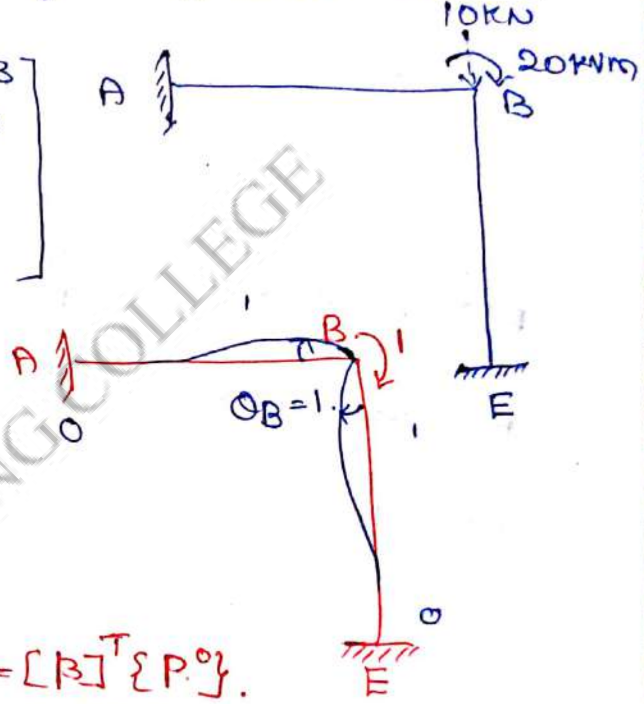
$M_{FAB} = -\frac{wl^2}{12} = -\frac{10\times4^2}{12} = -13.33$ kNm.

$M_{FBA} = \frac{wl^2}{12} = \frac{10\times4^2}{12} = 13.33$ kNm.

Force at C will be = downward force of 10 kN and a moment of 20 kNm at B.

Fixed end moments $\{P^0\} = \begin{bmatrix}-13.33\\13.33\\-10\\10\end{bmatrix}$

$\{F^f\} = 10\times2 = 20$



Transformation matrix [B]:

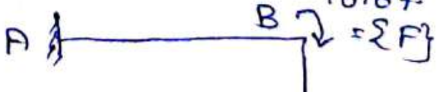
$[B] = \begin{bmatrix}0\\1\\1\\0\end{bmatrix}$

Fixed co-ordinate forces $\{F^0\} = [B]^T\{P^0\}$.

$= \begin{bmatrix}0 & 1 & 1 & 0\end{bmatrix}\begin{bmatrix}-13.33\\13.33\\-10\\10\end{bmatrix} = \{3.33\}$.

Forces applied at system co-ordinate $\{F^f\} = \{20\}$.

$\{F\} = \{F^f\} - \{F^0\} = \{20\} - \{3.33\} = \{16.67\}$

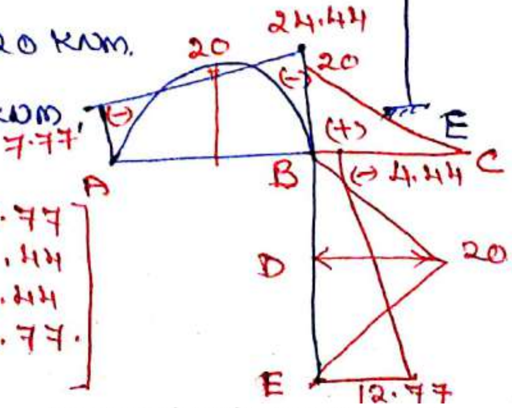


S/S span moments:

BM in span AB $= \frac{wl^2}{8} = \frac{10\times4^2}{8} = 20$ kNm.

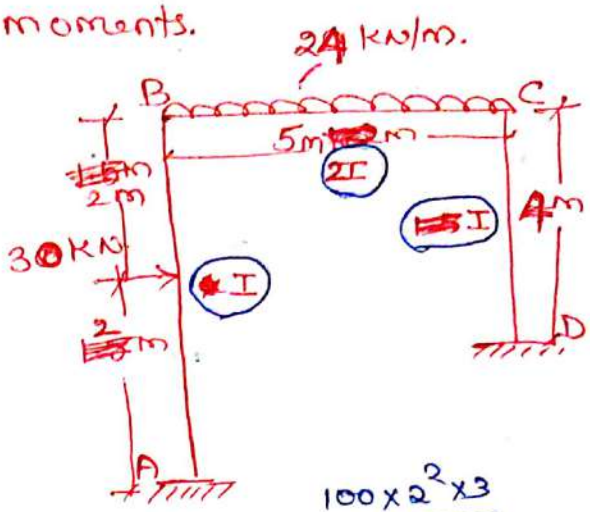
" " " BC $= -Wl = -10\times2 = -20$ kNm.

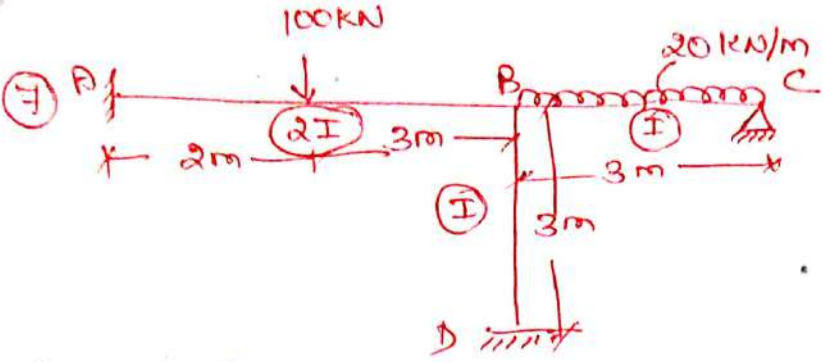
" " " BE $= \frac{Wl}{4} = \frac{20\times4}{4} = 20$ kNm.



$\{P^f\} = \{P'\} + \{P^0\} = \begin{bmatrix}5.55\\11.11\\5.55\\2.77\end{bmatrix} + \begin{bmatrix}-13.33\\13.33\\-10\\10\end{bmatrix} = \begin{bmatrix}-7.77\\24.44\\-4.44\\12.77\end{bmatrix}$

⑤ Analyse the foll frame by matrix stiffness method and obtain support / joint moments.





KI = 2 (rotation at B and C).

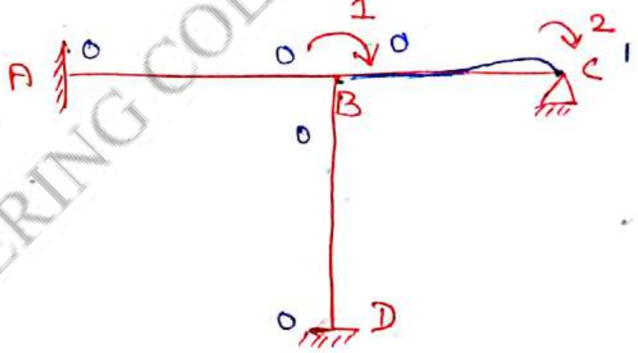
Fixed end moments:

$$\frac{100\times2^2\times3}{5^2}$$

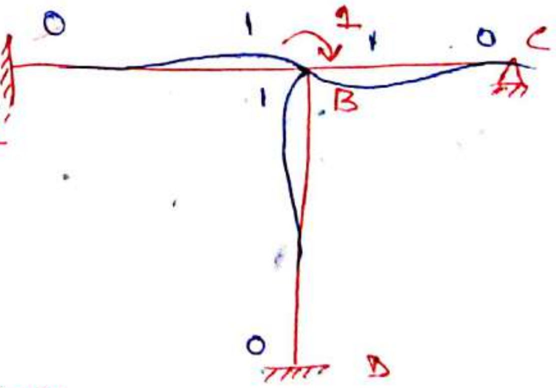
$$M_{FAB} = \frac{-Wab^2}{l^2} = \frac{-100\times2\times3^2}{5^2} = -72\,kNm, \quad M_{FBA} = \frac{Wa^2b}{l^2} = 48\ kNm.$$

$$M_{FBC} = \frac{-wl^2}{12} = \frac{-20\times3^2}{12} = -15\ kNm; \quad M_{FCB} = +15\ kNm.$$

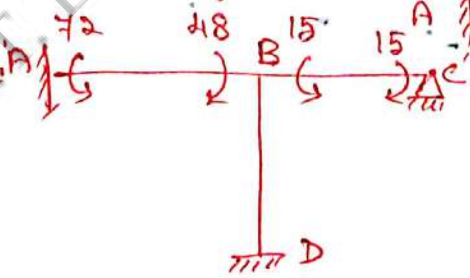
Transformation matrix [B]:



$$[B] = \begin{array}{c} \begin{array}{cc} \Delta_1 = 1 & \Delta_2 = 1 \end{array} \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array}$$



Fixed end moments



$$\{P^0\} = \begin{bmatrix} -72 \\ 48 \\ -15 \\ 15 \\ 0 \\ 0 \end{bmatrix}$$

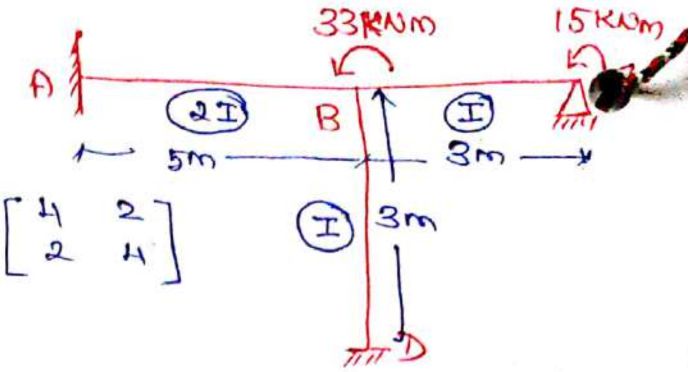
Fixed co-ordinate forces: $\{F^0\} = \{B\}^T\{P^0\}$.

$$= \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} -72 \\ 48 \\ -15 \\ 15 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 33 \\ 15 \end{Bmatrix}$$

Forces applied at system co-ordinate $\{F^f\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

$$\{F\} = \{F^f\} - \{F^0\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 33 \\ 15 \end{Bmatrix} = \begin{Bmatrix} -33 \\ -15 \end{Bmatrix}$$

Element stiffness matrix:



$$[k_i] = \frac{EI}{l}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

$$[k_1] = \frac{2EI}{5}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix};\quad [k_2] = \frac{EI}{3}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

$$[k_3] = \frac{EI}{3}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

$$[k] = \frac{EI}{15}\begin{bmatrix} 24 & 12 & 0 & 0 & 0 & 0 \\ 12 & 24 & 0 & 0 & 0 & 0 \\ 0 & 0 & 20 & 10 & 0 & 0 \\ 0 & 0 & 10 & 20 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20 & 10 \\ 0 & 0 & 0 & 0 & 10 & 20 \end{bmatrix}$$

System stiffness matrix: $[K] = [B]^T[k][B]$

$$[K] = \frac{EI}{15}\begin{bmatrix} 64 & 10 \\ 10 & 20 \end{bmatrix};\quad [K]^{-1} = \frac{15}{EI}\begin{bmatrix} \frac{1}{59} & -\frac{1}{118} \\ -\frac{1}{118} & \frac{16}{295} \end{bmatrix}$$

System displacements: $\{u\} = [K]^{-1}\{F\}$

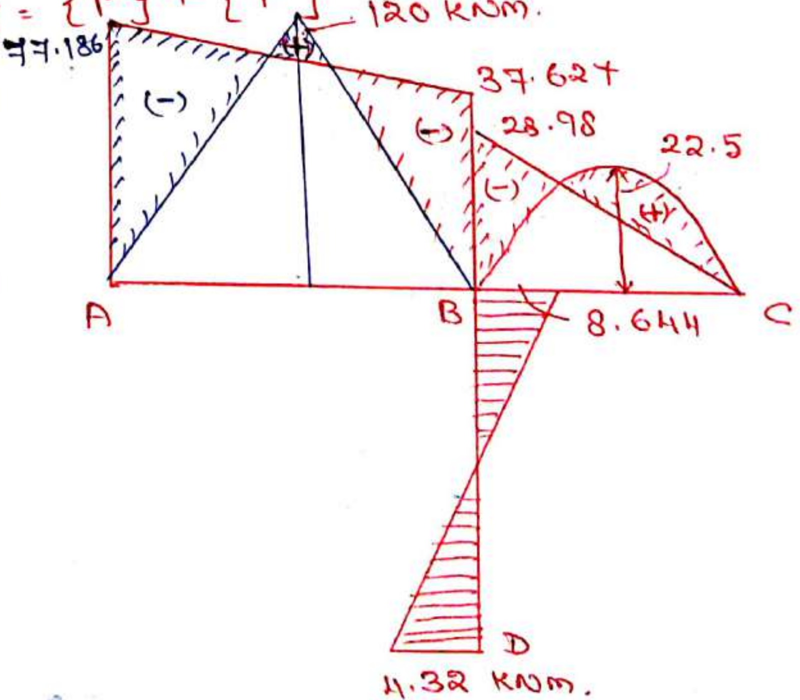
$$\{u\} = \frac{15}{EI}\begin{bmatrix} \frac{1}{59} & -\frac{1}{118} \\ -\frac{1}{118} & \frac{16}{295} \end{bmatrix}\begin{bmatrix} -33 \\ -15 \end{bmatrix} = \frac{15}{EI}\begin{bmatrix} -\frac{51}{118} \\ -\frac{63}{118} \end{bmatrix}.$$

Element forces: $\{P'\} = [k][B]\{u\}$

$$\{P^f\} = \{P'\} + \{P^0\}$$

$$\{P^f\} = \begin{bmatrix} -5.186 \\ -10.37 \\ -13.986 \\ -15 \\ -8.644 \\ -4.32 \end{bmatrix} + \begin{bmatrix} -72 \\ 48 \\ -15 \\ 15 \\ 0 \\ 0 \end{bmatrix}$$

$$\{P^f\} = \begin{bmatrix} -77.186 \\ 37.627 \\ -28.983 \\ 0 \\ -8.644 \\ -4.322 \end{bmatrix}$$



Bending moment diagram.

1. $[P^0] = [FEM]$

2. Find β matrix

3. $[F^0] = [\beta]^T [P^0]$

4. $[F^F]$ = forces applied at system co-ordinates.

5. $[k] = \frac{EI}{l} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

6. $[k] = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$

7. $[K] = [\beta]^T [k][\beta]$.

8. $\{u\} = [K]^{-1} \{[F^f] - [F^0]\}$

9. $\{\delta\} = [\beta]\{u\}$.

10. $\{P'\} = [k]\{\delta\}$.

11. $\{P^f\} = \{P^0\} + \{P'\}$.